Snapshots of the History of Mathematics

Lecturer: Eero Saksman
Instructor: Paola Elefante

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Content and aim

- The aim is to take a look at some themes or turning points in the history of math. The notes below do not cover the lectures, e.g. most proofs and anecdotes are left out...
- This is not a standard 'Course on the History of Math' ! The topics chosen reflect the lecturer's interests/random choices. We shall start by looking at some points in the early history of algebra, but at the moment there is no clear idea what we shall do next.
- The lecturer is not a math historian and has no real knowledge of the history of math! There is no quarantee (or even high probability) that what will be told is even close to up to date professional historic knowledge. The main motivation of the lectures is to try to awake some interest to the history point of view, and equally importantly, we hopefully have some fun with math at almost every lecture!

Mathematical prerequisities

- Some basic knowledge of algebra, number theory, and analysis courses will be needed. At some points of the course we will touch on more difficult topics, but one should get something out of also those parts even if some details require background e.g. in functional analysis.

Timetables and passing the course

- The lectures are on Tu 10-12 and We 10-12 (perhaps sometimes 9-12, but this will be announced separately). There are some weeks without lectures.
- ONE SHOULD CONTINUOUSLY FOLLOW THE COURSE HOME PAGE FOR POSSIBLE CHANGES E.G. IN THE LECTURE TIMES !!
- Those who want to pass the course should attend it regularly and prepare (perhaps in pairs) an essay that will be presented either in the class or in the instruction hour (Fridays 10-12). These essays are instructed by Paola Elefante. She has a preliminary list of topics, but one can suggest an own choice.

I : 'EARLY HISTORY' OF ALGEBRA AND NUMBER THEORY

I.1. Algebra in ancient Egypt

- We speak of time period around 2000 BC to 1800 BC.
- Both papyrus ('hieratic' (cursive) writing with pen and ink) and carving in stone were use for writing. Some ways to use expressing numbers in base 10 existed.
- Especially *Rhind Mathematical Papyrus* and *Moscow Mathematical Papyrus* contain math problems and solutions.
- Proportionality was important for Egyptians. Important for builders?
- Problem 7 of the Moscow papyrus. *In a triangle the ratio of height to the base is* 2 1/2 *and the area is 20. Determine both height and base.*

SOLUTION: $bh = 40$ and $h/b = 5/2$. Multiply to get $h^2 = 100$ so that $h = 10$ and $h = 4$.

I.1. Algebra in ancient Egypt: method of false position, peculiar way to write down fractions

- Example of use of 'method of false position from the Rhind papyrus:

What is quantity if by adding it to the seventh of itself you get 19.

SOLUTION: Guess that the answer is $x = 7$. Then you get 8 instead of 9. Hence the right answer must be

$$x = \frac{19}{8} \times 7 = 16\tfrac{5}{8}$$

- The Egyptian way of writing the above would have been more like

$$x = 16 + \bar{2} + \bar{8} := 16 + \frac{1}{2} + \frac{1}{8} \quad !$$

- More generally, they wrote all decimal parts as sums of mutually non-equal fractions with only ones as the numerator!

I.1. Algebra in ancient Egypt: Egyptian fractions

More examples:

$$\frac{2}{3} = \frac{1}{3} + \frac{1}{4} + \frac{1}{12} \ = \ \frac{1}{2} + \frac{1}{6}.$$

- Actually, it is nontrivial that one can write all rational numbers in $(0, 1)$ this way !

LEMMA . *Let* $2 \leq a < b$ *be integers. Then there are natural numbers* $2 < n_1 < n_2 < \ldots < n_\ell$ *so that*

$$\frac{a}{b} = \frac{1}{n_1} + \frac{1}{n_2} + \ldots + \frac{1}{n_\ell}.$$

Proof. On the blackboard.

I.1. Algebra in ancient Egypt: Egyptian fractions

- Exercise: Show that if $3|n$, then $\frac{2}{n}$ can be expressed as a sum of two different Egyptian fractions.
- Exercise: Same as above assuming $5|n$.

I.1. Algebra in ancient Egypt: Egyptian fractions

Some results and open questions on Egyptian fractions:

- Any fraction $\frac{a}{b}$ has representation with maximum denominator $\leq O\left(\frac{b \log^2 b}{\log \log b}\right)$ (G. Tenenbaum and H. Yokota 1990, or with at most $O(\sqrt{b})$ terms (M. Vose 1985).with

- Erdös and Graham conjectured that if $N \geq \phi(r)$, then any r-coloring of the set $\{1, \ldots, N\}$ contains a one-color subset that represents number 1. This was proven (with $\phi(r) = C^r$) by E. Croot (2003).

- Open question: Does the 'odd greedy algorithm' terminate?

- Erdös-Straus conjecture: Can one always write $\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$?

I.2 Algebra in Mesopotamia

- Now we move to Babylon (and other cities) is Mesopotamia, around 2000 B.C. to -1700 B. C.
- Mostly clay tablets. Clearly 'mathematician scribes'.
- The great achievements are some kind of decimal system in base 60 and, especially, solution of general equations of degree 2!
- Lots of quadratic problems have survived, mostly geometric, some word problems. Solutions not proven or 'deduced', many examples of 'recipes'.
- Example. Assume that the difference of the sides of the rectangle is 3,5 and area 7,5. What are the sides?

SOLUTION: the idea is to observe (geometrically) that we know

$$A + (\frac{a-b}{2})^2 = b^2 + 2 \times b \times \frac{a-b}{2} + (\frac{a-b}{2})^2 = (\frac{a+b}{2})^2.$$

Hence $(a+b)/2$ is determined, and rest is easy.

I.2 Algebra in Mesopotamia

- Babylonian mathematicians had to divide many cases depending on the sign of the coefficients.
- Method of false position was also used in treating linear systems...

I.3. Geometric algebra (?) and number theory in ancient Greece

• The famous period of Greek mathematics started originates around 600 B. C. The great inventions include the invention of the concept (and need) of proof of stated theorems, great theory of classical geometry (e.g. the Pythagorean theorem,) axiomatic system and some contributions to e.g. number theory. Euclid's Elements collects, adds and systematizes lot of the math from the period 600 BC to (?) 300 BC Some main contributions, like Apollonius and Archimedes come from slightly before 200 BC

• The Pythagorean theorem was probably known (without proof?, perhaps special cases only, used probably for practical purposes) in Mesopotamia and Egypt.

• There is a huge list of famous mathematicians: Thales, Pythagoras, Euclid , Eudoxus, Archimedes, Apollonius, ..., Diofantes,...

I.3. Geometric algebra (?) and number theory in ancient Greece

- At this point we skip most of the geometry and just discuss some (geometric-)algebraic and number theoretic contributions.
- On the other hand: some of the geometry can be directly interpreted as giving constructions for some algebraic quantities, like constructions of the product ab or $\sqrt{a}$. Also, theorems like $(a+b)^2 = a^2 + b^2 + 2ab$ are included in 'Elements' in geometric form.
- Archimedes solved some 3:rd (or 4:rd) degree equations e.g. when trisecting a given angle by extending the classically allowed for equipments (using a marked ruler or assuming that the intersections of conical sections can be constructed).
- Some authors suggested that e.g. Apollonius might have used some kind of algebra to obtain his beautiful results on conic sections etc., but this is doubtful.

I.3. Ancient Greece: number theory in 'Elements'

- Elements contains (in a geometric form) some remarkable theorems and understanding of some key facts of number theory:

– The Euclidean algorithm for deducing the greatest common divisor.

– Irrationality of e.g. $\sqrt{2}$.

– Proof that if n is a Mersenne prime, then $n(n-1)/2$ is an even perfect number (converse proven by Euler)

– Construction of infinitely many (all?) Pythagorean triples.

– Proof that there is an infinitude of primes

- We shall take a closer look at some of these.

I.3. Ancient Greece: number theory in 'Elements'

• THEOREM (Euclid). *There are infinitely many primes.*

Proof. Assume all primes consist of $p_1, p_2, \dots, p_\ell$. Consider

$$N = p_1 p_2 \cdots p_\ell + 1.$$

Then N is a product of primes, especially it is divisible by one of $p_1, \dots, p_\ell$, which yields a contradiction.

I.3. Ancient Greece: number theory in 'Elements'

- THEOREM (Euclid (most of it)). *All positive integer solutions of* $x^2 + y^2 = z^2$ *are given by the formulae*

$$x = 2tuv, \;\; y = t(u^2 - v^2),, \;\; z = t(u^2 + v^2), \quad \text{where} \quad t, u, v \in \mathbb{N}, \quad u > v.$$

Proof. Blackboard

- Triplets like above (x, y, z) are called Pythagorean triplets. Both Egyptians and Mesopotamians listed some of them (also for practical purposes?).

I.3. Ancient Greece: perfect numbers'

- Let $p \geq 2$. If the number $q := 2^p - 1$ is a prime, it is called is a Mersenne prime.
- Most of the biggest explicit primes known are Mersenne primes.
- An integer $n \geq 2$ is called a perfect number if it is the sum of its divisors smaller than itself. E.g. $6 = 1 + 2 + 3$ or $28 = 1 + 2 + 4 + 7 + 14$.

THEOREM (Euclid-Euler). *An even number n is perfect if an only if* $n = q(q+1)/2$, *where q is a Mersenne prime.*

Proof. Blackboard.

I.3. Ancient Greece: perfect numbers'

- Open question: Are there infinitely many Mersenne primes ? We know 48 of them today.

$$2^{257885161} - 1$$

is the largest known prime today.

- Are there any odd perfect numbers ?

I.3. Ancient Greece: exercises

Exercise: Show that points of intersection of given two circles, or line and a circle, or two lines can be expressed via arithmetic expressions, amended with allowing taking square roots, of the data (radii and centres, or equations of the lines).

Exercise: Try to show 'a la Euclidean algorithm' that a $\sqrt{2}$ is not rational.

Exercise: Assuming that you can 'construct' intersection points of parabolas and hyperbolas, show that you can 'solve' a general 3:rd degree equation.

I.3. 'Arithmetica' of Diophantus

- Diophantus was born around 200 AD, and lived a long life. His exact age is known because of the famous problem-poem couple of years later:

'Here lies Diophantus,' the wonder behold.
Through art algebraic, the stone tells how old:
'God gave him his boyhood one-sixth of his life,
One twelfth more as youth while whiskers grew rife;
And then yet one-seventh ere marriage begun;
In five years there came a bouncing new son.
Alas, the dear child of master and sage
After attaining half the measure of his father's life chill fate took him.
After consoling his fate by the science of numbers for four years, he ended his life.'

I.3. 'Arithmetica' of Diophantus

- Diophantus lived in Alexandria. His master piece 'Arithemetica is often called the first real algebra book.
- The book series consisted of 13 books, of which 10 are known to us (IV-VII were found only in 1970's as a 900 BC arabic translation.
- The books contains various problems, i.e. equations (or equation systems), where typically one has has more variables than constraints. Of course it also contained (most of) the algebra known before him, but it lead to new level of various algebraic problems. Diophantus was looking for solutions in positive rationals.
- Diophantus developed own, rather modern notation for polynomials, and considered also higher powers. It is instructive to take a look how he writes $x^3 - 5x^2 + 8x - 1$...

I.3. 'Arithmetica' of Diophantus

- The impact of 'Arithmetica' for future algebra and math was powerful! Arabic culture translated it, and e.g. gave name al-jabr to algebraic manipulation (described by Diophantus) where one coins together similar terms in an equation. This gave finally rise to the term 'algebra'
- A problem from 'Arithmetica': Find numbers x, y so that

$$\frac{x}{x^2 + y^2} = k,$$

where k is a given number.

SOLUTION: Substitute $x = ty$, where t is arbitrary rational number....

I.3. The Cattle problem of Archimedes

- Lessing found in 1773 a manuscript addressed to Erastothanes and other Alexandrian mathematicians. It is now believed to be written by Archimedes. The letter contains a poem of 44 lines. The poem challenges them to count the number of cattle in the Herd of the Sun by stating a Diophantine equations system to be solved (in verbal description).
- The problem leads to a famous (and important!) equation in number theory, Pell's equation, and the smallest solution has 206 545 digits !!
- Pell's equation. is of the form $x^2 - my^2 = 1$, to be solved in integers. Here $m \geq 2$ in not a square. We will look later more comprehensively at the history of solving Pell's equation.

I.3. Alexandria - the ancient city of science

- Alexandria was founded by Alexander the Great 331 BC, and developed to a flourishing city, a cultural centre by one of his friends (and generals), Ptolemy I. Both were students of Aristotle. Ptolemy founded a museum and a library, with scientists discussing medicine, theter, rhetoric, mathematics and astronomy.
- Alexandria was the schooling venue for Euclid, Eratosthanes, Archimedes, Apollonius, Hero, Ptolemy, Diophantes, Hypatia,...
- The science was at high level, e.g. Eratosthanes measured rather accurately the size of earth!
- Let us take a look at how Archimedes (in Syracusa) was connected to the rest of the world and how his math was preserved...

I.3. Alexandria: Hypatia's murder

- The flourishing time of Alexandria as The center for science was brought to an end when Hypatia was murdered in year 415 AD (at age $\sim$ 60). She was the daughter of Theon of Alexandria, a mathematician. Apparently Hypatia studied philosophy (including mathematics) in Athens.
- It is not known if she produced many pieces of original mathematics, but by (e.g. by her teaching and her comments on existing texts) she appears to have been a leading force in science and philosophy in Alexandria during her time there. Often she is mentioned as the first female mathematician in the history.
- There is a movie on Hypatia, which we could watch later on...

I.4. Some algebra in Medieval China

- Important early books: *Suan she she* (circa 200 BC) or *Jiuzhang suanshu* (Nine Chapters on the Mathematical Art).
- Contain e.g. linear equations and systems of them, also version of Gaussian elimination!
- Later, towards 500 AD , approximative solutions of polynomial equations (using binomial type formulas).
- Master Sun already solved (200 AD) 'Chinese remainder' type problems, the general method was described by Qin Jishao (1247 AD), resembling the modern organisation of the solution. Also the individual congruences $ax \equiv b \pmod{q}$ were solved by a version of the Euclidean algorithm.

I.5. Algebra in India

- For example, 'Aryabhata' (written by Aryabhata) from $\sim$ 500 AD gives rules in Sanscrite verses how to solve problems (mainly of astronomical type).
- Other important names Brahmagupta, Baskara...
- Solutions of 2:nd degree equations we considered, explanations mainly in geometric terms. Indeterminate equations also appear.
- An algorithm, called pulveriser for solving linear equations in integers was given. Closely related to the Euclidean algorithm.
- Surprising amount of work for solving Pell's equation! Both understanding of how to combine solutions to obtain new ones and workable algorithms (without proof) for finding the fundamental solution.
- Many formulas for sums like $\sum_{j=1}^{n} j^a, \quad a = 1, 2, \ldots.$

I.6. Algebra in Medieval Islam

- Already in 700's the Islamic empire contained whole Middle-East (up to India's border), North-Africa and most of Spain.
- Mathematics (and other sciences) flourished especially during the period $\sim$ 700–1000. The books of Euclid and Diophantos were soon translated, and ideas from India and China were also imported.
- Al Khwazizmi ($\sim$ 780-850) wrote an important text which contained 'al-jabr' in its title. His text was to a manual for solving equations. The proofs of e.g. solution recipes for 2:nd order equations were accompanied with geometric justifications. His book was translates as "Al-Khwarizmi on the Hindu Art of Reckoning", which gave rise to the term 'algorithm'.
- Other mathematicians of renown include Ibn Turk and Abu Kamil, the latter consider fairly complicated equations with solutions involving nested roots.

I.6. Algebra in Medieval Islam

- In 1000-1100 several mathematicians, e.g. al Shamaw'al (1130-1180), developed some algebra for polynomials, like the formula $x^{n+m} = x^n x^m$ (including negative powers) and established complicated formulas for dealing with roots like

$$\sqrt[3]{A} + \sqrt[3]{A} = \sqrt[3]{3\sqrt[3]{A^2 B} + 3\sqrt[3]{AB^2} + A + B}$$

- Al Samaw'al's book 'Shining book of Calculation' contained binomial formulas (the Pascal rule for the coefficients was clearly present)
- The famous mathematician and poet Omar Khayyam (Al-Khayyami), among other things, classified 3:rd degree equations and showed how they can be solved geometrically as intersections of conic sections. Especially, he expressed the lament that there is no algebraic formulas for solutions of cubics!
- All in all, the high period of Islamic science and mathematics preserved and advanced in many ways the knowledge of algebra.

I.7. West in 13-14:th century: Fibonacci

- Leonardo of Pisa, aka Leonardo Fibonacci (1170 – $\sim$ 1240) travelled and studied widely different cultures. His magnum opus 'Liber abaci' (Book of calculations) contains much what arabic world had done before. He also excelled in number theory.

- Examples of Fibonacci's results:

1. $1 + 3 + 5 + \ldots (2n - 1) = n^2$.

2. *The system* $x^2 - n = z^2$ 1 $x^2 + n = z^2$ *has integer solutions if and only if* n *is 'congruous number', i.e.* $n = ab(a + b)(a - b)$ *(in case* $a + b$ *even) or* $n = 4ab(a + b)(a - b)$ *(in case* $a + b$ *odd).*

Exercise. Choose in 1. above $(2n - 1)$ to be and odd square. How does this help you to create Pythagorean triples a la Fibonacci?

I.7. West in 13-14:th century: 'Summa arithmetica' and others...

- Other books include Paolo Gerardi's 'Libro di ragioni' (Book of problems). It contained flawed (i.e. only approximate) formulas for cubic equations.
- Luca Paciola's 600 pages book 'Summa Arithmetica was among the first printed mathematics books (1494). He used some kind of short hand notation (no real symbols yet). He worked in Venice and in Rome.
- In very early 1500's in Germany Rudolf and Stiefel developed more symbolic notations for handling equations.

II : SOLVING HIGHER DEGREE EQUATIONS – FROM CARDANO TO GALOIS

II.1. Solution formulas for 3:rd degree equations: del Ferro

- A big breakthrough occurred in algebra in Italy when Scipione del Ferro (1465-1526) from Bologna found solution formulas for the 3rd degree equation (recorded in a notebook, passed to his son-in-law, della Nave).

$$x^3 + px = q.$$

- Later on, he passed the knowledge to his student Antonio Maria del Fiore.
- Why did Ferro not publish his findings? Apparently at that time, in order to keep your position at the university, it was useful to have some secret recipes in your sleeve...!!

II.1. Solution formulas for 3:rd degree equations: contest with Tartaglia

- Del Fiore challenged Niccolo Fontana (∼1500-1557), aka Tartaglia, from Venice, to a public mathematical duel! However, on February 12, 1535, Tartaglia found a solution of cubics himself.
- In the contest Tartaglia posed more varied forms of cubic equations, and del Fiore was unable to solve most of them, whereas Tartaglia solved all the problems of del Fiore.

