Department of Mathematics and Statistics
Stochastic processes on domains
Excercise problem sheet 1
(To be returned by Friday 06.02.2015)

1. Let $\Omega = [0,1)$ be the unit interval and denote $I_j = [j2^{-n}, (j+1)2^{-n})$ for $j \in \mathbb{Z}$ and $n \in \mathbb{N}_+$. Show that the finite family $\mathscr{G}_n := \{ \, I_{j_1} \cup \ldots I_{j_n} : \; 0 \le j_1 < \ldots j_n < 2^n \, \}$ is a σ-algebra on Ω.

2. Let $\Omega = [0,1)$ and $\mathscr{G}_n$ as in 1. Let $\mathscr{F} = \mathscr{B}[0,1)$, $\mathbf{P}$ be the Lebesgue measure on $(\Omega, \mathscr{F})$ and $\xi : \Omega \to \mathbb{R}_+$ be a Borel measurable.

Show from the definition of conditional expectation that

$$\mathbf{E}\,(\xi \,|\, \mathscr{G}_n)\,(\omega) = \sum_{j=0}^{2^n-1} [\,\omega \in I_j\,] 2^{-n} \int_{I_j} \xi(\omega')\,\mathrm{d}\omega'$$

3. Let $f\colon \mathbb{R} \to \mathbb{R}_+$ be a simple, measurable function

$$f(x) = \sum_{k=1}^{n} a_k [\, x \in A_k \,].$$

Let Y be bounded and positive random variable, X a real-valued random variable and assume that X is $\mathscr{G}$-measurable for some sub-σ-algebra $\mathscr{G} \subset \mathscr{F}$. Show that the conditional expectation $f(X)Y$ with respect to $\mathscr{G}$ is

$$\mathbf{E}\,(f(X)Y \,|\, \mathscr{G}) = f(X)\mathbf{E}\,(Y \,|\, \mathscr{G})$$

almost surely.

4. Let $f\colon \mathbb{R} \to \mathbb{R}_+$ be a bounded, positive and measurable function (but not necessarily simple) and otherwise assume the same as in 3. Show that the claim of 3. holds in this case as well by using monotone convergence theorem.

A random variable $X = (X_1, \ldots, X_d)$ is a d-dimensional Gaussian random variable with *zero mean*, if its *characteristic function* $\varphi\colon \mathbb{R}^d \to \mathbb{C}$ is

$$\varphi_X(\lambda) := \mathbf{E}\,\exp\left(i\langle\, \lambda\,,\, X\,\rangle\right) = \exp(-\tfrac{1}{2}\langle\, \lambda\,,\, \Sigma\lambda\,\rangle)$$

for some positive definite symmetric matrix $\Sigma = (\mathbf{E}\, X_i X_j\,)_{ij} \in \mathbb{R}^{d\times d}$. Here $\langle\, x\,,\, y\,\rangle = x_1 y_1 + \ldots x_d y_d$. A d-dimensional Gaussian random X variable with *zero mean* has a *density function* if the covariance matrix Σ (the matrix in 5.) is invertible. Then the density function is

$$q(x) = (2\pi)^{-d/2} |\Sigma|^{-1/2} \exp(-\tfrac{1}{2}\langle\, x\,,\, \Sigma^{-1} x)\,\rangle$$

where $|\Sigma|$ is the determinant of the matrix Σ. If $(X_1, X_2, \ldots, X_d)$ is a d-dimensional Gaussian random variable and $\mathbf{E}\, X_1 X_j \;= 0$ for all $j \neq 1$, then X_1 is independent from $(X_2, \ldots, X_d)$.

5. Show the Lemma 2.2 (i) from Lecture notes (page 14).

6. Show the Lemma 2.2 (ii) from Lecture notes (page 14).

7. When $d = 1$ and $X = B_t$, use integration by parts to show that

$$\mathbf{E}\, B_t^{\,2N} = t^N (2N-1)!! := t^N (2N-1) \times (2N-3) \times \ldots 3 \times 1$$

for every $N \geq 1$.

8. Using 7. show the Lemma 2.2 (iii) and (iv) from Lecture notes (page 14).

9. A π-system on a set S is a family $\mathscr{I} \neq \emptyset$ of subsets of S such that $\forall A, B \in \mathscr{I}$: $A \cap B \in \mathscr{I}$. Show that the set $\mathscr{I}_1 = \{\; (-\infty, x] :\; x \in \mathbb{R}\; \}$ is a π-system on $\mathbb{R}$.

10. Let $f\colon \mathbb{R} \to \mathbb{R}_+$ be a simple, measurable function

$$f(x) = \sum_{k=1}^{n} a_k [\, x \in A_k\,].$$

Show that the time stationary Markov property for (X_t) with respect to $(\mathscr{F}_t)$ implies that

$$\mathbf{E}_x (f(X_t)\,|\, \mathscr{F}_s) \;= \mathbf{E}_{X_s}\, f\,(X_{t-s})$$

11. Let $f\colon \mathbb{R} \to \mathbb{R}_+$ be a bounded, positive and measurable function (but not necessarily simple) and otherwise assume the same as in 10. Show that the claim of 10. holds in this case as well by using monotone convergence theorem.

The 12. is the original 10. This will be also be part of the next excercise sheet.

12. Let $(P_{t,x})$ be as in Lemma 3.13 in Lecture notes (page 22). Let

$$\mu^x_{(t_1,\ldots,t_n)}(A_1,\ldots,A_n) = \int_{A_1} P_{t_1,x}(\,\mathrm{d}x_1)\ldots\int_{A_n} P_{t_n-t_{n-1},x_{n-1}}(\,\mathrm{d}x_n)$$

Show that family of measures $\{\ \mu^x_{(t_1,\ldots,t_n,t_{n+1})} :\ x \in \mathbb{R}^d, 0 \leq t_1 < \cdots < t_n\ \}$ satisfies the consistency condition for Kolmogorov Extension Theorem and deduce that therefore, there exists a stochastic process (X_t) such that $\mathbf{P}_x(X_t \in A) = \mu^x_t(A)$.