On Fermat Last Theorem

by

Oluwatosin Ishmeal

Student Number: 014442586

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Professor Eero Saksman and Paola Elefante

Department of Mathematics,

University of Helsinki.

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Chapter 1

Historical Background

On August 17, 1601, in the small town of Beaumont-de-Lomagne in France, a male child was born to a wealthy leather merchant and the second consul of Beaumont-de-Lomagne. That child by name **Pierre Fermat** would grow up to be the creator of a problem that would last 350 years after his death. Pierre also had a brother and two sisters with whom he grew up with in the same town of birth. As a teenager, Pierre attended the University of Tolouse, France and finally he moved to Bordeaux at age 19 in 1620. While living in Bordeaux, his interest in mathematics began to grow and just 9 years later in 1629, he was able to write his first major mathematical paper called **Restoration of Apollonius's Plane Loci** which he gave to M. Prades, one of the mathematicians at Bordeaux. Certainly while Fermat was in Bordeaux, he was in constant contact with Beaugrand, himself another good French mathematician and it was during this time that Fermat was able to produce important work on Maxima and Minima which he gave to Etienne d'Espagnet who also shares the same mathematical interest with him.

Now, how did his name changed to **Pierre de Fermat**? Sometime around 1630, Pierre moved to Orleans where he studied Law at the University and receieved a degree in Civil Law, thus being able to purchase the offices of councillor at the parliament of Tolouse. So by 1631, Pierre was a lawyer and a government official in Tolouse, also because of the office which he now held, he was entitled to change his name from **Pierre Fermat** to **Pierre de Fermat**. For his entire lifetime, Pierre lived and worked in Tolouse, though he was occassionally working from his hometown and the nearby town of Castres. Certainly, Fermat was preoccupied with mathematics,and he kept his mathematical friendship with Beaugrand and was able to start one with **Carcari**, another great French mathematician. Around 1636, Carcari left Tolouse to Paris to become the Royal Librarian and met another famous French mathematician, **Mersenne**. Carcari's description of Fermat's discoveries in falling bodies aroused Mersenne interest so much that he decided to write directly to Fermat and hear his opinion about falling. On 26th April, 1636, Fermat replied to Mersenne letter and also pointed out those he percieved to be the errors Galileo had made in his description of free fall.

Soon, things started to get tough for Fermat in connection with major French mathematicians. It all started when Beaugrand sent Fermat a copy of Descartes **La Dioptrique**. During this time Fermat was working with **Roberval** and **Pascal** over methods of integrations and their application to center of gravity, thus Fermat paid little or no attention to the paper he was just sent. Finally, because of pressure for his views by Mersenne on La Dioptrique, Fermat did give his opinion and that began the start of his troubles with Paris mathematicians. Fermat opinion on La Dioptrique was summed up by describing it as 'groping about in the shadows'. Fermat claimed that Descartes had not correctly deduced his laws of refraction since it was evidently embedded in the assumptions he made.

This statement uttered by Fermat towards Descartes paper did not go down well with Descartes, he was extremely displeased. In addition, he viewed Fermat's work on maxima and minima and tangent as diminshing the importance of his own **La Geometrie** which he was most proud of. Fermat was forced to stay away from Paris mathematicians from 1643-1654 due to repeated problem with Descartes, Desargues and Mersenne, one of which was because Fermat referred to Beaugrand as friend after Mersenne was angered by Beaugrand's misuse of the pre-publication of Descartes's essays. Moreover, since he was already a councillor in Tolouse, work-related pressures made it really difficult to devote the much needed attention towards his mathematical career. Another important reason was **The Fronde**, a contemporary civil war in France by which Tolouse was greatly affected. Nevertheless, during this turbulent times, Fermat worked on Number Theory and he is often called the founder of the Modern Algebraic Number Theory. But as it happens with every great mathematician, there came a moment of breakthrough in his career and this is what we will discuss in this essay. Fermat inspiration for his theorem was gotten from reading an edition of the Arithmetica written by Diophantus, that was translated into Latin by Claude Bachet. The quadratic Diophantine equation is given as: $x^2 + y^2 = z^2$, and the solution are given by the pythagorean triples. Around 1637, Fermat wrote in the margin of his Diophantus's Arithmetica:

'It is impossible to seperate a cube into two cubes or a fourth power into two fourth powers, or in general, any power higher than the second, into two like powers. I have discovered a truly marvellous proof of this, which this margin is too narrow to contain.'