II.1. Solution formulas for 3:rd degree equations: enter Cardano

- Tartaglia revealed his solution to Girolamo Cardano (∼1501-1576), while visiting in 1539 the latter. Actually, Cardano was planning to write a new treatise on algebra. Tartaglia gave the recipe in a form of a poem, and under the strong oath of secrecy!
- Cardano was a true renaissance man. He worked also on physics, medicine, gambling, philosophy,...
- He was quite respected as a physician and a skillfull gambler! He published the first treatise on probability, and used the rule 'ratio of favourable events to all possible events' to define probabilities. He e.g. argued for the education of deaf people, and invented the 'cardan shaft' for joining two rotating axis in an angle with respect to each others.

At some point Cardano was jailed for producing the hosorcope for Jesus Christus, later on was happily in the service of the pope...

II.1. Solution formulas for 3:rd and 4:th degree equations: 'Ars Magna'

• In 1540, Lodovico Ferrari ($\sim$1522-1565), Cardano's student and secretary, found the way to solve 4:th order equation. Ferrari died as a rich math professor in Bologna, possibly poisoned by his own sister using arsenic.

• Cardano faced the problem how to publish all this, since there was the oath for Tartaglia. However in 1543 Cardano and ferro travelled to Bologna and happened to meet della Nave, who revealed to them the notebook of del Ferro. Then Cardano felt himself released from the oath and started to write his great book on algebra, 'Ars Magna, that came out 1545.

• Ars Magna exposed, besides classical stuff, the new solutions for all kinds of 3:rd and 4:rd degree equations. Still geometric proofs for formulas, but overall the book is a watershed in the all time literature on algebra.

• Let us then follow del Ferros and Ferrari's footsteps and find formulas for the solutions.... (on the blackboard).

II.1. Problems raised by Cardano's formulas

- How to choose the branches of the cubic roots (there cannot be 9 roots for a 3rd degree equation!).
- Casus irreducibilis: if there are three real solutions, the solution formulas involve complex cubic roots!!
- Cardano used negative numbers ('fictitious' ones) in manipulation, although was mainly searching positive solutions for final solutions. Even more interestingly, he made some use complex numbers (at least square roots of negative quantities) to get to results. There is the following famous sentence in Ars magna.

... putting aside *the mental tortures involved*, multiply $5 + \sqrt{-15}$ by $5 - \sqrt{-15}$ making 25 - (-15), which is $+15$. Hence this product is 40....

Exercise. Show that $\sqrt{x + iy}$ can be expressed in terms of real roots of the real and imaginary part x, y (it can actually be shown that the same is not true for cubic roots!)

Exercise. Show that the cubic equation $(x-1)(x-2)(x-3) = 0$ leads to the case of casus irreducibilis.

II.1. Cardano's formulas recorded

- Let us record here (not just on the blackboard) the famous Cardano formulas for the equation $x^3 + px + q = 0$:

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$$

Just for fun, note how it looks for the general case $ax^3 + bx^2 + cx + d = 0$:

$$x = -\frac{1}{3a}\left(b + \sqrt[3]{\frac{2b^3 - 9abc + 27a^2d + \sqrt{(2b^3 - 9abc + 27a^2d)^2 - 4\left(b^2 - 3a\right.}}{2}}\right.$$

$$\left. + \sqrt[3]{\frac{2b^3 - 9abc + 27a^2d - \sqrt{(2b^3 - 9abc + 27a^2d)^2 - 4\left(b^2 - 3ac\right)^3}}{2}}\right)$$

II.2. Towards complex numbers: Bombelli

- Rafael Bombelli (1526-1572) lived in central Italy. His book 'L'Algebra' used complex numbers freely and gave the standard rules how to manipulate them! Moreover, he 'rediscovered' Diophantus and developed further notation and algebraic manipulation towards modern style. Still, his notation is far from that today....

Example. Take the equation $x^3 = 16x + 4$.

- Direct substitution to Cardanos formulas yields

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}.$$

How to find the values of the cubic roots??

- What follows was not an easy step to Bombelli either. He writes: '...An extravagant thought, according to many. I myself was for a long time the same opinion. The matter seemed to rest more on sophistry than truth, but I searched until I found proof.'

II.3. Towards complex numbers: Bombelli

- Bombelli wrote $\sqrt[3]{2+\sqrt{-121}} =: a+\sqrt{-b}$. By taking the cube Bombelli obtains the pair

$$2 = a^2 - 3ab \quad \text{and} \quad \sqrt{-121} = (3a^2 - b)\sqrt{-b}.$$

Multiplying the definition of $a+\sqrt{-b}$ by its conjugate we obtain $a^2+b^2 = \sqrt[3]{4+121} = 5$, and substituting this to the first equation yields $4a^3 - 15a = 2$. The latter has the solution $a = 2$ and we finally obtain $b = 1$ so that

$$\sqrt[3]{2+\sqrt{-121}} = 2+\sqrt{-1}.$$

Finally, the desired solution is $\quad x = (2+\sqrt{-1}) + (2-\sqrt{-1}) = 4.$

Exercise. Find the two other roots of the above 3rd degree equation.

Exercise. Use Ferrari's formulas to derive all the solutions of the 4rd degree equation $x^4 - 8x + 6 = 0$.

II.3. Towards complex numbers (?): Viete

- The French mathematician Francois Viete (1540–1603) wanted to advance new thinking in algebra by wanting to create a new art, not 'spoiled by barbarians', where one solves problems '*freely by score*' (not just individual ones). He makes interesting remarks about the difference between 'analysis' and 'synthesis'
- Viete famously developed better notation but his symbols still had geometric dimensions. For example, he writes

$$\ell.c.\ell.\overline{\overline{B\text{plneplaneplane} + Z\text{solidsolid}} + Zsolid} \quad = \quad \sqrt[3]{\sqrt{c^2+d^2}+d}.$$

On the other hand he used letters for unknowns, which was rarely done after Diophantus.

II.3. Towards complex numbers (?): Viete

• Viete made the important observation ('Viete's formulae') that the roots x_1, x_2 of the equation $x^2 + px + q = 0$ satisfy

$$x_1 + x_2 = -p \quad \text{and} \quad x_1 x_2 = q.$$

Similarly for the equation $x^3 + rx^2 + px + q = 0$ one has

$$x_1 + x_2 x_2 = -p, \quad x_1 x_2 + x_2 x_3 + x_3 x_1 = p \quad \text{and} \quad x_1 x_2 x_3 = -q.$$

He also generalised this to higher orders.

• Viete was lawyer by profession and served several Kings by his knowledge. Also his life was fairly rich and several nice stories are attached to it. E.g. there is a famous sort where he loved in a couple of minutes 45:th degree equation posed by the Dutch mathematician Adrian van Roomen. He also gave a solution to the famous problem of Apollonius on three circles.

II.3. Towards complex numbers (?): Viete

- Viete developed trigonometric formulae and (importantly for our story) applied them to treat the difficult 'casus irreducibilis' case of 3rd degree equations

- This part of Viete's work was not based on Moivre's formula for $n = 3$ but instead to reducing the casus irreducibilis cases to equations of the form $4x^3 - 3x = a$, where $|a| < 1$. Let us take look at this on the blackboard...

Exercise. Show that if Cardanos formulas lead to a solution involving cubic roots of honestly complex numbers, then the equation can be reduced to the form treated by Viete.

- Thus Viete was not a real advocate of complex numbers, whereas Girardi (1540–1603) was supporting there use and naturalness, Also, he found independently Viete's formulas for the roots and gave the first statement (conjecture) that the fundamental theorem of algebra should be always valid.

II.4. Geometry turns algebra: the work of Descartes

- As is well known, Rene Descartes (1596–1650) was one of the most famous philosophers and developer of analytic geometry methods.
- Due to works of Galilei times were changing towards what kind of mathematics was needed for applications

In analytic geometry Descartes was clearly anticipated by Pierre de Fermat.

- The approach of analytic geometry gave new geometric ways to look at algebra (graphs of polynomials, algebraic curves), and also the other way round.
- Descartes noticed that if polynomial has a root, this gives a linear factor to the polynomial. Moreover, he gave a rule for counting and upper bound for the number of positive roots (without proof).
- Descartes also supported the truth of the fundamental theorem of algebra.

II.4. Newton: symmetric functions

- Isaac Newton (1642-1727) was one of the greatest mathematicians and physicists of all times, and we return to him later on. This time we simply take a look at his work in algebra. Part of that was published in 1707 in 'Algebra Universalis' (written by his successor – Newton was not happy of this rewrite of his lectures).
- Newton classified (almost completely) the cubic algebraic curves.
- He had great interest in symmetric functions of n variables $x_1, \ldots, x_n$ and proved important special case of the following fundamental theorem:

Theorem. *Let $g(x_1, \ldots, x_n)$ be a polynomial that is invariant under any permutation of the variables. Then one can write*

$$g(x_1, \ldots, x_n) = h(e_1, \ldots, e_n),$$

where e_j:s are the elementary symmetric polynomials:

$$e_j(x_1, \ldots, x_n) := \sum_{1 \leq i_1 < i_2 < \ldots < i_j \leq n} x_{i_1} \ldots x_{i_j}.$$

PROOF. Blackboard...

II.4. Newton: symmetric functions

- One should observe that the polynomial with the roots $x_1, \ldots, x_n$ takes the form

$$(x - x_1)(x - x_2)\ldots(x - x_n) = x^n - e_1 x^{n-1} + e_2 x^{n-2} - \ldots + (-1)^n e_n.$$

- Newtons recursive formula for expressing sums $\sigma_k := x_1^k + \ldots x_n^k$ can be proven by using the above observation.
- Example .
$x_1^2 + \ldots x_n^2 = (x_1 + \ldots + x_n)^2 - 2(x_1 x_2 \ldots x_{n-1} x_n) = e_1^2 - 2e_2.$
- Exercise . Express $x_1^3 + \ldots x_n^3$ in terms of the elementary symmetric polynomials.
- Further remarks on the previous Theorem: The representation is unique. If polynomial g has integer coefficients, so does h.

II.5. Solving equations in the 1700's: Lagrange resolvent's

- Many contributions by de Moivre, Euler, Bezout, Vandermonde, Lagrange,...For our story the most interesting works are those by Vandermonde and (especially) Lagrange.
- Both Vandermonde and Largange (independently) invented the notion of resolvent in order to solve equations. It is now called Lagrange resolvent since Lagrange developed it further and in a more general situation.
- Alexandere Theophile Vandermonde (1735–1796) was originally a violinist who started to work seriously also on mathematics (and chemistry) much after he was 30 years old. He wrote an important treatise in 1770 (published in 1774), where the following basic ideas are contained (at least in special cases) for the first time:

– It is important to look how various quantities composed of the roots $x_1, \ldots, x_n$ of the given n:th degree equation
transform under permuations of the roots

– Hence, one tries to look for expressions E that take only a small number (say m) of different values under all permutations, whence one can expect that it satisfies an equation of lower degree m

II.5. Solving equations in the 1700's: Vandermonde and rules of attribution in math

• Vandermonde is known for determinants that bear his name. As often in mathematics, this attribution is misleading. This reminds of the following fairly general rules for Theorems of Definitions attributed to some mathematician:

– If a very famous mathematician used (or invented later on independently) a result of a less familiar name, the theorem is attributed to the more famous one.

– If the above rule does not apply, there is some other error in the attribution.

II.5. Solving equations in the 1700's: Lagrange resolvent's

- Joseph Louis Lagrange (1736–1813) was (the) leading mathematician right after Euler, with many lasting contributions to analysis, calculus of variations, mechanics, number theory and algebra. He was born in Italy, worked in Berlin between the age 30-50, moved then permanently to Paris.
- In order to gather the idea of Lagrange's resolvents (from the famous paper 'Reflexions sir la theory algebrique des equations') let us assume that x_1, x_2, x_3 are the roots of the cubic equation $x^3 + px + q = 0$ and let us look (all math discussions on the blackboard!) at the following quantity:

$$\tau := (x_1 + \omega x_2 + \omega^2 x_3)^3.$$

- By applying this method Lagrange found natural approaches to the equations of degree ≤ 4. However, for 5:th degree equation the resolvent type equations he found was at least of degree 6!!

II.5. Solving equations in the 1700's: Ruffini

- Paolo Ruffini (1765–1822) was and italian mathematician and philosopher working in Modena. He was the first one to seriously try to prove the insolvability (by radicals) the equations of 5:th degree.And he almost succeeded!In fact, his proof give fundamental knew insights in the role of the permutation group (some of this was already seen in Lagrange's work).

- Ruffini's many papers on the subject (1799-1810) were thought to be incomplete in his time (and they are). However, we shall take a look at one his central ideas that (in my opinion) could be combined by a part of later more successful proof by Abel to yield an easier argument than Abel's for the insolvability.We shall discuss the proof and application of the following two results on the blackboard:

LEMMA. *The symmetric group S_5 contains elements U, V that satisfy:*

$$U^3 = V^3 = \mathrm{Id}, \quad (VU)^5 = (VU^2)^5) = Id$$

THEOREM. *If the power $p(x_1, x_2, x_3, x_4, x_5)^\ell$ (here $\ell \geq 2$) invariant under the permutations U and V, then also p itself remains invariant.*

II.6. Gauss and solving equations in *Disquisitiones Arithmeticae'. Cauchy.

- Carl Friedrich Gauss (1777–1855) was generally called 'Prince of Mathematians' already during his lifetime. We shall discuss some of his work later on. Here we just briefly note what his famous book 'Disquisitiones Arithmeticae' contained touching our subject
- Gauss showed that one can always solve the cyclotomic equation $x^{p-1} + x^{p-2} + \ldots + x + 1 = 0$ by adjoining successively roots. Moreover, he showed that one can do this by using only square roots if and only if

$$p = 2^k q_1 \ldots q_\ell,$$

where q_j:s are distinct Fermats primes (of the form $q = 2^{2^m} + 1$).
- In proving this and some other results Gauss showed how to argue rigorously with 'field extensions', and group theoretic ideas were hidden here and there.
- Augustin-Louis Cauchy (1789–1857) was a versatile and prolific French mathematician. For our story it is important that he published an extensive study of finite permutation groups S_k and their subgroups in 1815.

II.7. Abel and insolvability of 5:th degree equations.

- Niels Henrik Abel (1802–1829) was a Norwegian mathematician. His early years we not very easy, but his genius in mathematics was revealed soon after 1817 when he got a new young math teacher Holmboe, who had deep understanding of mathematics himself. Abel started to read famous mathematicians papers on mathematics.
- In 1823 Abel visited mathematicians in Kopenhagen. Later on he visited Berlin and paris. At some point he believed in having found a solution of the general 5:th degree equation, but found a mistake in his solution. After the visit to Kopenhagen he continued to work on 5:th degree equations, and also started to study elliptic integrals.
- In 1824 Abel published a short note explaining his argument for insolubility of the quintics. Two years later he published a longer more extensive version of the same proof in the new important mathematical journal, 'Crelle's journal', where he also published many of his papers on elliptic functions. Later on Abel studied criteria for solvability of general equations. This work was left incomplete at his premature death at 26 years age.

II.7. Abel and insolvability of 5:th degree equations.

• Structure of Abel's proof on nolsolvability will be discussed at blackboard...

• Abel's main results on elliptic functions, eg. of type

$$\int \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

include the discovery of the double periodicity of the inverse functions (will be discussed at the blackboard...). Another deep result is the famous addition theorem for Abelian integrals. He was also very worried of the status of rigour in mathematical analysis, and made some steps to improve the situation (will be discussed later on in the course).

• It took time (not very many years though!) until the larger community noticed the great value of his work. Couple of days after he died in Oslo at age 26, amongst real financial difficulties, in tuberculosis, a letter received from Germany telling him that he was secured a professorship in Berlin.

II.7. Galois and general theory of solvable equations.

- Evariste Galois (1811-1832) lived a short and turbulent life, but produced a new theory (Galois theory) for solvability of equations that carries his name, and nowadays is connected to many different mathematical areas. He started his mathematics studies at 15 and published his first work (on continued fractions) in 1819.

- Galois was republican, and in addition was not somebody who kept his mouth quiet. This led him prisoned several times, and he found that in the entrance exams to Ecole Polytechnique the examinations were not able to follow his reasoning, and he was not clear when expressing himself. As an outcome he did not get in. In addition, his fathers suicide had a deep effect on him. Also, his manuscripts on the general theory of equations was not accepted by the French Academy, according to (partly anecdotal) certain stories

– the first memoir was lost by Cauchy

– the second memoir was lost since Fourier died

– the third memoir was returned by Poisson, who stated that some proofs were unclear.

II.7. Galois and general theory of solvable equations.

- This history seems too unfair to the Academy, his work was written in very unclear manner, and actually it is safe to assume that Cauchy was supporting him, and even Gauss's 'Disquisitiones Arithmeticae' was not accepted by the academy 30 years before.
- The death of Galois was tragic: he was challenged to a duel, and during the previous night of his death he tried to put together a letter scetching the main ideas of his theory. At some point he writes in the margin in desperation: I have no time...
- It took quite long time until in Liouville in 1843 declared that the theory works (it was published in Liouville's version in 1846)!

II.7. Galois and general theory of solvable equations.

• We will discuss very shortly at the blackboard the following (not exactly stated, and more in Galois' own style) items of Galois theory:

1. What is a Galois resolvent?

2. What is the Galois group of a given equation?

3. How does one read the solvability of an equation from its Galois Group?

4. Why is the general 5:th degree equation not solvable according to Galois theory?

II.8. The fundamental theorem of algebra.

- As discussed before, Girard was among the first ones to support the claim that every polynomial has as many roots (possibly multiple) as the degree of the polynomial. However, his opinion was not universally accepted : as late as 1702 Leibnitz claims that polynomial $x^4 + 1$ does not decompose in to lower order factors!
- Euler and D'Alembert tried to give a proof (as did Lagrange, Laplasce,... nearly everybody!), but all these contained gaps.
- The first proof that was accepted complete in its time was that of Gauss in his dissertation 1799.We shall discuss his proof and its faults in the modern eye on the blackboard...
- Later on (1816) Gauss gave 2 new proofs that are basically sound in todays standards. What is remarkable is that his second proof is quite algebraic by nature, the ides third one basically correspond to one found often in complex analysis textbooks today.
- In the meanwhile, Jean-Robert Argand (1768–1822) gave (1814) the first completely rigorous proof (to be discussed at the blackboard...) and even for polynomials with complex coefficients. His proof is also quite

III: FROM FERMAT TO NOETHER – NUMBER THEORY AND ROAD TO ABSTRACT ALGEBRA

- Aim of Chapter III is to trace the slow evolution of the modern concepts of algebra, like rings, ideas, general fields. Of course we shall not forget group theory, one of whose main impetus in the 1880's came from Galois theory of soluble polynomial equations that we just discussed
- Our aim is to use as a guiding line the extensions of the concept of whole numbers , especially in connection with Fermat's last theorem. Also (higher) reciprocity laws will be mentioned.
- We shall take a closer look at the work of Fermat, otherwise will be somewhat selective but try to mention in passing the most important events in number theory up to around 1850.

