Department of Mathematics and Statistics
Stochastic processes on domains
Excercise problem sheet 2
(To be returned by Friday 20.02.2015)

Note. In the Problems 1-10 the j,k and n are always integers.

1. Let $(P_{t,x})$ be as in Lemma 3.14 in Lecture notes (page 23). Let

$$\mu^x_{(t_1,\ldots,t_n)}(A_1 \times \cdots \times A_n) = \int_{A_1} P_{t_1,x}(\,\mathrm{d}x_1) \ldots \int_{A_n} P_{t_n - t_{n-1}, x_{n-1}}(\,\mathrm{d}x_n)$$

for every $x \in S$, for every $n \in \mathbb{N}_+$, for every $t_1 < \cdots < t_n$ and for every $A_1, \ldots, A_n \in \mathscr{S}$. Let $(A_n) \subset \mathscr{S}$ be a sequence of sets in $\mathscr{S}$. Define $B_{k,j}, C_{k,j} \in \mathscr{S}$ as $[\,B_{k,j}\,] = [\,A_k\,][\,k \neq j\,] + [\,S\,][\,k = j\,]$. and $[\,C_{k,j}\,] = [\,A_k\,][\,k < j\,] + [\,A_{k-1}\,][\,k > j\,]$. Let $\pi_{j,n}(t_1, \ldots, t_n) = (s_1, \ldots, s_{n-1})$ where $s_k = t_k[\,k < j\,] + t_{k-1}[\,k > j\,]$. Show that the measures $\mu^x_{(t_1,\ldots,t_n)}$ are consistent, i.e.

i) show that

$$\mu^x_{(t_1,\ldots,t_n)}(B_{1,j} \times \cdots \times B_{n,j}) = \mu^x_{\pi_{j,n}(t_1,\ldots,t_n)}(C_{1,j} \times \cdots \times C_{n-1,j})$$

for every $n \in \mathbb{N}$, for every $1 \leq j \leq n$.

2. Show that the transition probability operator of Brownian motion $P_t^{(B)}$ (see pages 21–22) satisfies

$$\lim_{t \to 0} P_t^{(B)}\, f(x) = f(x)$$

for every $x \in \mathbb{R}^d$ and every bounded and continuous f. (Hint: change of variables so that t appears only in the argument of f and dominated convergence).

3. Show that the transition probability operator of Brownian motion $P_t^{(B)}$ (see 2. above) satisfies

$$\lim_{t \to 0} \|\, P_t^{(B)}\, f - f\,\| = \lim_{t \to 0} \sup_{x \in \mathbb{R}^d} |P_t^{(B)} f(x) - f(x)| = 0$$

for every $x \in \mathbb{R}^d$ and every $f \in C_\infty(\mathbb{R}^d)$. (Hint: show that there is large R such that outside ball of radius R the difference is small and 2.)

Some theory. Suppose (X_t) is a Markov process and suppose there exists a *shift operators* $\theta_s\colon \Omega \to \Omega$ for every $s > 0$ with the property $X_t \circ \theta_s(\omega) = X_t(\theta_s(\omega)) := X_{t+s}(\omega)$ for every $t \geq 0$ and for every $\omega \in \Omega$. This way we can easily express that general idea of *forgetting the past* and the *restarting the clock.*
Note. In Problems 4, 5, 6 and 7 the triplet $(X_t, \mathscr{F}_t, \{\mathbf{P}_x\})$ is a Markov process. The algebra $\mathscr{A}_\infty$ and the σ-algebre $\mathscr{F}_\infty$ are defined in the proof of Lemma 3.14 on page 24. The Dynkin system is defined in the proof of Lemma 3.14 on page 25.

4. Let $s < t_1 < \cdots < t_n$. Show that the time stationary Markov property for (X_t) with respect to $(\mathscr{F}_t)$ implies that

$$\mathbf{E}_x(f_1(X_{t_1}) \dots f_n(X_{t_n}) \,|\, \mathscr{F}_s) = \mathbf{E}_{X_s}\left(f_1(X_{t_1-s}) \dots f_n(X_{t_n-s})\right)$$

for every $x \in S$ and for every f_j bounded and measurable function with $j = 1, \dots, n$. You can assume that this is know for $n = 1$.

5. Let $0 < t_1 < \cdots < t_n$ and let $A = \{X_{t_1} \in A_1, \dots, X_{t_n} \in A_n\} \in \mathscr{A}_\infty$. Show that the time stationary Markov property for (X_t) with respect to $(\mathscr{F}_t)$ implies that

$$\mathbf{E}_x([\,A\,] \circ \theta_s \,|\, \mathscr{F}_s) = \mathbf{E}_{X_s}[\,A\,] \tag{0.1}$$

for every $x \in S$. (Hint: try writing $[\,A\,] \circ \theta_s$ as a form of Problem 4. from the above definition of the shift operator)

6. Let $\mathscr{G}$ be the family of sets

$$\mathscr{G} = \{\ A \in \mathscr{F}_\infty :\ A \text{ satifies the identity (0.1) } \}$$

Show that $\mathscr{G}$ is a Dynkin system and that it coincides with $\mathscr{F}_\infty$. (Hint: see proof of Lemma 3.14).

7. Let Z be a $\mathscr{F}_\infty$-measurable bounded random variable. Show that the time stationary Markov property for (X_t) with respect to $(\mathscr{F}_t)$ implies that

$$\mathbf{E}_x(Z \circ \theta_s \,|\, \mathscr{F}_s) = \mathbf{E}_{X_s} Z$$

for every $s \geq 0$. (Hint: first simple Z, then the general case).

8. Suppose Z is an integrable real valued random variable and $(\mathscr{F}_t)$ is a filtration. Show that a process (X_t) which is *defined* as

$$X_t := \mathbf{E}\,(Z \,|\, \mathscr{F}_t)$$

is a martingale with respect to the filtration $(\mathscr{F}_t)$.

9. Let τ_1 and τ_2 be $(\mathscr{F}_t)$-stopping times. Show that

$$\tau_1 \wedge \tau_2, \quad \tau_1 \vee \tau_2, \quad \text{and} \quad \tau_1 + \tau_2$$

are $(\mathscr{F}_t)$-stopping times. (Hint: try to express the conditions as unions and intersections of conditions involving only τ_1 and τ_2. Also discrete time versions are fine.)

10. Show that the $\mathscr{F}_\tau$ is a σ-algebra, when τ is a $(\mathscr{F}_t)$-stopping time.

11. Let $(\mathscr{F}_t)$ be a filtration. Show that the $(\mathscr{F}_{t+})$ is right-continuous filtration.

12. Show that a random variable τ is a $(\mathscr{F}_{t+})$-stopping time if and only if for every $t > 0$ it holds that $\{\tau < t\} \in \mathscr{F}_t$. (Hint. $\Longrightarrow$ consider events $\{\tau \le t - 1/k\}$ and $\Longleftarrow$ consider events $\{\tau < t + 1/k\}$.)