Today, Fermat is best remembered for this wonderful and significant conjecture which states that;

$$x^n + y^n = z^n \text{ has no integer solution } (x, y, z) \text{ when } n > 2$$

The fact that Fermat himself did not publish this proof has left so many thinking how his original proof would have looked like and what way would the proof had gone and what mathematical concept would have been applied or used in the proof. Sadly though, Fermat died without publishing this proof and many mathematicians implied that he never actually proved this theorem, and thus the margin note became known as Fermat Last Theorem, as it was the last of his asserted theorem to remain unproven. A period of over 300 years with intense research of finding what Fermat originally claimed proof might look like led to the discovery of Commutative Ring Theory and Number Theory. Fermat also made notable contributions to other mathematical areas such as Analytical Geometry and Differential Calculus, he was even regarded as the inventor of Differential Calculus due to his work on finding tangent to curves and their maximum and minimum points. When Fermat discovered this theorem in the 17th century and kept his proof, the French mathematician posed a challenge to the future generations who would want to research deep into the Theories of Number and other related fields. True, at first sight, the theorem feels like an easy ride to prove, but the finest of mathematicals mind have been kept busy for a long time without any resounding result.In the next chapter, we will see how mathematicians have tried to find a proof to this seemingly difficult theorem.

Chapter 2

A quest for solution

Over the years, a lot of great mathematical minds of the past have been involved with this theorem in a quest for solution but only a few have been able to make major contribution towards the general solution of this problem.
Around 1794, **Sophie Germain** herself another French mathematician at Ecole Polytechnique who had to disguise as a man to conduct research forbidden to women, with her enormous confidence and determination was able to prove the case for any odd prime p when $2p+1$ is also prime. Such prime numbers are called Germain's numbers.
Also, in 1770, **Leohnard Euler** was able to prove the case $n = 3$, even though the proof was widely regarded as incomplete. **Evariste Galois** another French mathematician was working on Fermat Last theorem on the night before his infamous duel that claimed his life on May 31, 1832. As interesting as this theorem sounds, finding a solution could be frustrating due to years of work without significant result and that is what happened in the case of the Japanese mathematician Yutaka Taniyama who commit-

ted sucide after being depressed due to his inability to prove the Taniyama Conjecture which would imply the Fermat last theorem by way of Ellipctic and Modular curves. Then finally came a break through in 1995 with over 350 years of intense research and life time careers. An English man born on April 11, 1953 by the name Andrew Wiles with interest in Mathematics as a kid pursued his childhood dream,and was finally able to provide a proof for the Fermat Last Theorem. Due to his strong determination and a passion for his childhood dream, he was able to succeed where others had failed, but this came only after years of frustration, disappointment, sleepless night and hard toil, but at the end it was worth the effort. Fermat Last Theorem is a truly remarkable story of how mathematics most chanllenging problem 'was made to yield its secrets in a thrilling tale of endurance, ingenuity and inspiration', said Simon Singh in his book 'Fermat Last Theorem'