III.1. Fermat's work in number theory.

• Pierre Fermat(1601(??)–1665) was the leading mathematician of his time (still an amateur!) mathematician who did pathbreaking contributions to number theory. His main profession was that of a lawyer. Moreover, he almost discovered the analytic geometry and had methods similar to the notion of derivative to determine maxima and minima of functions. In addition, together with Pascal he was one of the founders of probability theory

• Fermat showed his great genius especially in his many theorem and (supposedly..) proofs of highly non-trivial and basic problem in number theory. In a letter to Huygens he explains that basically all his proofs are based on method of descent.Unfortunately he communicated very little on is proofs – mainly stated theorems and gave hints on how to approach then in his many letters to e.g. famous le Pere Mersenne (1588–1648).

III.1. Fermat's work in number theory.

- Fermat's accomplishments in number theory include:

– Fermat's little theorem: $a^p - a$ is always divisible by a if p is prime.

– A prime $p = x^2 + y^2$ if and only if it is of form $p = 4n + 1$.

– Complete classification of numbers that are sums of two squares (and count for different representations).

– Theorem stating that every natural number is a sum of four squares

– Solution of Pell's equation.

– Some examples of results for quadratic forms or higher degree equations.

III.1. Fermat's work in number theory.

– Theorem stating that equation $x^n + y^n = z^n$ has no nontrivial solutions if $n = 3$ or $n = 4$.

• We shall now see how Fermat's method of infinite descent works in a famous example

THEOREM. *A right angled triangle with integer sides cannot have area that is a square (of an integer).*

Proof. (on blackboard...)

III.1. Fermat's work in number theory.

- Fermat is also famous for his 'Last Theorem': Equation $x^n + y^n = z^n$ has no nontrivial solutions if $n \geq 3$. This was historically a very important claim, as we know (and shall see later on)
- It is a pity that Fermat did not write up his knowledge, in his final letter to Huygens he writes: '...Such is in brief the tale of my musings on numbers. I have put it down only because I fear that I shall never find the leisure to write out and expand properly all these proofs and methods... Maybe posterity will be grateful for me to having shown that the ancients did not know everything, and this account may come to be to be regarded by my successors as the 'handing on of the torch', in the words of the great Chancellor of England [Bacon] following whose intention and motto I shall add: many will pass away, science will grow.'

III.2. Euler and Lagrange: the search for the proofs Fermat never published.

- We all know Leonhard Euler (1707–1783) and many of his theorems and concepts in mathematics...
- Euler basically worked out the proofs of most of Fermat's claims. The list of his contributions contains e.g.:

– Proof that every prime of the form $p = 4n + 1$ is a sum of 2 squares.

– Generalization of Fermat's little theorem.

– Work on quadratic forms.

III.2. Euler and Lagrange: the search for the proofs Fermat never published.

– A proof (almost correct) of Fermat's last theorem for $n = 3$. This is interesting for us, as he needed to know when $p^2 + 3q^2$ with $(p, q) = 1$ is a cube. For this end he wrote

$$p^2 + 3q^2 = (p + q\sqrt{-3})(p - \sqrt{-3})$$

and claimed that it follows that both factors on the right must be cubes! This was the first occurrence of the use of complex integers!Euler argument was not correct, but it might have given food for thought for later generations...

III.2. Euler and Lagrange: the search for the proofs Fermat never published.

- Lagrange extended and continued Euler's work. His most important results in this field contain

– Proof that every positive integer is a sum of 4 squares.Euler simplified Lagrange's proof in a remarkable way.

– Proof of the general solvability of Pell's equation.

– First steps towards a more general theory of quadratic forms.

– Theorem to the end that the continued fraction of a given real number x is periodic if and only if x is a quadratic surd.

- One should also mention the important body of work in number theory due to Adrian Marie Legendre (1752–1833).

III.3. Gauss and 'Disquisitiones Arithmeticae'

- ' Disquisitiones Arithmeticae' was young Gauss' magnum opus and belongs to the most impressive mathematical monographs written ever, – We cannot discuss the whole content here, we basically and mention some important spots for our story (the algebra part we already discussed before, e.g. the construction of a regular 17-gon with ruler and compass.)

- Gauss proved the unique decomposition of natural numbers to primes, he understood the need to do it!

- He put together the known number theory in an elegant way, especially invented the notion of congruence (was basically using the ring Z_n or the field Z_p).Disquisitiones also contains the very first (valid) proof of the famous Gauss law of quadratic reciprocity (to be discussed on the blackboard, called by Gauss 'Theorema Aureum)

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{(p-1)(q-1)/4}.$$

Gauss found this result independently from Euler and Legendre and gave the first valid proof while 18 years old.

III.3. Gauss and 'Disquisitiones Arithmeticae', higher reciprocity laws

- The crown of 'Disquisitiones' is the general theory of binary forms and representation of arbitrary integers via them.In a remarkable 'tour de force' Gauss proved basic laws of composition of binary forms In principle he proves that the composition of (properly) primitive forms yields a group structure (this time it is not evident as often, e.g. the proof of associativity is (in)famously hard!)
- The last Section (published only after Gauss' death) of 'Disquisitiones' enters the study of higher degree congruences in a remarkably moder way
- Gauss also obtained the cubic and biquadratic reciprocity laws. Here he introduced the rigorous use of algebraic integers for the first time. Especially, he introduced the Gaussian integers: complex numbers of the form $a + ib$, where a and b are ordinary integers. We shall discuss these (on the blackboard) and use them as a toy example later on.

III.3. Sophie Germain: important results for Fermats last theorem

- Sophie Germain (1776–1831) obtained the most important results of general type (i.e. valid for a large class of exponents) on Fermat's last theorem.
- Germain was born in a fairly well educated and economically well-off family. While she was 13, the French revolution started in earnest and for the next years she remained home for most of the time reading e.g. mathematics books from his fathers library. She later told that the sort about the death of Archimed fascinated her and she decided to learn more mathematics, reading also works by Newton and Euler.At first her parents tried to prevent her from this, but (according to a famous anecdote) after even making her room cold during the night to prevent her on mathematics it did not prevent her from working feverishly on her equations. Finally, the parents gave up and supported her to some extent.

III.3. Sophie Germain: important results for Fermats last theorem

- When 18 she sent written essays to the newly established 'Ecole Polytechnique', but used the name of a former student 'Monsieur Antoine-August Le Blanc.'Lagrange became curious of this talented student and demanded to meet him (her), but did not withdraw his support after observing that 'he' is 'she'.
- In early1800's Germain was in correspondence with Gauss, still using the pseudonym 'Monsieur Antoine-August Le Blanc.' In 1807Napoleon was conquering the area around Braunschweig (were Gauss was born as was still living before moving permanently to Göttingen). Germain was afraid that Gauss would share the destiny of Archimed and was able to organise a French General Pernety to verify that Gauss is safe.This turned out to be the case, but for the interrogator Gauss answered that he does not no anybody named Sophie Germain...Finally Germain had to reveal her true identity to Gauss, and the latter wrote a famous letter to Germain:

III.3. Sophie Germain: important results for Fermats last theorem

"But how to describe to you my admiration and astonishment at seeing my esteemed correspondent Monsieur Le Blanc metamorphose himself into this illustrious personage who gives such a brilliant example of what I would find it difficult to believe. A taste for the abstract sciences in general and above all the mysteries of numbers is excessively rare: one is not astonished at it: the enchanting charms of this sublime science reveal only to those who have the courage to go deeply into it. But when a person of the sex which, according to our customs and prejudices, must encounter infinitely more difficulties than men to familiarize herself with these thorny researches, succeeds nevertheless in surmounting these obstacles and penetrating the most obscure parts of them, then without doubt she must have the noblest courage, quite extraordinary talents and superior genius. Indeed nothing could prove to me in so flattering and less equivocal manner that the attractions of this science, which has enriched my life with so many joys, are not chimerical, [than] the predilection with which you have honored it."

III.3. Sophie Germain: important results for Fermats last theorem

- Besides number theory, Germain did important work in elasticity (partly in collaboration with Legendre and Lagrange). The goal was to find the equation of a vibrating plate, and after many trials she finally found the equations of the right type, i.e. of the form

$$(\Delta)^2 u = a(\frac{d}{dt})^2 u.$$

- Germain died before obtaining the honorary doctorate in Göttingen University, arranged by Gauss.

III.3. Sophie Germain: important results for Fermat's last theorem

- In Fermat's last theorem (=FLT) one may assume that the exponent n is a prime, and x, y, z have no common factors. It used to be divided into two cases: case I is such that none of x, y, z is divisible by n. Case II is the complement.

THEOREM (Germain) Consider $x^p + y^p = z^p$, where p is a prime. If $q := 2p + 1$ is also a prime, then case I of FLT is true (i.e., if there is a nontrivial solution, one of x, y, z has to be divisible by p

PROOF. (On the blackboard).

- By using a slightly more general form of her theorem, Germain was able to show that case I is true for FLT for all $p \leq 100$.
- One says that p is a Sophie Germain prime if $2p + 1$ is also a prime. It is conjectured that there are infinitely many of them, but this is not known.

III.4. Interesting events in the Paris Academy in 1847

• We shall next describe certain incidents in the Paris Academy starting in March 1, 1847.For that end we first introducel some of the main actors:

– Gabriel Lamé (1795–1870) was a French mathematician, renown for his resolution of FLT for $n = 7$ in 1839. Actually, in the realm n prime and $n \geq 5$ only the case $n = 5$ had been previously resolved (due to efforts of Germain, Dirichlet, the final piece of the puzzle put into its place by the 73 years Legendre in 1825!) He also worked on elasticity and studied e.g. the efficiency of the Euclidean algorithm.

– Cauchy) we already know...

III.4. Interesting events in the Paris Academy in 1847

– Ernst Kummer (1810–1893) was a German mathematician working at the time in Breslau. Besides number theory he worked e.g. in ballistics and hypergeometric functions. He had especially close connection to Dirichlet, who was married to a sister of Kummer's first wife. Kummer was a school teacher when Jacobi and Dirichlet, after reading a paper of him in 1840, realised his great gifts and helped him to finally obtain a position in the Breslau University. We shall discuss his main contribution to number theory shortly.

– Peter Gustav Lejeune Dirichlet (1805–1859) did groundbreaking contributions e.g. to number theory, Fourier series, rigour in analysis, etc. He was a close friend of Jacobi and Liouville, and he was admired as s teacher by e.g. Eisenstein, Kronecker, Lipschitz, Christoffel, Heine, Dedekind and Riemann.

III.4. Interesting events in the Paris Academy in 1847

Dirichlet was elected as the successor of Gauss in 1855 when the latter died. His main contributions in number theory include the great theorem on primes in arithmetic sequences (the real starting point of analytic number theory!),class field formula, Dirichlet's theorem of approximation of irrationals by rationals, the theorem of units in algebraic number fields, study of the (Dirichlet) divisor problem...

– Joseph Louis Liouville (1809–1882) was a French mathematician, whom we met already as he was the one who refived Galois' work in 1840. He had great impact to mathematics in many ways, e.g. he founded Journal de Mathématiques Pures et Appliquées in 1846. His name is familiar from many basic concepts in math: Liouville's theorem in complex analysis, first existence proof of transcendental numbers, Sturm-Liouville theory (one of the first true collaborations in the history of math), theory of integrable systems etc.

III.4. Interesting events in the Paris Academy in 1847

So what happened in Paris?

• In the beginning of March Lamé announced (excitedly) that he has solved the FLT in general! The main idea was to consider the factorisation (n odd)

$$z^n = (x+y)(x+\omega y)\dots(x+\omega^{n-1}y), \qquad \text{where} \quad \omega^n = 1.$$

Of course he planned to use the trick (familiar to us by now) that the factors on the right are all prime to each others, and hence n:th powers. This was in the Eulerian spirit, since the factors on the right are complex numbers. Lamé admitted that he cannot take the whole glory of the idea, since Liouville had suggested it to him in a casual conversation some months before.

• After Lamé had given his presentation, Liouville himself took the floor. He expressed strong suspicions for the alleged proof and commented that many mathematician before had used complex numbers in arithmetic before. Especially, Liouville observed that unless unique factorisation into 'primes' is established, there is no hope even to start the proof!

III.4. Interesting events in the Paris Academy in 1847

- Then Cauchy spoke and mentioned that he had in 1846 presented an idea to the academy that could lead to the proof, but he had had no time to develop these ideas...
- In the following weeks Lamé admitted that Liouvilles critique was right, but he strongly believed that one can prove the unique decomposition into primes.In Mach 15 Wantzel claimed to have proven this (he did it for $n = 1, 2, 3, 4$, and noted that rest must go in a similar way). Cauchy criticised him and quickly wrote several papers attempting to give a proof.
- In March 22 both Lamé and Cauchy deputed secret packets with the Academy... And thereafter both published several notes on the question..

III.4. Interesting events in the Paris Academy in 1847

- In May 24 Liouville dropped the bomb: Kummer send a copy of his memoir from 1844, which contains examples of the failure of the unique factorisation!! Moreover, in 1846 he had published a shorter note on ideal numbers that can retain uniqueness of factorisation into primes and was soon to publish (in Crelle's journal) an extensive paper on e.g. applications to the FLT.
- In Paris (after the striking news) Lamé fell silent, but Cauchy was more stubborn.Cauchy e.g. stated ' What little [Liouville] has said [about Kummer's work] persuades me that the conclusions which Mr Kummer has reached are, at least in part, those to which I find myself led by the above considerations'. He stated that he is ready to applaude Kummer's work if the latter has gone couple of steps further [than himself], and ...' for what we should desire the most is that friends of science should come together to make known and propagate the truth'. Of course he then for some time continued to write more on the topic with sparse references to Kummer...

III.4. Interesting events in the Paris Academy in 1847

- Kummer had originally been using algebraic integers to study higher reciprocity laws. There is a story, perhaps true, that Kummer first applied this idea to both FLT and the higher reciprocity, but Dirichlet noted to him that a main problem could be the possible non-uniqueness in the prime factorization. Also it is believed that Gauss originally told Dirichlet that he first tried to simplify his great theory of binary forms by using algebraic numbers, but met too many difficulties (non-uniqueness again?) and abandoned this approach. In any case it is rather clear that Gauss had knew that cases of non-uniqueness exist. Also Eisenstein laments in 1844 that everything would go beautifully if there would be uniqueness in the factorisations to primes, but there isn't!

III.4. Algebraic integers, primes and irreducibles

DEFINITION (i) An algebraic field K is any subfield $K \subset \mathbb{C}$ so that $[K : \mathbb{Q}] < \infty$. Equivalently, $K = Q(\alpha)$ (the field generated by the complex number $\alpha \in \mathbb{C}$) such that α is an algebraic number (i.e. satisfies a polynomial equation with integer coefficients).

(ii) An algebraic integer λ is a complex number that satisfies a monic polynomial equation with integer coefficients.The ring of integers of the algebraic field K is the set of all algebraic integers contained in K, often denoted by O_K.One of course has to check that O_K is a ring!

Example. We already know Gaussian integers that is the set $O_{Q(i)}$.

Example . Quadratic fields $K(\sqrt{a})$, where $a \in \mathbb{Z}$ is not a square.

DEFINITION (i) $u \in O_K$ is unit if also $u^{-1} \in O_K$.

(ii) $\lambda \in O_K$ is irreducible if $\lambda = \alpha\beta$ (with $\alpha, \beta \in O_K$) implies that either α or β is a unit.

(iii) $\lambda \in O_K$ is prime if $\lambda | \alpha\beta$ (with $\alpha, \beta \in O_K$) implies that either $\lambda | \alpha$ or $\lambda | \beta$.

III.4. Algebraic integers, primes and irreducibles

LEMMA (i) Every $\lambda \in O_K$ factorizes to a product of irreducibles.

(ii) Every prime is an irreducible.

THEOREM factorisation to irreducibles is unique if and only if every irreducible is prime.

- If the condition of the above theorem is satisfied (e.g. Gaussian integers !), life is easy and the sandard reasoning using divisibility remains valid.However, this is not always true:

Example. Consider $K = Q(\sqrt{-5})$. Its integers are $K_O = \mathbb{Z}[\sqrt{-5}]$. We may write

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}).$$

The numbers $\{2, 3, 1 \pm \sqrt{-5}\}$ on the right are irreducibles but they are not primes! (Exercise) !!

III.4. Kummer and FLT

- Kummer observed that in the case of $K(e^{2i\pi/23})$ the unique factorisation does not hold in O_K. However, he introduced the notion of ideal numbers, which can be used to restore the unique factorisation.In more technical terms one embeds $K = K(\alpha)$ into a larger field $K' = K(\alpha') \supset K$, and this is done in such a way that elements in O_K factor uniquely in the larger ring $O_{K'}$.
- Of course Kummer's way to explain ideal numbers was much much more complicated that what we use here!

The final result of Kummer is the following:

THEOREM (Kummer) FLT is true for prime exponent p if p is regular, i.e. $p|h_p$, where h_p is the class number.

- Via this theorem Kummer obtained FLT for all $p \leq 100$ apart from exponents 29, 57, 69 !

III.4. Quadratic fields

- The (quadratic) field $K(\sqrt{d})$ is Euclidean if and only if $d \in \{-1, -2, -3, -7, -11\} \cup \{2, 3, 5, 6, 11, 13, 17, 19, 21, 29, 33, 37, 41, 57, 73, 97\}$.

Especially, in these cases the prime number decomposition is unique.

- In general, one can rephrase the uniqueness of prime number decomposition in terms of the class number $h(d) = h(K(\sqrt{d}))$. The uniqueness holds if and only if $h(d) = 1$.

GAUSS CONJECTURES ON THE CLASS NUMBER: (i) $\lim_{d \to -\infty} h(d) = \infty$. True (Heilbronn 1934).

(ii) For $d < 0$ one has $h(d) = 1$ only for $d \in \{-1, -2, -3, -7, -11, -19, -43, -67, -163\}$. True .(Dorian Goldfeld 1976)

(iii) There are infinitely many positive d with $h(d) = 1$. Still open.

III.5. Dedekind and the modern notion of an ideal

- Richard Dedekind (1831-1916) was a German mathematician, who is famous for e.g. his radically abstract approach to defining real numbers, contributions to ring theory in connection with algebraic numbers, theory of algebraic function fields etc..He was the last doctoral student of Gauss (was also born in Braunschweig, also worked therein).
- After long work towards trying to have clear exposition of Kummer's theory of ideal numbers Dedekind ended up (around 1876) to the concept of an ideal of a ring as we know it today. And if product of ideals of O_K for a given algebraic number field K is defined in a natural way, Dedekind was able to show that decomposition f of an ideal in O_K into prime prime ideals is unique! (to be discussed on the blackboard).