We will now see how mathematicians before Wiles tried to prove Fermat Last Theorem. Fermat claimed that there are no integers x, y, z such that

$$x^n + y^n = z^n; \qquad n > 2$$

Though Fermat did provide a sketch of a correct proof for a particular case $n = 4$, it was not until over a century before Euler proved the case $n = 3$ which curiously turned out to be even more difficult than $n = 4$. Three mathematicians such as: Sophie Germain, P.G.L Dirichlet and Gabriel Lame proved the cases $n = 5, 7$, but none of their methods could prove the general case. Around 1847, Gabriel Lame and Augustin Luis Cauchy both made the claim that they are on the verge of proving the Fermat Last Theorem using the same method employ by Sophie Germain. However, their proof which was based on Unique Factorization Method. This proof was wrong because they assumed incorrectly that the root of unity which is a complex number can be factored uniquely into primes which are similar to integers, and their error was pointed out by **Ernst Kummer** in April 1847 to the Academy of Science in Paris. Kummer later went ahead to develop the conjecture that proved many more particular values of n. As time went by and no concrete result was imminent, the theorem began to catch the attention not only of mathematicians but also rich people in the society and in 1908, **Paul Wolfskehl**, a German industrialist with interest in mathematics, promised the prize of 100,000 marks to anybody who could solve the mystery entangled in Fermat Last Theorem. As a result of this, more and more proofs started to come in, most of which were also not deep founded or full of errors of quite elementary nature. Years keep going by and it looked like there would never be a solution to this problem. During this course of time, a whole lot of interesting fields in Number Theory were developed which were part of the quest of finding a result to this great mathematical problem. Finally, in 1995, by this time Fermat was just more than a theorem, it had become a life curse to a lot of people, and one of them was finally able to break through. As mentioned in the previous chapter, British mathematician Andrew Wiles was finally able to present a proof of this theorem in June 1993, though it was not instantly accepted and had to go through some adjustment to correct some minor errors which was noticed in September 1993, but at last the proof was published in May 1995 and this set the whole world running and for the first time, the news media had a main story of a mathematical revelation. Wiles proof which was a result of his genuis builds heavily on previous results by other mathematicians who could not succeed in their quest for a solution to the Fermat Last Theorem. Another significant fact about Wiles proof his how he was able to connect deep results from varieties of subejct areas majorly in mathematics. Few years back though in 1956, two Japanese mathematicians **Yutaka Taniyama** and **Goro Shimura** in their rigor formulated a conjecture that: 'every elliptic curve over the rationals is modular', but they were unsuccessful to prove this conjecture. While reading about the conjecture in 1984, the German Mathematics **Gerhard Frey** noticed that if this conjecture could be proved then it would implies the Fermat Last Theorem. As noticed by Frey, proving the Fermat Last Theorem would required two steps: First, that if an elliptic curve were to be constructed using a set of numbers that were a solution of Fermat's equation, the resulting curve could not be modular. Second, it was necessary to prove the modularity theorem or at least to prove some sub-class of cases that included the Frey's equation. Frey started working on this but was unsuccessful in his attempt like Taniyama and Shimura. Eventually, few years later, an American mathematician by the name **Kenneth Ribet** was able to prove the first part of Frey's proposition and this was a major step in Wiles final proof of Fermat Last Theorem which was based on the seond part of Frey's proposition.

Generally, it was thought that proving the Taniyama and Shimura conjecture was just as hard as proving Fermat Last Theorem which had been around for over 350 years, but Wiles thought differently. After going through the proof of the Taniyama and Shimura conjecture by Ribet, Wiles decided to also prove the conjecture using his own way and that kept him from discussing his ideas with other mathematicians, because as said by Wiles himself; 'you can't really focus yourself for years unless you have undivided concentration, which too many spectators would have destroyed'. Hence, only his wife knew about what he was doing. Though this was not an easy decision as it took Wiles a period of 7 lonely years which was filled with long periods without progress but he was determined and this finally 'paid off' when he was able to announce the proof the theorem in 1993 during a conference in Cambridge titled: **'Modular forms, Elliptic curves and Galois Representation**, but due to a critical error in the proof which was discovered during peer review, the proof could not be published until 1995, when it was publised in **Annals of Mathematics, Volume 141**. True, this was a truly remarkable demonstration, even if Fermat had an amazing proof, it was certainly nothing like this which was produced by Wiles. It was indeed a mathematical breakthrough due to the fact that the concept of Wiles proves which was deeply related in Elliptic curves and Modular forms had not been known of at time of Fermat. Thus, Wiles was duly acknowledged by the mathematical community and received the Wolfskehl prize, but due to the fact that the corrected proof was accepted in 1994 and Wiles was born in 1953 (41 years), he could not receive the highest mathematical honor **The Field Medals** which has an age limit of 40 years, instead a Silver plaque was given to him in recognition of his achievements by the International Mathematical Union in 1998. He also won a host of other mathematical awards.