III.5. Dedekind and the modern notion of an ideal

- This gave the natural approach to ideal numbers. Dedekinds free use of arbitrary subsets was unconventional and very modern. Moreover, later on Dedekinds approach led (in the hands of Noether) to the nowadays standard basic structures in ring theory and in field theory.
- (On blackboard) We shall give a short description of the ideals in connection with the algebraic number theory. Also, we shall take a look at how Dedekind defined real numbers (Dedekind cuts)...

III.6. Slow process of abstraction in algebra in the 1800' s

- We already learned how the central notion of an ideal was arrived at. Next we shall list some of the major milestones in the road towards abstract algebra in the 1800's

GROUP THEORY:

- Gauss and his 'implicite' group theory.
- Cauchy's theory of permutations, especially Galois theory of solvable equations, finally Jordan's papers on permutation groups
- Kronecker's definition of almost 'abstract' groups and fields
- Geometry: e.g. n-dimensional theory of linear spaces by Grassman,projective geometry (Monge etc.),finally Klein's famous Erlangen program: 'Geometry is defined by the group of isometries it allows for...'Lie's theory of transformation groups,theory of invariants (origins in geometry) by Cayley etc...

III.6. Slow process of abstraction of algebra in the 1800' s

- Cayley was close to formulate a real abstract notion of a group.
- The abstract group notion was in a sense present in v. Dyck's, Kronecker's, Hölder's and Weber's works in 1860-1880's. Especially Heinrich Weber (1842–1913) was instrumental in this respect. The abstract presentation found its way to textbooks around 20 years later on.

III.6. Slow process of abstraction in algebra in the 1800' s

RING AND FIELD THEORY:

We have already discussed the role of algebraic number fields or integers in the emerging notions of a 'field' and 'ring'.Other influences in the 1800's include:

- Early algebraic geometry (polynomials of several variables), invariant theory (Cayley&Sylvester, Gordan, Hilbert...).
- Complex numbers, Hamilton's quaternions, other hypercomplex numbers.
- Matrix theory of Cayley etc. Cayley noticed the associativity and non-commutativity of the matrix multiplication, as well as the existence of zero divisors.His proof of the famous Cayley-Hamilton theorem was though by the 'old-fashioned' induction...
- Kronecker's 'constructive' definition of extension fields (even irrationals...),Fields of algebraic functions (Dedekind and Weber,...),Henselt's p-adic numbers,...

III.6. Slow process of abstraction in algebra in the 1800' s

- For axiomatic definitions a decisive impetus overall was provided by David Hilberts (1862-1943) famous work on axiomatic systems of classical geometry, Grundlagen der Geometry (1899). Also his abstract approach to invariants and finite bases of ideal in general were very influential.

- Weber essentially gave (arguably) the first definition of an abstract field 1893. The first one to really use them in the abstract sense was Ernst Steinitz (1871–1928) in fundamental paper from 1910 ! Steinitz

– Distinguished fields of characteristic 0 and those of characteristic p, and also observed that the 'prime' fields are Q and Q_p with p prime.

– Developed the theory of transcendental extensions,algebraic closures,and general fields for which Galois theory applies.

- Emmy Noether (1882–1935) and (slightly later on) Emil Artin (1898–1962) really started the era of abstract algebra. We shall speak more on Noether's life and work soon. Let us just mention that Artin made important contributions to solidify purely abstract approach to algebra, and worked e.g. on algebraic number theory (higher reciprocity laws), Galois theory and class field theory.

III.6. Slow process of abstraction in algebra in the 1800' s

• Finally one could say that abstract algebra had broken through and became universally accepted via the publication of B. L. van der Waerden's (1903–1996) 'Moderne Algebra'. One should recall Dieudonne's reaction to hearing a lecture by van der Waerden...

III.7. The abstract algebra of Emmy Noether

- Emmy Noether (1882–1935) was (according to estimates by many of her contemporaries) the most important female mathematician in the history before her of during her time.
- We shall soon discuss more here life, first we shortly point out her main contributions to algebra.
- Noether wrote a dissertation on invariants with Gordon in 1907.
- In 1921 Noether published the paper 'Ideal theory in Rings. It introduce the famous ascending chain condition for any increasing sequence of ideals.Also she defined for the first time the rings in the modern standard way (Sono's work on 1917 went unnoticed).The paper simplified and unified a lot hat was done before in separate cases by more complicated means.
- In a paper from 1927 Noether characterised the rings R for which every ideal is a unique product of prime ideals (now called Dedekind domains). Her conditions require that R is an integral domain that is integrally closed in its field of quotients, and bot R and R/I satisfy the ascending chain condition (here I is any ideal of R).

III.7. The abstract algebra of Emmy Noether

- Noether did also important work in non-commutative algebra, and especially its applications to commutative algebras, 'foreshadowing cohomological theories'.
- Many mathematicians have later on spoken on the impact that abstract approach, personified in Noether (she gave often almost improvised lectures, where new thing were sometimes proven on a spot) made on them while they e.g. visited Göttingen and discussed with her or heard seminars by her.
- We shall describe later on in connection with 'Calculus of Variations' Noether's famous result on symmetries and conserved quantities in physical systems. Now we discuss slightly more her life.

III.8. Life of Emmy Noether

- Emmy Noether was born in Erlangen in 1882. His father (Max Noether) was also a great mathematician, working in early algebraic geometry.She appears not to have been specially interested in mathematics as a youngster but already displayed an unusually quick mind. She was also quite talented in languages.
- Noether studied at the University of Erlangen. She graduated in 1903 (?) and spent the winter semester 03–04 in Göttingen. She heard lectures e.g. by Svhwartzschild, Minkowski, Klein and Hilbert.Then she returned to Erlangen and made her thesis under supervision of Paul Gordan (1837–1912), known as 'King of Invariants'. Noether's thesis was heavily computational and in the already somewhat old-fashioned style of Gordon.

III.8. Life of Emmy Noether

- The years 1907-1915 Noether spent in Erlangen, though being in touch with many other mathematicians through travels and correspondence.Especially important for her development was Ernst Fischer (1875–1954), who was the new professor at Erlangen. Noether herself stated that under the influence of Fischer her methodology in the theory of invariants gradually changed towards the abstract approach due to Hilbert. Soon she was a leading researcher in this direction.

- In spring 1915 Noether moved to Göttingen upon the invitation by Klein and Hilbert. The latter people wanted her help in understanding invariants of physics, especially those of just emerging general relativity theory of Einstein. Noether was perhaps the leading mathematician actively working on invariants, and she had in 1914 increased the scope of Hilbertian techniques in her important paper 'Fields and systems of rational functions'. Hilbert used to say jokingly that "... a specialist of invariants [himself] turned to the student [Noether] of the King of Invariants [Gordan] for help!.

III.8. Life of Emmy Noether

- As an outcome Noether produced her celebrated Noether's theorem: any differentiable symmetry of the action of a physical system has a corresponding conservation law.This result (proven in 1915, published as her habilitation thesis in 1918) belongs now to fundamental results in physics and also in calculus of variations.We shall take a closer look at this theorem later on when dealing with calculus of variations.
- Noether writes jokingly in a letter: '...a team in Göttingen was carrying out calculations of most complicated kind for Einstein – although none of us understands what they are for.... Klein writes ... You know that Fräulain Noether is continually advising me in my projects and that it is really through her that I have become competent to the subject.... Hilbert, in turn: Emmy Noether, whom I called upon to help me with such questions as my theorem on the conservation of energy..." .

III.8. Life of Emmy Noether

- In 1915 Noether was not able to obtain habilitation (granted only for male candidates!). All appeals to the university senate and Minister of Culture were rejected, despite of Hilbert's struggles. An argument against her was that young men could not see a woman teaching them, upon which Hilbert objected: "... I cannot see why the sex of the candidate should be an argument against admission as a Privatdozent – after all this is a university not bath house."
- Noether was able to teach under Hilberts name, and from 1919 on under her own name. in 1922 she became "nicht beamteter ausserordentlicher Professor", and from 1923 on started to get some money from her lectures. Even after that lifestyle was very spartan and simple: her life was completely dominated by mathematics and her fellow mathematicians. All material things were secondary.
- She got important honours towards the end of 1920's, for example she was elected as an editor for Mathematische Annalen and gave a plenary talk in 1932 ICM.

III.8. Life of Emmy Noether

- Noether was a legendary lecturer: opinions of the differ as much as one can imagine! Apparently many listeners were afraid of her abstract approach (in van der Waerden's words: "She could only think in concepts, not in formulas, and precisely herein lay her strength. It was the very nature of her mind that compelled her to invent conceptual forms that were suitable as carriers for mathematical theories."She was not organised, was happy to improvise. Once she had thought that perhaps she could improvise the proof of a well known theorem in an abstract way. It did not quite work out on the blackboard. Whereupon she was overcome with rage, stumping on the chalk on the floor and crying out: "I'm forced to do it in the way I do not want!"In general she was known as a strong, outspoken, and independent personality.
- Noether was loved by the circle of her students and she was really unselfish with them. She wanted van der Waerden to publish his work although he had found (important) results that were already contained in her lectures the same semester!She did not care of outward appearance.Famous is the story of her raincoat.

III.8. Life of Emmy Noether

- Noether seems to have been a natural mathematics genius who was happy to share mathematics constantly. She loved to talk and correspond constantly and develop constantly her ideas whether eating or walking among friends, or lecturing.
- When Hitler came to power she had to leave her position.After all, she was a Jew and had some leftist connections. At this situations she was concerned only of others with similar destiny. She moved to Princeton and took a position in Bryn Mawr College. According to Hermann Weyl (1985-1955) she was at the summit of her creative power, when in 1935 she died in a sudden complication of an operation that was thought to be a success.

IV: NEWTON, HOOKE AND LEIBNITZ: THE BIRTH OF ANALYSIS AND MATHEMATICAL PHYSICS

- In this section our aim is to take a look at some facets of the work of Isaac Newton. We shall also meet (amongst others) Robert Hooke and Gottfried Wilhelm Leibniz, and try to understand how Newton was lead to writing his masterpiece 'Principia...'

IV.1. Precursors of analysis revisited

- Remember that Andreas Hartmann already discussed some of the developments in mathematics that were predecessors to Newton's and Leibniz's calculus. We will be thus be short here and just illustrate by some examples.
- The exhaustion method of Greek mathematics used upper and lower approximations to compute lengths or (especially) areas. In a sense, assuming that e.g. the area and its basic properties are mathematically well defined this method is exact even in the modern sense, and in rigor it was surpassed only in the time of Weierstrass'.However, one cannot call it 'calculus', since each case had to dealt separately with completely different means. It originates with Antiphon ($\sim$ late 5:th century BC), Eudoxus (408–355 BC), and Archimed was the supreme master of this method.He obtained a number of nontrivial results by this manner, some requiring lot of ingenuity in the process of comparison. Here we look only at one of the simplest of his results:

THEOREM *The area under a segment of a parabola is 1/3 of the area of the corresponding rectangle.* Proof. (Archimed) on the blackboard...

IV.1. Precursors of analysis revisited

- Archimed's exhaustion of the parabolic segment was generalised by several mathematicians like. Fermat showed his great shrewdness also here and was able to integrate areas under all the curves $y = x^a$, quad $a > -1$.

THEOREM *Let $a > -1$. The area under $y = x^a$, for $x \in [0, M]$ equals $(a+1)^{-1} M^{a+1}$.*

Proof. (Fermat 1636) on the blackboard...

- John Wallis (1616–1703) systematised his predecessors work in his *Arithmetica Infitorum* (1656). Eg. he computed $\int_0^1 x^k dx$ for integer values of k using formulas for sums $1^k + \ldots n^k$. He invented the use of the real line and the symbol ∞ ! Wallis had often bad sleep and sometimes did mental calculations while awake in his bed. One night he calculated in his head the square root of a number with 53 digits. Next morning he dictated the 27-digit square root of the number, still entirely from memory. His feat that was considered outstanding. Hence Oldenburg, who was the Secretary of the Royal Society, sent a colleague to investigate how Wallis did it. It was considered important enough to merit discussion in the Philosophical Transactions of the Royal Society of 1685 !

IV.1. Precursors of analysis revisited

• Grégoire de Saint-Vincent (1584–1667), a Flemisch mathematician, and Ahlfons Anton de Sarasa (1618–1667), a Jesuite, observed at the end of 1640's that the area under a hyperbola satisfies a functional equation that allows to conclude that it is given by logarithm. Subsequently Saint-Vincent and Nicholas Mercator (1620–1687) obtained the Taylor series of logarithm

$$\log(1+x) = x - x^2/2 + x^3/3 - \dots,$$

actually known already to Newton. Of course they applied the geometric series of Viete (1597).

• What came to determining tangets or normals to curves, many researchers before Newton and Leibniz had there own approaches how to do it:

– Archimedes used geometrical approach to e.g. draw tangents to spirals.

– Fermat solved extreme value problems by demanding that $f(x_0+h) - f(x_0) = O(h^2)$.

IV.1. Precursors of analysis revisited

– Descartes found tangents by demanding that the line and the curve have only one point of contact.

– Huygens did fantastic thing by using just geometry and some analysis.

• WHAT WAS STILL MISSING: clear understanding that integration and differentiation are inverse to each other, and calculus rules for these operations !

IV.2. Newton's anni mirabilis

- Isaac Newton (1642–1727) was born at Woolsthorpe Manor into a fairly humble family of farmers - apparently he was the first newton who was able to write his name. His father died soon after. Newton attended a Garmmar School between age 12–17. After this his mother tried to make him a farmer. Newton showed considerable ingenuity in constructing mechanical devised, bit hated farming. With the help of his mothernal uncle he was admitted to Trinity College at Cambridge University in 1661. During the first university years he studied e.g. Aristotle, Descartes, Galileo, learning e.g. Kepler laws. Isaac Barrow was perhaps the most important teacher to him.

- Because of thread of plague, the University was closed more than a year, and Newton returned to home in Woolsthorpe for two years 1665-1667. During this time he worked feverishly on mathematics and some physics problems. The outcome of his investigations was such that the time is called *anni mirabilis*. He was practically working day and night, forgetting to eat or wash.

IV.2. Newton's anni mirabilis

- Among the fruits of the anni mirabilis are:

– his experimental theory of colour: invention of prisms and that white light is composed of different colours.

– binomial theorem for rational fractional powers (bold guesswork!)

– power series of logarithm

– first steps in differential and integral calculus : the fundamental theorem (1665) and calculus of 'fluxions' (derivatives) and 'fluents' (integrals). A summary of this is contained in his personal tract *De analysis per equations numero terminatorum infinitas* (1669).

– extensive use of power series and operations for them (rather than modern rules of calculus).

— Newton used fairly clumsy notation for integral in his calculus (contrary to Leibniz). On the other hand, he writes $f(x+dx)$ in the form $\dot{f}o + f(x)$, i.e. in the modern language $o = dx$ and $\dot{f} = f'$.

– Instead of limits he spoke of 'ultimate ratios',...

IV.3. Lucasian professor at Cambridge

– Newton started to work towards the laws of motion. He had predecessors in Galileo Galilei 1564–1642, Descartes. The work of the great Dutch scientist Christiaan Huygens (1629–1695) on centrifugal force was especially important. However, his later understanding of planetary motions was not yet there...

– at this stage Newton published nothing of the mathematics he had invented. If he would have done it, many troubles later on would have been avoided!

• After returning to Cambridge Newton was elected as a fellow of Trinity.In 1669 he was named as a successor of his teacher Isaac Barrow (1630–1677), who admired Newton's gifts and had something else or mind for himself. This chair ('Lucasian chair') has been held by many famous names: e.g. Barrow, Newton, Waring, Larmor, Babbage, Stokes, Dirac and (quite recently) Hawking.

IV.3. Lucasian professor at Cambridge

- Barrow was also one of those who had methods to draw tangents, and also (apparently in 1664) Barrow found independently a proof of the fundamental theorem of calculus! It is somewhat unclear when his happened, but he published it is a book in 1670 (Leibniz owned a copy of the book before his own invention of calculus, but stated that he read it only later on). Some historians think that Newton got e.g. the basic idea behind the fundamental theorem from Barrow's lectures or private tutoring.
- A professor at Cambridge should have actually taken holy orders and become an ordained Anglican priest. However, he was able to avoid this by a special permission of the king.
- During early 1670's Newton lectured on optics and developed his theory further. He was never married and continued fairly solitary life, although he had friends and contacts.

IV.3. Lucasian professor at Cambridge

- A most remarkable aspect of man Newton was that he was seriously, and in an scientific manner interested in things that are nowadays considered completely unscientific! It is often estimated that he used much more time in his studies of BC Alchemy and Chronology (the latter according to the Bible). !! Especially, he worked towards

– The Philosopher's Stone, an alchemical substance supposed to be capable of turning basic metals such like lead into gold.Newton was tireless in his laboratory trying on different mixtures and heating procedures. He made a careful bookkeeping of all what he did so that if the goal were to be suddenly achieved, the idea would not be lost.The number of trials he made was immense. Tirelessly Newton in collecting manuscripts on alchemy, that are almost impossible to read for non initiated.

– Corrections to the chronology of (Biblical) history: Newton believed that some mistakes had crept in in some accounts of historical dates (both nonbiblical and biblical), and he tried to correct them. In 1728, soon after his death, his book "The Chronology of Ancient Kingdoms" came out.

IV.3. Lucasian professor at Cambridge

–Temple of Salomon was a source of interest for Newton, because he thought that Salomon had some divine knowledge while planning the temple, and e.g. Newton tried to find some secrets about the proportions of the temple

– Newton made extensive estimation of the date of Apocalypse.

- The seriousness and where extent of the above activities made John Maynard Keynes to claim that ""Newton was not the first of the age of reason, he was the last of the magicians".Apart from the above mentioned opus on chronology he did not publish any of his work on occult.

- When years passed Newton gave lectures to usually a very small audience, sometimes consisting only of couple of listeners, sometimes none at all! Usually he gave at most one lecture a week.

IV.3. Hooke's criticism of Newton's theory of light

- Robert Hooke (1635–1705) was the permanent curator of the Royal Society, founded in 1660, probably the oldest scientific society that has been functioning continuously. Hooke was a tremendously many-sided scientist, who did important work in mechanics (Hooke's law), gravitation theory (we will discuss this later on), microscopy (important monograph 'Micrographia' in 1665), biology (he noticed that living things consist of cells), palaeontology (understanding that e.g. fossil wood could be used to understand what kind of life existed in prehistoric times), astronomy (tried to use parallax to measure distance to stars), ...

- Hooke's main duty was to demonstrate (!) or communicate 3-4 new laws of the nature in at the weekly meetings of the society. This arduous task he committed for 40 years! He was sometimes keen to take part of the glory of the new findings to himself, but apparently he had a truly creative mind himself, but lacked the time to concentrate on his findings. As a public person he was very outspoken as apt to get involved in conflicts.

IV.3. Hooke's criticism of Newton's theory of light

- In January 1672 Newton was selected as a member of the the Royal Society, especially because of the reflecting telescope he invented. This made him happy. Soon after he sent his paper on theory of light to the Society. This he himself thought his greatest achievement up to that time. The society set a committee to read the paper, with Hooke as the writer of the report (this as natural since Hooke was a specialist in the field).
- In his report Hooke made standard positive remarks, but took a quite critical stand on Newtons experiments with the prism and Newtons ideas of light as small corpusceles (and atomic nature of single coloured light rays). Soon after on Newton also met resistance from the continent also. In fact, many (including Huygens and Hooke) supported the theory of light as waves especially since the wave hypothesis explained beautifully the Snell-Descartes-Fermat law of reflexion.
- Newton tried strongly to defend his theory and experiments, but with rather weak results.

IV.3. Hooke's criticism of Newton's theory of light

- This episode was perhaps of crucial importance for him:

– his innate inability to take criticism (which abounded in those times) strengthened and he became more and more reluctant and perfectionist in matters related for publishing any of his work.Especially he had originally planned to publish his early theory of calculus at the same time with the theory of light, and he abandoned all this ideas. Also, other factors could have played in this: After the plague and the terrible fire of London (1666 – big portion of the town burned, but it was rebuilt in a couple of years according to plans of Christopher Wren (1632–1723), who was a main architect, but also notable mathematician-physicist-astronomer, also Hooke was very active) the price of paper went up for a long time, and printing became very difficult for scientific works.

– In any case Newton was mentally quite involved with the issue of defending his theory of light, and with occult science until early 1680's. Hooke's and others criticism here on Newton's hypothesis of light's constitution might have caused him to later on, in connection with gravitation, to say his famous words: " I do not invent hypotheses ".

IV.3. Hooke's criticism of Newton's theory of light

- As we already know, Leibniz invented independently calculus in 1675 during his famous trip to Paris where he finally concentrated seriously to mathematics. Before that he also learnt how it is to be criticized by Hooke when in 1673 he presented his famous calculator to the Royal Society, and of course it did not yet work properly.
- In 1676 Newton sent Leibniz a letter ('epistle prior'), where he listed some of his results in analysis. Mostly the letter contained his theory and applications of Taylor series. Newton carefully avoided giving hints for his main ideas and methods...
- Later in 1676 Leibniz visited London, where Collins showed him some works of Newton, e.g. Leibniz read parts of 'De Analysis' and 'Historiola', but had the books with him only for a couple of days (30 years later Newton assumed that Leibniz took them with him to Paris for a thorough study).Collins tried to get Newton to publish his results on analysis, but Newton assured Collins that his methods are superior to those of Leibniz.

IV.3. Hooke's criticism of Newton's theory of light

- Newton send a second letter to Leibniz, which the latter got only in 1677. In this letter Newton was even more praising for Leibniz's work, and it seems that during these years the relation between the creators of analysis was quite normal. Newton's letter still concealed his methods, but Leibniz's answer was more open.
- The first publication on new infinitesimal calculus was to be Leibniz's *Nova Methodus Pro Maximis et Minimis* in the Acta Eruditorum in October 1684, containing the calculus rules for differentiation. It was only 6 pages long and hard to read (Jacob Bernoulli called it 'enigma' rather than an explanation). The paper did not mention Newton's name. However, in his letter's from this time Leibniz mentions often Newton and mentiones that the latter had obtained some results independently. Of course he had no true knowledge of the scope of Newton's work at this point.

IV.3. Hooke's criticism of Newton's theory of light

- Newton did not pay so much attention as one would have guessed to Leibniz's publications around this time, since around this time he was not much involved in mathematics. But in 1684 he was suddenly back to business, perhaps second time his life as vigorously as in his youth. We will next describe what lead to this new activity and creation of his magnum opus 'Principia'.

IV.3. Hooke's letters, Halley's visit and the birth of 'Principia'

- Newton's and Hooke's relation was strained. But in 1679 Hooke wrote a conciliatory letter to Newton. He had nevertheless a high opinion of Newton as a mathematician, and he suggested that Newton should try to investigate with his mathematics, and jointly with experiments, some of the new ideas in Physics he (Hooke himself) and some others we expressing at the time. Hooke informed Newton on latest news in exact sciences. Especially, Hooke wanted to ask Newton's opinion on Hooke's conjecture on the law of gravitation.
- Newton replied in 4 days! He stated that at such a venerable age (37!) he did not want compete with younger minds, and was no more so attracted to philosophy. However, in turn he proposed an experiment for Hooke to be made to verify that Earth indeed is rotating.

IV.3. Hooke's letters, Halley's visit and the birth of 'Principia'

- Newton's cannon ball drop experiment was based on the idea (and open question at that time) that if one drops a cannon ball from a tower, then, on account of Earth's rotation, it does not drop exactly on the root of the tower, but one could measure the deviation. Assume the cannon ball is dropped on a tower of heigh h located at equator and there is no friction.

– Newton initially noted that the ball should drop a little bit of the root to the direction of rotation, but his argument was rather primitive, and wrong in a sense (to be discussed on the blackboard). It is actually not easy even to guess beforehand whether there is a deviation and to which side of the tower it is.

– Also Newton proposed to imagine what would happen to the cannon ball if it could continue its way inside Earth without resistance. He proposed that it would follow a spiral that approaches the centre of Earth.

IV.3. Hooke's letters, Halley's visit and the birth of 'Principia'

– Hooke in his reply gave a much better answer to the latter question, actually (by better insight, luck and some numerical computations??) guessing basically the correct result (to be discussed on the blackboard) !

– This shows how far Newton was from his later understanding of gravitation and its mathematics (all would change in 5 years). Namely, let us see on the blackboard how just from Keplers laws one can easily deduce and estimate the deviation.

– Hooke even tried to perform the measurement, and reported a positive report! However, his measurements were not exact enough and statistically satisfying as the true deflection for a tower of height 10 meters is of order 1 mm.

IV.3. Hooke's letters, Halley's visit and the birth of 'Principia'

• Hooke sent the report of his experiments with the cannon ball to Newton in 1680. Moreover, this letter contained remarkable insights: Hooke's hypothesis on the attraction of planets to each others by the famous inverse square law ! This idea was 'in the air', but Hooke was consistently putting it forward. And, more importantly (apparently referring to numerical tests) he reported that this should lead to elliptical orbits for planets (and he stated that the movement 'inside Earth' should be similar to elastic oscillations) !! Hooke knew that Newton's mathematical powers would be needed in checking that the inverse law leads to elliptic orbits. Newton newer answered Hooke, but apparently started to investigate and answered this question without passing the knowledge to others at this point. Hooke's letter's apparently put Newton in motion towards physics in midsts of his alchemy studies.

IV.3. Hooke's letters, Halley's visit and the birth of 'Principia'

- The next important event was Edmond Halley's (1656–1742) visit to Newton in 1984. Prior to that Halley had met with Hooke and Wren in a coffee shop (!) in London, and discussed amidst coffee and tobacco Halley's important question: what kind of a shape is a comet's path in the space? Hooke claimed they would be ellipses because the attractions follows the inverse square law. Wren tried to get Hooke to give a proof for this, even trying to make a bet on this, but Hooke demurred as he did not have a sound mathematical argument for this. Halley was positive for Hooke's answer, but he wanted a proof. So Wren told Halley that in Cambridge there is a professor named Isaac Newton who might be able to answer he question in an exact way.

Halley thought the matter was all important, and he took the uncomfortable trip to Cambridge, met Newton and posed the question. Newton's immediate answer was: the path is an ellipse, the force being the inverse square law.

IV.3. Hooke's letters, Halley's visit and the birth of 'Principia'

- Halley was 'struck with joy and amazement' after hearing Newton's immediate answer, and soon after Newton send Halley proof of this claim. Halley persuaded Newton to write more, and (surprisingly) Newton agreed on. Actually he started to work almost day and night, sometimes forgetting to eat, and just costuming wine. After less that 2 years 'Principia' was born. It was printed by Halley's own expense.

- Let us discuss mostly on the blackboard the main content of Principia. In 1670's (according to his private notes) Newton had been turning away from calculus and was more inclined toward using classical geometry in connection with infinitesimal arguments. In this Newton had also respected as a model Hughens masterly geometric treatment of many questions that would nowadays be presented through calculus. Hence, Principia is essentially free of calculus!

IV.3. Hooke's letters, Halley's visit and the birth of 'Principia'

• One should note that (like the inverse square law), many of the physics principles Newton introduces in Principia were already known, and were 'in the air'. For example, of Newton's laws of force, after Galilei, Hooke, Huygens and others (especially after Huygens important papers on the centrifugal force in 1659,1673), perhaps the most novel point is the law of force and counter force (action).Newton gives credit to many of his predecessors, but not in an appropriate way to Hooke.It is interesting note how he writes in a private letter to Halley: ..."Mathematicians, that find out, settle and do all the business, must content themselves with being nothing but dry calculators and drudges, and another that does nothing but pretend and grasp all things must carry away all the inventions as well as those that were to follow him as those that went before... And... I must now acknowledge in print I had all from him and so did nothing myself but drudge in calculating, demonstrating and writing upon the inventions of the great man."

III.3. Hooke's letters, Halley's visit and the birth of 'Principia'

- To demonstrate Newton's methods and results in Principia, let us take a look how one derives in his style the all important Kepler laws assuming inverse square attraction.

– 1. LAW: The radius vector of a planet sweeps equal areas in equal times.

– 2. LAW: The path of a planet is an ellipse with sun in one of the foci points.

– 1. LAW: The square of the time the planet needs for one round around sun is comparable to the third power of the average of the largest and shortest distance of the planet form the sun.

PROOF: On the blackboard...

- The math behind these results is nontrivial fact, and the proofs by Newton constitute an Archimedean feat in mastery of geometric methods. Principia contains lot more, estimates of Moons effect to tides, first non-trivial result on calculus of variations,...

IV.3. Master of mint and the war on calculus

- After Principia Newton's work on science was rather limited in comparison to his capacities in this field. We already discussed his work on algebra. Some of his work in analysis started to be published, but mainly through other writers, e.g. Wallis published some of his work in In science Newton mainly concentrated on refining the new editions of Principia, and starting around 1700 on the calculus war against Leibniz. Couple of times he still resumed for a moment just to show that 'the old lion still had the claws' (we will discuss this later on while discussing calculus of variations). Also in 1690's Newton had serious problems with his own health, being crippled weeks without sleep and sometimes paranoid thoughts. However, his healt became more stable at the end of the century.
- At the end of the century Newton wanted a higher official post for himself, and preferred to live in London. With the help of some of his friends in important positions he got a job in the Royal Mint Factory. Having shown his ability to effectively persecute counterfeiters (often to death) he became the 'Master of the Mint' in 1699, a job which he took an equally effective care of during the rest of his life.

IV.3. Master of mint and the war on calculus

- The unfortunate Calculus war launched in a mild form already in 1693, when Wallis published some of Newton's calculus and at the same time claimed that Newton had explained his method to Leibniz in the 1676 letter. It is true that in the continent Newton's calculus was mostly forgotten, and Leibniz's star was in steady rise until 1700 – many thought that he was the principal founder of calculus (at the same time admiring 'Principia' that contains no calculus!). At this time Leibniz and Newton were on good terms, in the 1690's both still wrote flattering letters to each others. At least Leibniz appears rather sincere here, when he heard the news of the 'new Master of Mint' he expressed his concern of the subsequent loss for science.

IV.3. Master of mint, and the war on calculus

- Finally the battle on calculus started earnestly when eager younger fans of Newton, Fatio and Keill (Newton's apes according to Bernoulli) started actively propagate the priority issue. Newton himself kept silent, but later research has revealed that he worked actively in background, essentially leading the operations against Lebniz.The real start was Fatio's article in 1699, where he cleverly indicated that 'Leibniz had a change to steal from Newton'. Leibniz defended his cause hoping that Newton, whom he admired would come to help, but the latter stayed silent.

IV.3. Master of mint and the war on calculus

- In fact, Leibniz himself wrote an anonymous review (typical for him) of Newtons 'Optics' that was finally published in 1704, and where in the appendix Newton finally published some of his calculus himself. In his review Leibniz wrote in a decorative style 'Instead of the differences of Leibniz, Newton applies and has always applied fluxions.... as also Honorary Fabrious in his 'Synopsi Geometrica' substituted progressive motion in the place of indivisibles of Cavalieri..." . This could be interpreted as saying that Newton stole from Leibniz. This probably was the decisive thing for Newton, and after this he had a definite goal in the war. Hooke died in 1703, and Newton was elected as the President of the Royal Society. His fame was rising all over the Europe as of no scientist before, and along with all the power he had he probably felt that his position was now secured...

IV.3. Master of mint and the war on calculus

- At some point Johann Bernoulli (by then the most important active mathematician) tried to help Leibniz with an anonymous letter, where he unfortunately made the counterattack by noting that perhaps Newton stole the calculus from Leibniz. This of course angered Newton further, who became more determined to make things clear, and he started (silently) push his case even more vigorously.
- Leibniz still did not believe that Newton himself was involved in any way, and he made a request for the Royal Society to clean his reputation. This was a very unfortunate move since Newton was the president, and (again silently) he organised a committee that appeared at least somewhat balanced to study the case. Its report *Commercium Epistolicum* was published in 1714 and it found Leibniz guilty! In reality, Newton was secretly leading the writing of the report. Newton also published an anonymous letter that condemned Leibniz's notation (one of the great inventions of the latter mathematician) ... "Newton is not confined to notation..." Finally it became clear to Leibniz that Newton himself was fully in the game, they even exchanged some letters.

IV.3. Master of mint, and the war on calculus

- Leibniz died in 1716. Even this did not do much to smoothen Newton's attitude towards Leibniz. Newton's national and international fame rose steadily, and when he died in 1726 he was buried, in a style reserved usually for royal people, in Westminster Abbey.

V: CALCULUS OF VARIATION AND WAVE EQUATIONS IN THE 1700's

V.1. Early history of calculus of variations

- According to antique texts on history, the founder of Karthago, Queen Dido arrived on the coast of North Africa (perhaps around 900-800BC). The area was ruled by the Berber king Iarbas. Dido asked him for just for a small piece of land until she could continue her journeying, only the area as could be encompassed by an oxhide. Dido cut the oxhide into thin strips which anabled her to encircle an entire nearby hill, and this was to be the birth of Karthago
- Mathematically, we are facing a typical problem of calculus of variations. What is the largest planar area that can be surrounded by a curve of given length? The answer was known to Greeks, and we shall return to this question later on.

V.1. Early history of calculus of variations

- Hero of Alexandria ($\sim$ 10-70) claimed that under several reflections light chooses the path that takes the shortest time. Ibn al-Haytham aka Alchaen ($\sim$ 965-1040, mathematician in Kairo) extended this to cover also refraction. Fermat restated the principle and shows that Snell's law of refraction can be obtained as a mathematical consequence of the principle! Of course Fermat's proof belongs more to the standard calculus of extreme values, but he spoke of lights behaviour as a general variational principle that nature obeys.

- In Principia Newton gave (without proof) the solution to the first variational problem where the problem is non-trivial and the answer is hard to guess. The problem he studied consist of determining the shape of a bullet (or ship's bow) of least resistance. His approach of proof was later found amongst the notes Newton left behind. It basically coincides with the approach layer employed by Leibniz, Bernoulli brothers, and (even later on) by Euler. We shall take a closer look at that in connection with Euler's work on calculus of variations.

V.1. The Brachistocrone problem

• The question is simple: determine the curve between two given points A to B so that a mass point that moves along the curve just under the gravitational force of Earth travels as quickly as possible.

• The problem was posed by Johann Bernoulli in *Acta Eruditorum* in June 1696 as a challenge to fellow mathematicians. Perhaps one motivation was to try to check whether Newton could solve it – secretly hoping that the continental mathematicians could outdo the writer of Principia.

• Assume that the starting point is the origin, and the end point is $(a, -b)$ with $a, b > 0$. Mathematically the problem is then equivalent to determining function $y : [0, a] \to (-\infty, 0]$ such that $y(0) = 0$, $y(a) = -b$ and the integral

$$\int_0^1 \frac{\sqrt{1 + y'(x)^2}}{\sqrt{-2gy(x)}} dx$$

takes as small value as possible (derivation on the blackboard...) Of course one needs to assume some smoothness from y.

V.1. The Brachistocrone problem

- Actually Galileo had stated in *Two New Sciences* that the answer should be an arc of an circle. As we shall see (also observed by the young Huygens), his intuition was wrong.
- Solutions we gives by Jacob Bernoulli (1655–1705), Leibniz, (partially (?) also by L'Hôpital (1661–1704), E. W. von Tschirnhausen (10 April 1651 – 11 October 1708)),and (anonymously) by Newton. The story says that when Newton's solution reached Johann Bernoulli he noted 'I recognize the lion by his paw'! It took Newton only one evening after a long day's work to solve the problem.
- The solutions of others were somewhat similar to the basic method mentioned above (that we shall describe later on). However, Johann Bernoulli's solution used in a clever way Fermat's principle (to be described later on using the blackboard).
- Of course at this time everybody took it for granted that the problem has a unique solution.

V.1. The Brachistocrone problem

DEFINITION. A cycloid (generated by a circle of radius $r > 0$, rolling below the real axis, starting at the origin) is a curve of the form

$$x = r(t - r\sin(t)) \quad y = r(1 - \cos(t)) \qquad t \in [0, 2\pi].$$

Often one extends the curve by using all values $t \in \mathbb{R}$.

- The solution curve to the brachistocrone problem is the above cycloid, where $r > 0$ can be uniquely chosen so that the curve passes through the point $(a, -b)$ (Ercercise !).

V.2. Euler and Lagrange

- Euler apparently became interested in calculus of variations in the 1730's and published in 1744 the fundamental work *"The Method of Finding plane Curves that Show Some Property of Maximum or Minimum"*. The book contains the first general treatment of basic variational problem of the type

$$\min I, \quad \text{where} \quad I := \int_a^b F(x, y(x), y'(x))dx \qquad y(a) = y_0, \;\; y(b) = y_1.$$

- In modern notation Euler discretised the integral by setting $x_0 = a$, $x_N = b$ and $x_k = x_0 + kh$ for $1 \leq k \leq N-1$. Here $h = (b-a)/N$. Setting $y_k = y(x_k)$ and $y'_k = y'(x_k) \approx h^{-1}(y_{k+1} - y_k)$ one may approximate

$$I \approx h \sum_{k=1}^{N-1} F(x_k, y_k, \frac{y_{k+1} - y_k}{h})$$

V.2. Euler and Lagrange

- Euler set the differential of previous expression with respect to y_k zero, and obtained finally a differential equation (details at the blackboard)

$$(*) \qquad \boxed{\frac{d}{dx}\big(F_{y'}(x, y(x), y'(x))\big) = F_y(x, y(x), y'(x)).}$$

This equation is the famous *Euler-Lagrange equation*.

- Apparently Euler did not think finite approximations, but for him the integral is the sum of infinite many(!) terms that all are 'infinitesimal'. He perfumed freely computations (e.g. local minimising procedures) with infinitesimals. Moreover, he seemed to believe that his condition (*) was also sufficient. His paper from 1744 contained many other results and lot of examples, e.g. some early versions of multipliers for isoperimetric problems.