Till this moment, we have talked about Fermat's history and his contemporaries, the Fermat Last Theorem and also the quest to find a solution to this seemingly unsolvable problem and how Wiles was finally able to solve the mystery. In the next chapter, we will see what it's like to solve smaller cases $(3, 4)$ of the Fermat Last Theorem.

Chapter 3

Solutions to case $n = 2, 3, 4$ using Pythagorean triples

Now, let us take a look into what it took to solve Fermat Last Theorem using the idea of Pythagorean Triples. I must say that from here, we will do some simple mathematics which are quite easy to follow. The idea of Pythagoras will also be used in the chapter. Another important idea is that of coprime or relatively prime integers. First, we give the statement of Pythagoras Theorem and the definition of co-prime integers.

Pythagoras: In a right triangle,the square of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the other two sides. Mathematically, if x, y, z are the lengths of the sides, then $x^2 + y^2 = z^2$ is known as the pythagorean equation, where z is the hypotenuse and x, y are the other two sides.

Coprime: Two integers a, b are said to be coprime if they have no positive common divisor other than 1, and this is denoted by $(a, b) = 1$.

Now, let's go ahead and show that the pythagorean equation is really true and how we can obtain these triples, which would give us an insight into the proofs of the case $n = 3, 4$ of Fermat Last Theorem. One may assume that (x, y, z) are pairwise coprime integers, i.e. no two of these values are divisible by the same prime. So if p is a prime and p divides x, then p does not divides y or z. One can ask that why is this assumption important? Well, this assumption is important because it helps us to test the existence and uniqueness of any solution to this equation. Also, since the addition of squares of the other 2 sides gives the hypotenuse (z), it is safe to assume that z has to be odd, but suppose the inverse it true, i.e. z is even, then;

$$\exists \text{ another value } Z' = 2z$$

Also $\Rightarrow Z'^2$ is then divisible by 4 since

$$Z'^2 = (2 * z)^2 = 4 * z^2$$

Since x, y are coprime integers, then they must be odd and thus;

$$\begin{aligned} \Rightarrow \quad \exists \ X, Y \text{ such that } \ x &= 2 * X + 1; \\ y &= 2 * Y + 1 \end{aligned}$$

Note that $x^2 + y^2$ is not divisible by 4 because:

$$\begin{aligned} x^2 + y^2 &= (2 * X + 1)^2 \quad + \quad (2 * Y + 1)^2 \\ &= 4X^2 + 4X + 1 + 4Y^2 + 4Y + 1 \\ &= 4[X^2 + X + Y^2 + Y] + 2 \end{aligned}$$

Thus, the assumption that z is even is rejected, and z must be odd. Now that we know that z is odd, then either x or y must be even because an odd number is the addition of an odd and an even number. But now suppose that x is even (we can also assume y is even), then we have that;

$$\Rightarrow \qquad x^2 = z^2 - y^2 = (z+y)(z-y) \qquad \text{(difference of two squares)}$$

Note that $(z-y)$ and $(z+y)$ must be even, since z and y are both odd.
So we have that:

$$\exists \quad a, b \;\; and \;\; c \;\; such \;\; that:$$
$$x = 2a$$
$$z - y = 2b$$
$$z + y = 2c$$

this gives:

$$(2a)^2 = (2b)(2c)$$
$$\Rightarrow \qquad 4a^2 = 4bc$$
$$\Rightarrow \qquad a^2 = bc$$

But now how can we move forward? Let's assume that $(b, c) \neq 1$,
then $\exists$ d such that $d > 1$ and d divides both b, c. Then d divides both $b + c$ and $b - c$, but $z + y + z - y = 2b + 2c$ which gives;
$2z = 2b + 2c$
$\Rightarrow z = b + c.$
and thus d divides z.
Also, $z + y - (z - y) = 2b - 2c$ which gives;
$2y = 2b - 2c$
$\Rightarrow y = b - c.$
Also d divides y.

Thus, this is a contradiction since z and y are coprimes. So our assumption that $(b,c) \neq 1$ is rejected, and thus $(b,c) = 1$.

We know that both b and c themselves are squares because if $(b,c) = 1$ and $bc = z^2$, then $\exists x, y$ such that $b = x^2, c = y^2$, so $\exists$ p, q such that:

$b = p^2$

$c = q^2$ and thus

$z = b + c = p^2 + q^2$

$y = b - c = p^2 - q^2$

$x = 2a = 2pq$, $(since \ \ a^2 = bc \Rightarrow a = pq)$. So we get that:

$(p^2 + q^2)^2 = (2pq)^2 + (p^2 - q^2)^2$

So,picking any two integer p, q at random, (say $p = 2, q = 1$), we get:

$z = p^2 + q^2 = 2^2 + 1^2 = 5$

$y = p^2 - q^2 = 2^2 - 1^2 = 3$

$x = 2pq = 2 * 2 * 1 = 4$

and thus we get $3^2 + 4^2 = 5^2$.

Also, if x, y, z have a common divisor $d > 1$, then;

$z = d[p^2 + q^2]$

$y = d[p^2 - q^2]$

$x = d[2pq]$.

Note that this formula for generating Pythagorean triples was first developed by Euclid around 300 B.C. and is thus known today as the Euclid's formula

For example, if $p = 2, q = 1 \ \ and \ \ d = 2$,we get:

$z = (2)(5) = 10$

$y = (2)(3) = 6$

$x = (2)(4) = 8$

and

$6^2 + 8^2 = 10^2$.

This method adopted in this proof of the case $n = 2$ is 'not too difficult' to apply to the case of infinite descent which was a major instrument which Fermat employed when proving most of his theorems.

As defined by Wiki, 'a proof of infinite descent is a particular kind of proof by contradiction which relies on the facts that the natural numbers are well ordered and that there are only a finite number of them that are smaller than any given one'. One typical application is to show that a given equation has no solutions and this is exactly what is needed to be done in the proof of the Fermat Last Theorem. Thus if we choose any random natural number $n > 1$, then there are finite number less than n and if $n = 1$, then it is smallest of the natural numbers which contradicts the infinite descent.

Thus we are going to show that no solution exist for the case $n = 3$ of the Fermat Last Theorem, though this case is 'somewhat harder', it would be nice to see how things would turn out. We are still going to use some idea of the case $n = 2$.

Now, we have that;

$$x^3 + y^3 = z^3 \text{ has no integer solution for } x, y, z$$

First, assume that $\exists$ a solution to

$$x^3 + y^3 = z^3, \text{ where } x, y, z \text{ are pairwise coprime.}$$

$$\Rightarrow z^3 = x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

Thus, if we choose $x = a + b$ and $y = a - b$, then

$$z^3 = (a+b+a-b)[(a+b)^2 - (a+b)(a-b) + (a-b)^2]$$
$$z^3 = 2a * [a^2 + 3b^2]$$

At this point, it is safe to assume that:

1. $(a, b) = 1$
2. $(a+b)$ is odd
3. a, b are positive.