V.2. Euler and Lagrange

- Let us pause to make a modern derivation of Euler-Lagrange equations (*), assuming F is C^2, and y is an C^2-extremal (on the blackboard...)

EXAMPLE: Find the minimum of the integral $\int_0^1 (y'(x))^2 dx$ for functions y such that $y(0) = 0$ and $y(1) = 1$.

Solution: E-L-equations take the simple form $y'' = 0$, and we obtain the solution $y(x) = x$. One can in this simple case verify by Cauchy-Schwartz that the solution is an actual (global) extremal. In general this is difficult, and historically these kind of question arose only much later on....

Exercise: Apply Cauchy-Schwarz to show that $y(x) = x$ is the unique C^1-function that minimizes the above integral among all C^1-functions.
Hint: By the boundary conditions $\int_0^1 y'(x)dx0 = 1$.

Exercise: For given boundary values find the externals of the integral $\int_0^1 \big(f(y(x)) + y'(x)^2\big)dx$, where f is a given function. (Hint: use the Lemma on the next page).

V.3. Euler and Lagrange – Brachistocrone again

THEOREM: If the Brachistocrone problem has a solution that is C^2 on $(0, a)$, then this solution is given by a cycloid though the given points.

Proof. We shall solve (on the black board) the E-L-equations. For that it is useful to note:

THEOREM: If the integrand F does not depend on x (i.e. $F = F(y(x), y'(x))$), then the E-L-equations have the first integral

$$F(y, y') - y' F_{y'}(y, y') = C, \qquad \text{where } C \text{ is a constant.}$$

Proof. On the black board...

V.4. Lagrange and invention of calculus of variations

• Lagrange was just 19 in year 1755 when he sent his famous letter to Euler, describing his algorithmic use of the symbol δ. Thus, δ denotes the variation of a function under the consideration, or the variation of the integral as the function is varied. Lagrange observed that δ and the integral sign commute, and moreover $d\delta = \delta d$. By using these rules we obtain a shorter derivation of the E-L-equations (on the blackboard).

• The historian H. Goldstine believes that Lagranges approach was quite formalistic, and he probably did not really have in his mind even the broad idea of the modern proof that we saw previously. However, Lagrange's approach simplified and unified Euler's (and others) computations in a drastic manner. Euler himself was really taken by the young Italian's work, and supported him actively. Euler also coined the term 'calculus of variations inspired by Lagrange's notation.

V.4. Lagrange and the method of multipliers

- Euler already considered some problems related to calculus of variations involving side conditions in the integral form, often called isoperimetric problems . A typical problem could be a variant of the problem of Queen Dido: find a curve joining points $a < b$ on the real axis, having given length s and enclosing the largest possible area in the upper half plane. The problem is a special case of the following:

THEOREM: *Let F and G be C^2-smooth and assume that the C^2-function y satisfies $y(a) = y_0$, $y(b) = y_1$ minimizes (or maximizes) the integral*

$$\int_a^b F(x, y, y')dx$$

under the side condition $\int_a^b G(x, y, y')dx = c$, where c is a given constant (we omit some technical assumptions here). Then y satisfies the Euler-Lagrange equations for the integrand $F + \lambda G$ for some constant $\lambda \in \mathbb{R}$, and one has to pick the solution to these equations so that the two boundary conditions and the given integral constraint are satisfied.

V.4. Lagrange and the method of multipliers

Proof. (On the blackboard...) One makes use of the following lemma
LEMMA: Let h, g be continuous functions on the interval $[a, b]$ such that one has $\int_a^b \phi(x)h(x)dx = 0$ whenever $\int_a^b \phi(x)g(x)dx = 0$ (here ϕ is smooth and compactly supported on (a, b)). Then $f = \lambda g$ for some constant λ.

Solution of the (modified) Queen Didos' problem (on the blackboard).

V.5. Road to analytic mechanics in the 1700's

- After preliminary observations by Leibniz, Euler and Maupertuis (independently ?) proposed the principle of least action (which we do not discuss in detail). Again priority disputes (however, not between Euler and Maupertuis) followed... Euler was jot able to base mechanics on a single principle, but his extensive work on variational principles of mechanics was influential.
- By developing considerably further Johann Bernoulli's ideas, D'Alembert based statics on a single principle, called D'Alembert's principle of virtual work.

$$\sum_j (\mathbf{F}_j^{(a)} - \dot{\mathbf{p}}_j) \cdot \delta \mathbf{r}_j = 0.$$

V.5. Road to analytic mechanics in the 1700's

- After many important treatises on mechanics and calculus of variations, Lagrange finally obtained a coordinate free formulation of D'Alembert's principle. In his celebrated Mechanique analitique (1788) this is expressed as the equation

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_j}\Big) - \frac{\partial L}{\partial q_j} = 0,$$

where the q_j:s are independent (generalised) coordinates and L is the Lagrangian

$$L = T - V = \{\text{kinetic energy}\} - \{\text{potential energy}\}.$$

In 1840's Hamilton rephrased this as the famous (Lagrange's) principle of least action. Namely, the above equations are justhe Euler-Lagrange equations for integrand L. . Hence, nature acts in such a way that it tries to minimise the (integrated) difference between the potential and kinetic energy !!.

V.6. Some analytical issues in the 1700's

- The analysis kept its (sometimes hidden)roots in geometry alive until 1740's, when Euler's textbooks started to make their mark. Euler spoke on quantities without necessarily referring their geometric counterparts. Also, Euler spoke much more clearly about 'infinitesimal increments', and functions (like $f'(x)$) obtained as ratios of differentials took a more prominent role. The role of dependent variables became more clear. Power series (and Taylor series) were all important, and they were treated like 'infinite order polynomials'. Here one could just recall Euler's famous first proof of the formula $\sum_{k=1}^{\infty} k^{-2} = \pi^2/6$.
- Things took time to be invented and understood.For example:

– while solving the catenoid problem Johann Bernoulli in the 1690's he did not yet have the exponential function in his toolbox. The basic properties of e^x, especially in connection with complex arguments or differential equations were mostly discovered by Euler, who also disseminated the knowledge in his textbooks.

V.6. Some analytical issues in the 1700's

– During early 1710's Leibniz and Johann Bernoulli held different opinions how to define $\log(x)$ for $x < 0$. Leibniz thought it should be imaginary, while Bernoulli noticed that $d(-x)/(-x) = dx/x$, whence he thought that $\log(x) = \log(-x)$. Euler clarified the situation by noting that Bernoulli's argument just entails that the quantities differ by a constant $\log(-1)$, which he later on identified, and noticed in general the infinite many valued nature of the logarithm.It took long time from Bernoulli to admit that his position was less fruitful...

– Often mathematicians (especially Euler) were happy to use equalities like $-1 = 1 + 2 + 4 + 8 + \ldots$. An interesting question was how to sum the series $1 - 1 + 1 - 1 + 1 - \ldots$. Leibniz had an interesting probability based argument why the sum is $1/2$... Euler made a distinction between the 'value' and the 'sum' of a series. He was even able to compute the 'value' of the sum 1-1+2-6+24-120+...!

V.6. . Some analytical issues in the 1700's

– In general, 'functions' had to have analytic expressions. The domains of definition were not well defined. Complex and real arguments were easily treated on equal basis if needed. All this led some leading mathematicians to a long dispute in connection with the wave equation as we shall see soon.

– Euler's impact was remarkable also as a role model in publishing: his eagerness to publish his findings immediately and the new style in his textbooks was hugely influential. In contrast, the previous generation was still often hiding their tricks – Johannes Bernoulli and Leibniz kept their idea on partial integration for themselves more that 20 years!

– At the end of the century, while teaching in Ecole Polytechnique Lagrange tried to base all analysis on Taylor expansions, but this did not succeed well....

IV.5. The controversy about the wave equation and Fourier series

- Pythagoreans found many magic mathematical relations between the length of a spring and the height of the tone it produces, especially they famously found the series of overtones.
- Consider a string that has bee suspended between the points 0 and π. In the beginning of 1700-century one had accumulated some empirical evidence that the spring can vibrate in a mode corresponding a given overtone. Brook Taylor gave 1713 the first more accurate model for the movement of a vibrating string. Especially he gave the formula

$$u(x, t) = \cos(ct)\sin(x),$$

for the vibration of the fundamental tone.

- Jacob Bernoulli approximated the spring by a system of n mass points attached to each others like pearls, and he was able to derive Taylors result as a limit when the number of the mass points increases indefinitely.

V.7. The controversy about the wave equation and Fourier series

- In turn, Daniel Bernoulli (1700–1782), son of Johann Bernoulli, proposed in the 1730's that the higher modes obey the formula

$$u(x,t) = \cos(nct)\sin(nx), \quad n = 1, 2, \ldots.$$

he also noted importantly that linear combinations describe vibrations as well.

- D'Alembert obtained a breakthrough in 1747 when he showed that a vibrating string satisfies the wave equation

$$\frac{du^2}{dt^2} = c^2 \frac{du^2}{dx^2}$$

and noted remarkably that any combination

$$f(x - ct) + g(x + ct)$$

yields a solution. By this way one obtains also solution formula for a given general initial value problem. $u(x,0) = f(x), \quad \frac{du}{dt}(x,0) = 0,$ $u(0,t) = u(\pi,t) = 0.$

V.7. The controversy about the wave equation and Fourier series

- Euler got immediately excited on D'Alembert's equation and it's solution formula, and noted that it could be also used to describe realistic initial values where the string is at time $t = 0$ stretched only from one point and then released free, e.g. $u(x, 0) = \pi/2 - |x - \pi/2|$. This initial value is not differentiable at $\pi/2$. Euler thought that D'Alembert's solution formula produces a sensible result also in this case. D'Alembert was furiously against this and noted that the derivatives have to be well defined in wave equation, although the notion of derivative was not clear at this time. Form him the functions involved should be analytic. This caused a long lasting dispute between the two famous scientists.

V.7. The controversy about the wave equation and Fourier series

- In 1753 Daniel Bernoulli made the bold claim that every solution (say, of the initial value problem where initial velocity is zero and initial shape is given by function f) could be expressed as a superposition of overtones, i.e. in the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n \cos(nct)\sin(nx).$$

The coefficients c_n should be determined from the initial condition that takes the form

$$f(x) = \sum_{n=1}^{\infty} c_n \sin(nx), \quad x \in (0,\pi).$$

- Bernoulli also noted that the non-smooth initial values are covered by his solution. Of course D'Alembert was not able to accept this. In turn, Euler criticised Bernoulli's solution by noting that it was not clear whether every initial value could be expressed as sine series. Moreover, he complained the lack of formulae for the coefficients c_n.

V.7. The controversy about the wave equation and Fourier series

- Soon enough Clairaut found the formula

$$c_k = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)dx.$$

Also Euler found this formula, although he did not publish it. The use of Fourier series remained quite limited before Fourier's work in 1807.

- For Euler also *discontinuous* solutions (or initial values) were allowed.By this he meant functions whose is different on different intervals. E.g. function $f(x) := x$ for $x \geq 0$ and $f(x) = -x$ for $x < 0$ is discontinuous in this sense. Euler even argued that one may change such a function in an infinitesimal neighbourhood of 0 by an infinitesimal function that makes it smooth.

- Lagrange showed in 1759 that Jacob Bernoulli's limiting argument leads to results that agree with Euler's point of view, but his work did not gain universal acceptance.

V.7. The controversy about the wave equation and Fourier series

- Overall, the vaguest concept of a functions and the theory of Fourier series were not sufficenly developed during 1700's to be able to settle the controversy in any satisfactory manner.

V!: EMERGING QUEST FOR RIGOR IN ANALYS IN 1800's: PRE-WEIERSTRASS PERIOD

VI.1. Fourier and his series

- The theory of Fourier series started seriously in 1807 (1822) as Jean-Baptiste Fourier (1768–1830) published his famous treatise (book) Théorie analytique de la chaleur (The Analytic Theory of Heat). It contained several groundbreaking ideas:

– the basic claim that every function on $[0, \pi]$, whether continuous or not (!) can be expanded to a sine series! And many different applications of this statement .

– The continuous form (Fourier transform) $\widehat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi\xi x} f(x)dx$ and its inversion formula obtained as a limit of the corresponding statement for Fourier series.

– Many nontrivial applications of Fourier series and transforms to partial differential equations.Especially Fourier had derived the heat equation)

$$u_y = k\Delta u$$

and applied in an innovative way (to be described at the blackboard) his new methods in solving it in various situations.

VI.1. Fourier and his series

- For peers of Fourier it was a most fantastic observation that a sine series whose terms are analytic could converge to non-continuous function, even to a "function" that is non-continuous at infinitely many points, and also needs infinitely many steps in the piecewise definition – just look at the Fourier series of the 2π-periodic function $\mathrm{sgn}_\pi(x)$, where $\mathrm{sgn}_\pi(x) = -1$ for $x \in (-\pi, 0)$, $\mathrm{sgn}_\pi(x) = 1$ for $x \in (0, \pi)$, and $\mathrm{sgn}_\pi(n\pi) = 0$ for all $n \in \mathbb{Z}$.
- Fourier did not provide a proof of the convergence. Lagrange was impressed by the novelty of Fourier's ideas, but noticed that one is missing a lot in the rigour.

VI.1. Fourier and his series

- Fourier's ideas enlarged the notion of a function. In his own words:

In general the function $f(x)$ represents a succession of values of ordinates each of which is arbitrary. An infinity of values being given to the abscissa x, there are an equal number of ordinates $f(x)$. All have actual numerical values, either positive or negative or nil. We do not suppose that these ordinates to be subject to a common law; they succeed each other in any manner whatever, and each of them is given as it were a single quantity.

- However, in real examples and situations Fourier did not use such a general concept, and in fact in (attempts) of proofs usually assumed implicitly that the functions are continuous.

VI.2. Cauchy's 'Course d'Analyse

- Cauchy started to teach at École Polytechnique in 1815, and continued to do it it until 1830. Later on he reached in Torino, Prague, and finally from 1848 again in Paris at the Faculté des Sciences. The first version of his lectures, 'Course d'Analyse', were published in 1821, and through it, and the new enlarged/corrected printings that followed, they became the major impact in the new movement towards rigour.

- Cauchy's first definitions of e.g. limits or continuity appear surprisingly old-fashioned and vague to our eyes.

– LIMIT: When the values successively attributed to the same variable approach a fixed value indefinitely, in such a way as to end up by differing from it as little as one could wish, this last value is called the limit of all the others. (Cauchy 1821)

VI.2. Cauchy's 'Course d'Analyse

–CONTINUITY: Let $f(x)$ be a function of the variable x and suppose that for each value of x between the given limits this functions always takes unique and finite value. If, having a value x between these limits, one attributes to the variable x an infinitely small increase α, the function itself increases by the difference $f(x+\alpha) - f(x)$ which depends simultaneously on the new variable α and the value of x. This done, the function $f(x)$ will be, between two limits assigned to the variable x, a *continuous* function of these variable if, for each value of x intermediate between these limits the numerical value of the difference $f(x+\alpha) - f(x)$ decreases infinitely with α. In other words, the function $f(x)$ will remain continuous with respect to x between the given limit if, between these limits, and infinitely small increase in the variable produces an infinitely small increase in the function itself. One says furthermore that the function $f(x)$ is, in the neighbourhood of a particular value attributed to x, a continuous function of this variable, whenever it is continuous between two limits of x, however close, which contain that value of x.

VI.2. Cauchy's 'Course d'Analyse

- Moreover, perhaps in order to comply with the wishes of the times (and the students), Cauchy introduces in towards the end of his course the Leibnizian infinitesimals and used also them freely!
- The dubious nature of the above definitions is, surprisingly enough, often compensated by unexpected strictness in the actual proofs of Cauchy. Also ε appears!

THEOREM If the function $f(x)$ is positive for very large x and the ratio $f(x+1)/f(x)$ converges to a limit k when x increases indefinitely, then the expression $[f(x)]^{1/x}$ will at the same time converge to this limit.

Proof. Let us first assume that the quantity k, which is necessarily positive, has a finite value and let ε denote a number, as small as one wants. Since the increasing values of x make the ratio $f(x+1)/f(x)$ converge to the limit k, one may give the number h a value so large that for x equal or larger that h, the said ratio is constantly enclosed between the limits $k-\varepsilon \quad , \quad k+\varepsilon$ ...

VI.2. Cauchy's 'Course d'Analyse

- Cauchy's treatment of series was very innovative, and often rather strict even in modern standards. He made the great creative act by defining a Cauchy sequence. Of course at this point of time one did not have doubts that the limit value exists...
- A remarkable achievement of Cauchy was his treatment of the concept of integral. For Leibnitz et al it was most often defined as the inverse operation of differentiation. Fourier was not afraid of discontinuous functions, and he thus needed to define integral as the area under the graph of the function to be integrated. Cauchy in turn proved rather rigorously that the integral can defined as a limit of approximating sums using finer and finer subdivisions ! His proof assumes implicitly uniform continuity, and existence of limit points to Cauchy sequences. Cauchy's notion of integral was was an important first step towards abstract and rigorous notion of an integral.
- Cauchy also gave an ingenious and original proof of Newton's binomial theorem (this lead him to the famous 'Cauchy's functional equation).

VI.3. Gauss

- Gauss was famed for 'Gaussian rigour', and although he did very little work in 'pure' analysis, it appears that he had independently and earlier than Cauchy arrived at many similar ideas on rigour. He abhorred divergent series, and his treatment of hypergometric series (1813) is impressive in its careful insistence of convergence. In his notes from around 1800 one observes a very accurate definition of lim sup and lim inf for a series. Unfortunately (again), he did not publish these ideas.
- Gauss' 'Allgmaine Theorie des Erdmagnetismus (1838) contains a proof of the Gauss' theorem, which the historian John Archibald calls striking in the level of rigour.

VI.4. Bolzano

• Bernard Bolzano (1781-1848) also had a remarkable new insights (published 1817) to rigour. These include a verbal definition of continuity, that is very close to the modern one: ... the difference $f(x+\omega)-f(x)$ can be made smaller than any given quantity provided that ω can be taken as small as we please. Moreover, he gave a 'proof' of the existence of supremum for an arbitrary subset of reals, and used it to show the familiar Bolzano's mean value theorem.

• Bolzano's proof of the existence of supremum could almost be transferred to modern textbooks: if the nonempty set $A \subset \mathbb{R}$ is bounded from above, one can find a_k:s so that $a_k \in A$ but all points in A are smaller than $a_k + 2^{-k}$. Finally one observes that a_k:s form a 'Cauchy sequence'. However, Bolzano's proof of the convergence of a Cauchy sequence was unsatisfactory.

VI.4. Bolzano

- Bolzano worked a a priest in Prague, and later as a professor of philosophy. Bolzano startled many of his colleagues with his teachings of the social waste of militarism and the needlessness of war, and was finally dismissed from the university in 1819. For a long time he was exiled to countryside. He worked on many fields, besides mathematics in philosophy and logic. His posthumously published work ' Paradoxien des Unendlichen (The Paradoxes of the Infinite) (1851)' was well received by e.g. Cantor and Dedekind.

VI.5. The problem of non-uniform convergence: Abel's criticism

- Like Gauss, Abel was devastated by the free and unrigorous manner many mathematicians performed analysis in his time. In his own words from letters to Hansteen and Holmboe:

– I will apply all my strength to bringing more light into the vast darkness that unquestionably exists in analysis... Everywhere one finds the unfortunate method of concluding from special to the general, and it is very strange that after such a procedure there exists only few of the so-called paradoxes...

– On the whole divergent series is a devilry, and it is a shame that one dares to base any demonstrations on them... Can you think of anything more terrible that saying that $0 = 1 - 2^n + 3^n - 4^n + \dots$, where n is a positive integer. Risum teneatis amici.... Even the binomial formula is not rigorously proved...

VI.5. The problem of non-uniform convergence: Abel's criticism

- Abel started by using Cauchy's lecture notes that he admired as a basis. He gave a proof of the convergence of the binomial series, and in connection with that he observed that a trigonometric series of the type

$$\mathrm{sgn}_\pi(x) = \frac{4}{\pi}|big(\sin(x) + \frac{1}{3}\sin(2x) + \frac{1}{5}\sin 5x + \dots\big)$$

converges at every point to the right limit, every partial sum is continuous, but the sum is not continuous! This was in contradiction to a theorem of Cauchy in 'Course d'Analysis', which Abel pointed out. However, some historians think that Cauchy's definition somehow implicitly contained the needed assumption of uniformity.

- However, even if the various criticisms are taken into account, the time was note yet ripe for much more rigour than Cauchy's school provided: Abel himself (in his proof of the binomial theorem) missed the need of clearly establishing uniform continuity for power series.

VI.5. The problem of non-uniform convergence: Abel's criticism

- It took a long time, until the second half of the century, that people realized that one can write down very simple examples showing the the need of uniform convergence in order to obtain continuous limit functions, like setting

$$f_n(x) = \begin{cases} nx & \text{for} \quad x \in [0, 1/n], \\ 1 - \frac{n}{n-1}(x - 1/n) & \text{for} \quad x \in [1/n, 1]. \end{cases}$$

- Cauchy, and especially Stokes (1849) spoke about 'infinitely slow convergence' (i.e. non-uniform convergence). In any case, in early 1850's, the need of clearer statement of uniform continuity was 'in the air' . Criticism of this point in Cauchy's lecture notes came, at least indirectly, from his own students Briot and Bouquet, and Stokes.

VI.5. The problem of non-uniform convergence: Weierstrass

- It was Karl Weierstrass (1815–1897) who finally clarified the concept of uniform convergence and its ubiquitous role in analysis. His teacher Christoph Guderman (1798–1852) already noted the importance of the convergence 'in a uniform way', but did not yet possess a clear definition. Weierstrass himself used uniform convergence in a crucial manner in the proof of his famous theorem (1841) stating that if a sequence of analytic functions converge uniformly in a connected domain, then the limit is analytic and the series can be differentiated term by term. After he started 1856 to lecture regularly in University of Berlin for more than 30 years, the message of uniform convergence started to spread slowly among the new generations. We come back to the famous Weierstrass rigour and its influence later on.

VI.6. Dirichlet's 1829 paper on Fourier series

- Johann Peter Gustav Lejeune Dirichlet (1805–1859) was one of the truly great mathematician of the 19. century. He made deep contributions e.g. in analysis and number theory. What comes to rigour, perhaps Jacobi's words describe well his status: "If Cauchy claims to have proven something, it is often ok, if Gauss does it it is usually correct, but when Dirichlet says so I can trust it."
- Dirichlet studied in Paris 1822-25. There, besides getting interested in number theory, he met Fourier and Poisson who initiated him to the theory of PDE's and Fourier analysis. Apparently Fourier told him about the urgent problem of understanding convergence of Fourier series.
- Let us first discuss a little bit the previous attempts of proof of the convergence of Fourier series:

VI.6. Dirichlet's 1829 paper on Fourier series

– Fourier (1822) expressed the n:th partial sum at 0 as the integral $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_j(y)f(x)$, where

$$K_j(x) := \cos(jx) + \sin(jx)\frac{\sin(x)}{1-\cos(x)}$$

He argued that because of the rapidly increasing oscillations the part corresponding to the first term goes to zero. In the the latter term similarly one only needs to consider a neighbourhood of the origin, and there Fourier did not hesitate to set f equal to $f(0)$, which allowed him to bring the argument into a conclusion.

– Poisson (1820) studied the cosine series and replaced $\sum_{n=1}^{\infty} a_n \cos(nx)$ by $\sum_{n=1}^{\infty} \rho^n a_n \cos(nx)$, where $\rho \in (0,1)$. This led him to the famous Poisson kernel. However, his conclusion when finally setting $\rho = 1$ was far from rigorous.

VI.6. Dirichlet's 1829 paper on Fourier series

– Cauchy (1827) wanted to prove that the Fourier series converges. Perhaps it tell something of Cauchy that he without explanation assumed that f is analytic, and applied his new residue theorem on it to transform the partial sum into a more manageable form. Moreover he applied the following statement:

If $\sum_{n=1}^{\infty} a_n$ is convergent, then also $\sum_{n=1}^{\infty} b_n$ converges assuming that $\lim_{n\to\infty} a_n/b_n = 1$. "

EXERCISE: Find a counterexample to Cauchy's statement.

VI.6. Dirichlet's 1829 paper on Fourier series

- Dirichlet published in 1829 his fundamental paper on Fouries series. This paper is remarkable in several ways:

– The proof is rigorous in modern standards, it could be translated almost word by word to the modern class room!

– Dirichlet gives exact assumptions required on the function f. Thus, he proves that Assuming that f is piecewise continuous and monotonic its Fourier series converges at each point to the value $(f(x^+) + f(x^-))/2$.

– In the proof Dirichlet introduces the Dirichlet kernel and writes the n:th partial sum in the form

$$S_n f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin((n+1/2)(x-y))}{\sin((x-y)/2}dy.$$

VI.6. Dirichlet's 1829 paper on Fourier series

– Dirichlet gives a modern definition of the function, and notes that one cannot define integral of certain functions, and gives his famous example of such a function (below $c \neq d$)

$$f(x) = \begin{cases} c & \text{for} \quad x \in \mathbb{Q} \\ d) & \text{for} \quad x \notin \mathbb{Q}. \end{cases}$$

Thus f is nowhere continuous.It is perhaps the first example ever of a truly non analytically given function!

– Dirichlet states that he will later on generalise his theorem, which will require a generalization of the notion of integral. This he never did. Still in a letter to Gauss in 1853 he was optimistic in his project, and Gauss also expressed his belief that the Fourier series of a continuous function should converge (both were wrong!). Most likely Dirichlet passed on his dreams to Riemann, and by this manner Dirichlet indirectly contributed to the generalizations the notion of integral.

VI.7. Riemann's contribution to integral and Fourier series

- Bernhard Riemann (1926–66) was one of the deepest mathematicians of all times. His contributions include complex analysis, Riemann surfaces and manifolds. In these areas it was more his deep innovations that counted, not rigour partly because time was not ripe for it. However, he continued the program started by Dirichlet, that could be summarised simply by the two questions:

– when is a function integrable?
– when does its Fourier series converge?

- Riemann published in 1854 (for his Habilitationsschrift) his theory of integral, as we learn it today in first courses of analysis. He noted 'that the functions not covered by Dirichlet's analysis do not probably occur in nature, but also observed that generalisation was well motivated by the desire to understand better the fundaments of infinitesimal calculus, and for sake of possible applications inside mathematics!
- We shall recall the basic idea of Riemann's integral at the blackboard...

VI.7. Riemann's contribution to integral and Fourier series

- Riemann gave an example of an integrable function that is discontinuous in a dense set: $f(x) = \sum_{k=0}^{\infty} \frac{(kx)}{k^2}$, where (x) is the decimal part of x.
- He also proved an ingenious theorem, where the convergence of an arbitrary trigonometric series (not necessary a Fourier series, the only condition is that the coefficients tend to zero) is related to the existence of a limit of (certain kind of) a second difference quotient of the doubly integrated series. This results was later on the starting point of the investigations of Cantor, that lead ultimately to his set theory.
- Riemann's paper was known in small circles, but was published only in 1867 and had wide and strong impact after this. It was widely regarded as the most general possible notion of an integral!

VI.8. 'Die Weierstraßche Strenge'

- Karl Weierstrass (1815–1897) was born in Ostenfelde, Westphalia, Germany. As a youth he was rather wild character according to many stories, slightly opposite to the image of his later personality as peace loving, very human and well-anchored person. Part might be explained by his need to oppose a strong-willed father, who was eagerly trying to guide his children's lives even when they were middle-aged. His father put his talented child to Bonn's University in order to become a high civil official. The stories say that he was not satisfied with this, and spent his time with his pal's concentrating mostly on fencing and drinking. Probably also quite a bit time on math...
- In 1839, after unfinished years at the university, he was allowed to enroll at the Theological and Philosophical Academy at Munster, where his aim was to obtain a secondary schools teaching degree, and keep doing his mathematics at the spar time allowed for in the evenings.

VI.8. 'Die Weierstraßche Strenge'

- Luckily one of the teachers there was Gudermann, who became a source of inspiration for Weierstrass. Gudermann was a strong advocate of power series, and had some understanding of uniform convergence. As we know, Weierstrass became a master in both topics.
- In year 1841 Weierstrass both graduated as teacher and published his first math paper, as mentioned before. Gudermann was convinced of his students genius, but his recommendations were not taken seriously enough, and Weierstrass started as a teacher, first in Deutsch-Kronessa, and then in Braunsberg, altogether 15 years. Weierstrass was able to continue mathematics despite of huge work load. Finally his paper on Abelian functions was published in Crelle Journal, and he became immediately famous in the mathematical circles. Richelot (successor of Jacobi) himself travelled to Braunsberg to bring Weierstrass the diploma of a honorary doctorate in Königsberg. In a couple of years he became a professor in Berlin and a member of the Berlin academy.

VI.8. 'Die Weierstraßche Strenge'

• Weierstrass was a master teacher, both as a lecturer and thesis advisor. On the other hand, because of health problems, his way to lecture was most of the time the following: he was sitting near the blackboard and dictated to a chosen student what to write. . He behaved equally towards all people, and was very slow in publishing. The famous white wooden box containing his notes and unpublished results unfortunately was lost after his death. Many of his results were actually published by his students. He was the teacher of Sonya Kowalewskaja, Gösta Mittag-Leffler and many other famous mathematicians. His mathematical leadership was strongly felt in all the mathematics circles in Europe, in most areas of mathematics. This came through his students, hearer's of his lectures, and his rare but the more influential publications, and through his fame as the leading exponent of the new 'Weierstaßian rigour'.

VI.8. 'Die Weierstraßche Strenge'

- Weierstrass lectured in cycles of 4 semesters. Almost invariably the titles of his lectures consisted of the following:

– Theory of analytic functions
– Theory of Elliptic functions
– Applications of elliptic functions to geometry an mechanics
- Theory of Abelian functions

- Weistrass approach to the foundations of analysis was explained mainly during the first course, but other courses served as examples of how to apply this new philosophy. He e.g. introduced the strict use of $\varepsilon - \delta$ techniques, gave strict foundations to the real number system, gave precise definitions of differentiability, continuity, integral and other basic notions of analysis in a precise manner, and proved basic results with rigour. Weiestrass gave several important counter examples that provided fundamentally new thinking and directions in e.g. calculus of variations, and real analysis. We shall discuss these soon in more detail in connection with more specific topics.

VI.9. Emergence of pathological functions

- As mentioned before, Fourier found functions (=limits of continuous analytic expressions) that consist of infinitely many non-continuous 'bumps'.
- Cauchy (1823) produced an example of a function that is C^∞, but its Taylor series does not converge.
- Ampere (1806) gave a (flawed, of course) proof that any function has derivative apart certain particular isolated values of x!
- The previous result is not so unique: before 1870 in most textbooks of calculus it was stated (and often proved) that continuous functions must be differentiable in 'most points'. Often it was also stated (despite of Cauchy's example $f(x) = x\sin(1/x)$) that all continuous functions are piecewise monotonic. Lamarle and other mathematicians published proofs of various kinds, always concluding that points of non-differentiablility of continuous functions possess form a small set.

VI.9. Emergence of pathological functions

- Riemann claimed in his lectures that the (Riemann) function $R(x) : \sum_{k=1}^{\infty} \frac{sin(nx)}{n^2}$ is (basically) nowhere differentiable, or at least that its points of non-differentiability are dense.

- Hankel noted that $F(x) := \int_0^x \left(\sum_{k=0}^{\infty} \frac{(ky)}{k^2} \right) dy$ is not differentiable at the dense set of points where the integrand has a jump discontinuity (e.g. when $x = m/n$ with m, n relatively prime).

- Weierstrass released a bomb when he proved (presented to the Berlin Academy 1872, published by Paul du Bois-Raymond in 1875) the following theorem:

THEOREM: Assume $0 < a < 1$ and $b > (1 + 3\pi/2)/a$ is a positive odd integer. Then the function

$$f(x) := \sum_{n=0}^{\infty} a^n \cos(b^b \pi x)$$

is nowhere differentiable.

Proof. On the blackboard...

VI.9. Emergence of pathological functions

- A similar example was also given slightly later on independently by Darboux.
- These examples were shocking for big part of the community. Even Poincare called such thing 'bizarre'. However, for a group of younger mathematicians they provided stimulus for deeper studies of integrability and other related issues...
- The difficulty of the things is seen by noting that Weierstrass believed that there is are increasing functions that are nowhere differentiable. This is wrong by the well-know theorem of Lebesgue, however Weierstrass never claimed for a proof.

VI.9. Emergence of pathological functions

- Hardy proved in 1916 that the Weierstrass function is nowhere differentiable just under the condition $0 < a < 1$ and $ab \geq 1$.
- Hardy also showed that the Riemann function $R(x)$ is not differentiable at any irrational point (and also not in certain rational points). The picture was completed by Joseph Gerver, who proved in 1971 (he was a schoolboy by that time) that R is differentiable only at points of the form $n\pi/m$, where n, m are odd integers !
- One often states that Bolzano gave the first (geometrically defined) nowhere differentiable function. In fact, he gave 1830 the example but noted that it is non-differentiable only in a dense set. Actually, his aim was this to criticise a paper of 18 years old Galois, where the latter claimed that 'function's are differentiable apart from single exceptional points'...

VI.10. The unraveling of real numbers: existence of transcendentals

- We have met the irrationals (quadratic surds) of Greek mathematician, binary numbers of Leibniz, ... The quest for better understanding of the reals ($\mathbb{R}$) started in the first half of the 1800's. For example Cauchy had assumed that every Cauchy sequence converges, but some mathematicians started to ask for a 'definition' or real numbers, hopefully in terms of more easily understood quantities, like rational numbers or natural numbers.
- Recall that *n*:th degree algebraic numbers are solutions of an (irreducible) equation $a_n x^n + \ldots + a_0 = 0$, where $a_n \neq 0$. These numbers were in general thought to be the most 'accessible' of reals, although apparently in the 1700's the leading mathematicians suspected the existence of 'transcendental numbers' (the term was coined by Leibniz for transcendental functions).

VI.10. The unraveling of real numbers: existence of transcendentals

- The irrationality of some important numbers were established already in the 1700's. Especially, one has

THEOREM: (Euler1737) e is irrational.

Proof. Euler's proof was based on noting that the simple continued fraction of e has the form

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, \ldots.$$

However, we shall give the proof of Fourier (on the blackboard...).

THEOREM: (Lambert 1761) π is irrational.

Proof. Lamberts proof used continued fractions...

VI.10. The unraveling of real numbers: existence of transcendentals

- Finally in 1837 Liouville proved the existence of transcendental numbers. His proof is based on the following beautiful result, which shows that algebraic numbers have some limitation for the speed by which they can approximated by the rationals:

THEOREM (Liouville). Assume that x is an algebraic number of degree $n \geq 2$. Then there is a constant c (which may depend on x) such that for any rational number p/q (here $p, q \in \mathbb{Z}$, $q \geq 1$) one has

$$\left|x - \frac{p}{q}\right| \geq \frac{c}{q^n}.$$

Proof. (On the blackboard...).

COROLLARY (Liouville). The number $\sum_{n=1}^{\infty} \frac{1}{10^{n!}}$ is transcendental.

- Hermite showed in 1873 that e is transcendental (proof later simplified byHilbert and others. Lindeman finally verified in 1882 the transcendentally of π. Important improvements of Liouvilles inequality were obtained by Thue, Siegel and (finally) Roth.

VI.10. The unraveling of real numbers: Cantor's diagonal argument

• Cantor proved the existence of transcendentals by showing the following theorem that was very striking for his contemporaries:

THEOREM (Cantor 1874). The set of real numbers is not countable, whereas the set of algebraic numbers is. Hence there are 'more' transcendental than algebraic numbers. Proof. (On the blackboard...).

• As discussed in the lecture of Francesca Corni, the above fact and Cantor's work in Fourier series (to be discussed at the blackboard...) lead Cantor to the basic notions of set theory: cardinality and the well-ordered set of ordinals! Unfortunately we do not have more time to discuss the first steps of set theory or of logical foundations of mathematics whose study started in earnest in late 1800's (Frege, Cantor, Peano, Russell,...).

• Before Cantor several mathematicians had pondered somewhat the problem of infinite sets. Bolzano's book on paradoxes was mentioned before, and already Galilei has noted that paradoxically there is a bijection from $\mathbb{N}$ to its proper subset. Dedekind used this property as the defining property of infinite sets.

VI.10. The unraveling of real numbers: constructing the reals

- Bolzano made a good try to construct reals from rational numbers in 1830's. Hankel expressed in 1867 ('Theorie der komplexen Zahlensysteme') that real numbers should be viewed as 'intellectual structures', derivable from more elementary notions. In the book Hankel verified that $\mathbb{C}$ cannot be extended to a larger number field with retaining all the basic properties (a fact presumably known to Gauss, and contained in earlier lectures by Weierstrass).
- Weierstrass was perhaps the first one to construct successfully real numbers using rationals (or integers) as the starting point. His construction was based on certain 'aggregates' of rational numbers, and is not the easiest one to apply.The construction of Weierstrass was given in his lectures in the 1860's, but it was published only in the 1870's by his students.

VI.10. The unraveling of real numbers: constructing the reals

- Charles Meray (1835–1911) 1869–72 published a simpler construction of reals, basically considering equivalence classes of Cauchy sequences of rationals. However, he called non-rationals thus obtained 'fictitious numbers'.
- Cantor and Heine independently found the same approach as Meray, and Heine published their definition in Crelle journal in 1872. He tried to interpret the outcome by viewing the obtained reals as 'tangible signs'...
- Another interpretation was given by Dedekind, who published his construction in 1872, although it probably dates yo even earlier years than Weierstrasses first lectures on the topic! His famous Dedekind cuts are familiar to many of us from the basic math (to be discussed on the blackboard...). Dedekind had clear point of view that reals (and even natural numbers) are constructions of the human mind! This view was quite modern in its time. In a sense it is close to the famous dictum of Kronecker, but the points of views of these mathematicians were otherwise quite diametrical!

VI.10. The unraveling of real numbers: constructing the reals

- Further ponts of view were developed at the end of 1800's, like Johannes Thomae's (1840–1921) enforcement of Heine's ideas of arithmetics of reals as just a game with symbols with given rules, or Frege's important but somewhat problematic program to reduce natural numbers to axioms. In 1900 David Hilbert 1862–1943 gave an axiomatic definition of reals. For Hilbert the consistency of these axioms was guaranteed if the natural numbers are free of contradictions.
- The Historian Moritz Epple notes: ' What was a real number at the end of 1900's century? A intuitive, geometrical or physical quality, or a ratio of such quantities? Creations of human mind? An arbitrary set of signs subjected to certain rules? A purely logical concept? Nobody was able to decide this with certainty. Only one thing was beyond doubt: there was no consensus of any kind...'

VI.11. Calculus of variations in the 1800's

- Let us return to the calculus of variations. When discussing its development in the 19. century, we shall restrict ourselves to the single basic question that slowly emerged: For a 'nice' variational problem the Euler-Lagrange equation is a necessary condition for an extremal. But how do we know that extramals exists at all?
- Even before that one could ask how to differentiate whether a given extremal yields a minimum or a maximum. For that end, at the end of 1700's Legendre made a natural step by considering second variation of a variational integral, that for the standard problem can be written in the form (at the blackboard)

$$\delta^2 I = \int_a^b \left(f_{yy} w^2 + 2 f_{yy'} w' w + f_{y'y'} w'^2 \right) dx.$$

The integral does not change if one adds a term of the form $(w^2 v)'$, whence we obtain the form

$$\delta^2 I = \int_a^b \left((f_{yy} + v') w^2 + 2(f_{yy'} + v) w' w + f_{y'y'} w'^2 \right) dx.$$

VI.11. Calculus of variations in the 1800's

- If v can be so chosen that one gets an exact square, one can deduce that $\delta^2 I$ is positive. The conditon for this is $(f_y y + v')f_{y'y'} = (f_{yy'} + v)^2$, i.e. a differential equation of Riccati type. Legendre claimed that a necessary condition for the existence of the required v was simply that $f_{y'y'} > 0$ on the area of consideration. This was shown to be not sufficient by Lagrange.

- in 1837 Jacobi made deep contributions to the research direction initiated by Legendre. Especially, he was able to rephrase the solvability of the Riccati equation of Legende in terms of non-existence of so-called conjugate points, and this he connected in a surprising way to the whole fields of externals of the Euler-Lagrange equation.
- in 1870's Weierstrass finally obtained his necessary condition for extremity that under some circumstances is also sufficient! Hilbert's invention of the 'Hilbert invariant integral' simplified Weierstrass theory considerably, but we skip this part and instead take next a look at the interesting (and related) history of Dirichlet's principle.

VI.11. Calculus of variations in the 1800's: Dirichlet's principle

- We next take a look how leading mathematicians of the early 1800's used variational principles to prove solvability of difficult PDE's or integral equations, as the problems required non-explisite solutions that were out of the standard reach of the analysis of the time. We shall later on see how spectral theory also lead to problems far out of the reach of the methods of the time.
- Gauss was devoting most of 1830's in the study of magnetism. As a byproduct he an H. Weber built the first regularly used telegraph sending messages to each others using binary codes. Mathematically these studies led Gauss to developing the first really nontrivial results in potential theory. Especially, he used variational principles to obtain the existence of an equilibrium distribution of electricity on the surface of a given body.

VI.11. Calculus of variations in the 1800's: Dirichlet's principle

• In his lectures and published notes Dirichlet applied variational methods e.g. to prove the existence of harmonic functions with given boundary values. Namely, we have:

LEMMA Let $\Omega \subset \mathbb{R}^d$ is a bounded domain.The Euler-Lagrange equation for the minimisation problem

$$\text{(D)} \qquad \min \int_\Omega |\nabla u|^2 dx, \qquad u_{|\partial\Omega} = f$$

equals the Laplace equation $\Delta u = 0$ in Ω.

Proof. On the blackboard...

DEFINITION Dirichlets problem for a bounded domain $\Omega \subset \mathbb{R}^d$ and given continuous $f : \partial\Omega \to \mathbb{R}$ asks for the function $u \in C(\overline{\Omega})$, with $u_{|\partial\Omega} = f$ and $\Delta u = 0$ in Ω.

VI.11. Calculus of variations in the 1800's: failure of Dirichlet's principle

- In 1850's Riemann applied Dirichlets method to solve the Dirichlet's problem in connection with his proof of e.g. the Riemann mapping theorem! He coined the name Dirichlets principle to this method of e.g. solving non-trivial PDE's by noting that their solutions solve natural variational problems, and hence the existence of the solution can be taken for granted!
- However, in 1870 Weiestrass shocked the mathematical community by noting that certain variational problems that look very easy and 'tame' do not necessarily posses solutions:

EXAMPLE: the variational problem

$$\min I(u) := \int_{-1}^{1} \big(xu'(x)\big)^2 dx \qquad u(1) = 1, \quad u(-1) = -1$$

does not possess a solution. Details at the blackboard....

VI.11. Calculus of variations in the 1800's: failure of Dirichlet's principle

- One could ask why Gauss and Dirichlet did not notice this problem, since both apparently already were able to see (at least in certain connections) the fundamental difference between infimum an minimum. Gauss did this while he criticised D'Alemberts proof of the fundamental theorem of Algebra. Dirichlet in turn while commenting Steiner's ingenious proof of the isoperimetric inequality. Probably for both these mathematicians the reason was that while applying 'Dirichlet's principle', they both had only physical applications in mind, and in that situation the existence of the solution was evident on physical grounds.

THEOREM Of all plane curves of fixed length circle has the largest area.

Proof. We shall give Steiner's ingenious geometric argument and discuss its problematic points (On the blackboard).

- Perron's way to demonstrate the defect in Steiner's argumentation:

Claim: 1 is the largest positive integer. Proof: Assume x is the largest one. If $x > 1$, then $x^2 > x$, so x is not the largest. Necessarily $x = 1$.

VI.11. Calculus of variations in the 1800's: rigorous developments in Dirichlet's problem

- Weierstrass criticism of Dirichlet's principle had an increasingly deep impact on contemporary mathematicians. This was partly due to general increase in the level of rigour and to the (thus) missing proof of Riemann's conformal mapping theorem Soon partial salvations were achieved by several mathematicians and we take a look at some of these.
- Carl Gottfried Neumann (1832–1925) worked as a professor at Halle, Basel, Tübingen and Leibzig. He worked extensively on PDE's, Neumann boundary condition is named after him. In Dirichlet's problem he applied the method of boundary layers and was able in 1878 to solve the problem (via a 'Neumann series') for smooth enough and convex domains.
- Hermann Amadeus Schwarz (1843–1921) developed an iterative method called 'Alternierende Verfahren' which, roughly speaking, produces a solution for the union $\Omega_1 \cup \Omega_2$ of two smooth simply connected domains assuming that the solution exists for the domains Ω_1 and Ω_2 ! This produced many new examples, especially when connected with suitable use of explicit conformal maps.

VI.11. Calculus of variations in the 1800's: rigorous developments in Dirichlet's problem

- Henri Poincare (1854–1911) applied the method of balayage ('sweeping out) to solve the Dirichlet problem to a wide class of bounded domains in $\mathbb{R}^d$, the essential requirement being that at each point $x \in \partial\Omega$ there was a closed ball contained in Ω^c that contained point x in its boundary.
- That some conditions from the boundery are needed is easy to see just by considering the punctured ball $B(0,1) \setminus \{0\}$ (on blackboard...) ! Later Lebesgue and others found simply connected counter-examples in $\mathbb{R}^3$.
- Also the application of the very Dirichlet's principle even to nice domains restricts the possible continuous boundary values. This defect in the principle was found by Friedrich Prym (1841-1915) in 1871 (and independently by Hadamard in 1906). Namely, even for the unit disc $B(0,1)$ in R^2 one can construct continuous boundary values such that they do not have a continuous extensions to the unit disc with finite Dirichlet integral! (on blackboard...)

VI.11. Calculus of variations in the 1800's: rigorous developments in Dirichlet's problem

- Finally, Dirichlet's principle was restored by Hilbert in 1899. He showed (under nice assumptions on the boundary and on the boundary function) that one may apply Dirichlet's original principle and pick a subsequence of the original minimising sequence that converges to the desired solution. This is easily illustrated by modern methods of Sobolev spaces (on blackboard...) .Hilbert's work was the starting point for the 'direct methods of calculus of variation'.
- The almost decisive results for the Dirichlet problem were obtained by Oskar Perron (1880-1975) in 1923.The Perron method could be described as a radical improvement of Poincare's method.

VI.12. Integration etc. at the end of 1800's

- In the 1870's one defined the (outer content) of a set $S \subset \mathbb{R}$ as the infimum of the total length of (usually finite number of) intervals needed to cover the set. Hermann Hankel 1839–1873 considered (after Riemann) the oscillation $S_f(x)$ of a given function $f : \mathbb{R} \to R$ at x (recall that $S_f(x) := \limsup_{y \to x} |f(y) - f(x)|$). For bounded functions he observed that one can characterise Riemann integrability by the condition that the set $\{S_f(x) \geq \sigma\}$ has zero content for all $\sigma > 0$.
- Hankel believed that equivalently one could demand that S_f is nowhere dense. This confusion was caused by he overlooked the existence of Cantor sets with positive measure, an hence of positive content! The same basic mistake was repeated and caused confusion in the works of many mathematicians of his time, like Harnack and Du Bois Raymond. Instead, one almost generally believed that every nowhere dense closed set $F \subset \mathbb{R}$ has the property that some derived set $F^{(k)}$ vanishes !

VI.12. Integration etc. at the end of 1800's

- In 1875 H.J. S. Smith (1926–83), the Sevilian professor of geometry at Oxford, constructed (in the modern terminology) Cantor sets of positive Lebesgue measure.. He also disproved Harnack's statement on existence of the limits $f(x^-), f(x^+)$ for any Riemann integrable function.

- Ulisse Dini (1945–1918) made a deep study of upper and lower (and left and right) derivatives of functions. Among many other things, he proved the following form of the fundamental theorem of calculus:

THEOREM (Dini 1878) Assume that $f : [a, b] \to \mathbb{R}$ has derivative at every point and f' is Riemann integrable. Then $f(b) - f(a) = \int_a^b f'(x)dx$.
Proof. (On the blackboard...) Remains true also for the

'Dini-derivatives'.

A fundamental question (Dini 1875) Assume that $f : [a, b] \to \mathbb{R}$ has derivative at every point and f' is bounded. Is f' Riemann integrable? Dini conjectured that the answer must be negative.

VI.12. Integration etc. at the end of 1800's

Answer to Dini's question was provided by Vito Volterra (1860–1940) (who for this purpose independently constructed Cantor sets of positive measure):

EXAMPLE (Volterra1884) There is $f : [a, b] \to \mathbb{R}$ that has derivative at every point, and such that f' is bounded but fails to be Riemann integrable.

- After Volterra's example, the limitations of Riemann integral started to bother some mathematicians, most notably Weierstrass who tried several times to extend it in the1880's without real success. Other reasons included need to understand convergence in L^2-norm for Fourier series related to well-known Bessel's inequality (e.g. Harnack). Here the immediate problem is that natural limit functions need not to be Riemann-integrable. Also the question when a convergent trigonometric series is a Fouries series (Fourier's claim) seemed to require enlarging the notion of integral.

VI.12. Integration etc. at the end of 1800's

EXERCISE· Construct a non Riemann-integrable bounded function that is a pointwise limit of Riemann-integrable functions.

- Cantor and Schaefer constructed (in modern language) Cantor functions, which added to the puzzlement over more general questions around the Fundamental theorem of calculus.

- Many mathematicians tried to extend in various ways Riemann integral to unbounded functions. Among them were Harnack, Hölder, and Du Bois Raymond. Also rectifibility of curves and its relation to integral was discussed. Camille Jordan (1838–1922)proved in 1881 that functions of bounded variations can be characterised as differences of two increasing functions , which was important for later developments.

VI.12. Integration etc. at the end of 1800's: Jordan measure

- Multidimensional Riemann integral could be defined, but problems arose e.g. with inegrations over more complicated domains. Giuseppe Peano (1858–1932) defined in 1883 inner and (resp.)outer measures (contents) for plane domains (sets) by defining the inner content as the supremum of areas of inscribed (resp. enclosing) polygons. He made an important step by characterising the one-dimensional (Riemann) integral of given function f as the measure under the graph of f.
- Jordan defined in 1892 Jordan (outer and inner) measures for plane domains or sets by covering or inscribing with unions of small cubes instead of polygons. He noticed that a necessary and sufficient condition for measurability of a domain is that its boundary has inner measure zero, or (assuming that Ω is enclosed compactly in a cube Q) by

$$m_{out}(\Omega) + m_{out}(Q \setminus \Omega) = m(Q),$$

which is close to Lebesgues definition of measurability. Moreover, Jordan verified the finite additivity of his measure.

VI.12. Integration etc. at the end of 1800's; Peano and Jordan measures

- In addition, Jordan used arbitrary measurable partitions instead of intervals when he defined the Riemann integral; thus the interval (or cube) I is partitioned as $I = \bigcup_{k=1}^{\ell} A_k$, and one considers upper and lower sums corresponding to this partition! Again, we recognise a hint to the direction of (later) Lebesgue style...

VI.12. Integration etc. at the end of 1800's; some highlights for the Riemann integral

• Before going to real generalisations of the Riemann integral, let us collect some of the highlights of the theory of Riemann integral that were obtained at the end of 1800's. The proofs of all the 4 theorems mentioned below were quite complicated:

– A 'Fubini' theorem for Riemann integral necessarily invokes upper of lower Riemann integrals:

THEOREM (Jordan 1892) Assume f is a bounded function that is Jordan integrable over the bounded set$E \subset \mathbb{R}^2$. Then, if P is the projection to x-axis, and $E_x := \{y : (x, y) \in E\}$,

$$\int_E f(x,y)dxdy = \int_{PE} \left(\overline{\int}_{E_x} f(x,y)dy \right) dx = \int_{PE} \left(\underline{\int}_{E_x} f(x,y)dy \right) dx.$$

VI.12. Integration etc. at the end of 1800's; some highlights for the Riemann integral

– Osgoods theorem on bounded convergence theorem or the Riemann-integral:

THEOREM (Osgood1897) *Assume that the f_n:s are a sequence of continuous and uniformly bound functions on $[a, b]$ that converge point wise towards the continuous limit f. Then the Riemann integrals satisfy $\int_a^b f_n \to \int_a^b f$ as $n \to \infty$.*

– The difficult proof of this theorem used a lemma which brings Lebesgues theory to mind: If $F \subset \mathbb{R}$ is a closed nowhere dense and bounded set that is the union of an increasing sequence of such sets F_n, then the contents satisfy $m^*(F_k) \to m^*(F)$ as $k \to \infty$.

VI.12. Integration etc. at the end of 1800's; some highlights for the Riemann integral

THEOREM (Arzela 1885,1900) *Same as Osgoods, but with continuity replaced by the Riemann integrability!*

In turn Arzela's proof used the Lemma (which essentially says in modern language): If $U_n \subset [0, 1]$, $n \geq 1$ are open and $m^*(U_n) \geq a$, then there is at least one point that belong to infinitely many of the sets U_n.

THEOREM (Du Bois Raymond 1880's) *A bounded, everywhere convergent trigonometric series whose sum is Riemann integrable is a Fourier series.*

– His proof was a virtuosic piece of analysis!

VI.12. Borel and his measure

- Emil Borel (1871–1956) was a French mathematician who also worked in the French cabinet as a minister in 1925. His inspiration for the 'Borel-measure came from a problem (to be explained on the blackboard...) related to the problem of finding relation between analytic functions that are defined by the same series both in and outside of the unit disc, but which do not extend analytically over the boundary.

- Borel's definition from 1898 states that a subset of $A \subset [0,1]$ that is a union of (at most) countably many intervals has the measure that is given by the sum of the lengths of the intervals. Then one sets $m(A^c) := 1 - m(A)$. Next one iterates this procedure indefinitely : if disjoint sets $A_1, \ldots$ from previous step are given, one sets $m(\cup_{k=1}^{\infty} A_k) := \sum_{k=1}^{\infty} m(A_k)$, and obtains also the measure of the complement (in general one invokes transfinite induction). This produces a countably additive measure, called nowadays the Borel measure.

- Borels argument was only later on properly organised and written in a rigorous manner...

VI.12. Borel and his measure

- Borel did not consider integration, his definition was not clearly complete, and he applied his measure only (in a somewhat vague way) in defining come probabilities (a forerunner of Kolmogorov's later general definition).
- Rene-Louis Baire (1874–1932) introduced the Baire hierarchy for functions, which created a system of functions that is closed for taking limits and contained all continuous functions. Again the idea was based on transfinite induction. Also Baire's work was an important for Lebesgue's later work.

VI.12. Finally: Lebesgue measure

- As we all know, Henri Lebesgue (1875–1941) developed his measure and integral in the very beginning of the 20. century. He was impressed early on on Baire's results, and admired Jordan's work on integration as well as Du Bois Raymond's work on Fourier series. We will not discuss in detail his theory as it is the thing we learn in the course of measure theory and integration.
- One should note that there were others that were arriving at a similar generalisation of the integral, like W.H. Young.
- Let us just make a quick list of basic results Lbesgue obtained in the 5 publications 1899–1900 and in his thesis from 1902, which gave an account of the new theory:

– definition and basic properties of Lebesgue measure (e.g. countable additivity, approximation by Borel sets)

– the famous definition of Lebesgue integral. The basic convergence theorem (monotone, bounded, dominated).

– applications to Fourier series: e.g. Du Bois Raymond theorem without any integrability assumption.

VI.12. Finally: Lebesgue measure

– the generalisation of the Fundamental theorem of analysis: f is the (Lebesgue) integral of its "derivative f' " if and only it is absolutely continuous, and then f' is obtained from f as standard derivative that exists almost everywhere.

• Lebesgue theory was considered rather difficult, but after small initial resistance it spread quickly among leading (and not too old) analysts. Simply, it was the tool that theory of Fourier series, the emerging theory of Hilbert spaces and early functional analysis urgently needed.Some notable (early) milestones:

– Fubini theorem (1907)

– Riez-Fischer theorem (1905) (L^2 and ℓ^2 isomorphic) , completeness of L^p-spaces and duality (Riesz 1908).

– General Lebesgue-Stieltjes integral (1913) du to Rado...

FIN!