   Observe that: $2a * (a^2 + 3b^2)$ is a cube

Next, we need to find the $gcd(2a, a^2 + 3b^2)$. Suppose f is prime that divides both $2a$ and $a^2 + 3b^2$,then we know that $f \neq 2$, since $a^2 + 3b^2$ is odd because $a + b$ is odd. Now, suppose that $f > 3$, then $\exists \ A, B$ such that $2a = fA$ and $a^2 + 3b^2 = fB$. Since f is odd, 2 must divide A, we have that $a = fA/2$, so $\exists$ a value $H = A/2$ and $a = fH$. Combining the equations, we get:

$$3b^2 = Bf - a^2 = Bf - f^2H^2 = f(B - fH^2)$$

Now f doesn't divide 3 since it is greater than 3 and so by Euclid's Lemma, it must divide b and this contradicts (a, b) being coprime since it also divides a. Thus, the $gcd(2a, a^2 + 3b^2) = 1 \ \ or \ \ 3$.

Note: Euclid's lemma states that given a prime p that divides a product ab, then either p divides a or p divides b.

Suppose it is 1, then the 2 factors of z^3 are coprime and this implies that 3 doesn't divide a and these two factors are cubes of two smaller numbers say p and q, i.e. $2a = p^3$ and $a^2 + 3b^2 = q^3$. Thus since $a^2 + 3b^2$ is odd, then q is odd and therefore q satisfies $q^3 = a^2 + 3b^2$. But we can write q in terms of two coprime integers u, v such that $q = u^2 + 3v^2$,this can be shown (but the details are left out) so that we get,

$$a = u(u^2 - 9v^2)$$
$$b = 3v(u^2 - v^2)$$

Since a is even and b is odd, then u is even and v is odd. Thus $p^3 = 2a = 2u(u + 3v)(u - 3v)$. The factors $2u, (u + 3v), (u - 3v)$ are pairwise coprime since 3 cannot divide u. Thus each of them must individually equal cubes of smaller integers, i.e.

$$2u = i^3$$
$$(u + 3v) = j^3$$
$$(u - 3v) = k^3$$

which eventually yields $i^3 + j^3 = k^3$, and by the argument of infinite descent the original solution x, y, z is impossible. A slightly different approach can also be use to show that if $gcd(2a, a^2 + 3b^2) = 3$, then there exist a smaller solution of the form:

$$2u + k^3$$
$$u + v = l^3$$
$$u - v = m^3$$

which yields $k^3 + l^3 = m^3$ and by infinite descent the same original solution x, y, z was impossible. Thus we can conclude that no solution exists to the Fermat Last Theorem of the case $n = 3$.

Now, let us see the case $n = 4$. This case is the most natural of the cases considered as it arises naturally from the solution of the Pythagorean Triples. Restating the case $n = 4$, we have:

$$\exists \text{ no integer solution for } x^4 + y^4 = z^4$$

We prove this case of the Fermat Last Theorem by showing that:

$$x^4 + y^4 = z^2$$

has no solution in the positive integers.

Suppose that (x^2, y^2, z) are pairwise coprime and x, y, z is the solution to $x^4 + y^4 = z^2$, with z as the smallest possible. Recalling from our proof of case $n = 2$, we know that $\exists\ p, q$ such that:

$$x^2 = 2pq$$
$$y^2 = p^2 - q^2$$
$$z^2 = p^2 + q^2$$

for coprime p, q that are not both odd. The second equation implies that y, p, q forms a pythagorean triples with odd y, so we may write:

$$y = a^2 - b^2$$
$$p = 2ab$$
$$q = a^2 + b^2$$

for coprime integer a, b that are not both odd. Thus, the last equation implies that q, a, b are pairwise coprime and from $x^2 = 2pq = 2(2ab)q = 4abq$, we can deduce that; $a = u^2, b = v^2, q = w^2$, for some pairwise coprime integers u, v, w. But subtituting these in the equation for q yields: $u^4 + v^4 = w^2$ and this contradicts the minimality of z or the assumption that z is the smallest possible integer, and thus the original solution was impossible. Thus, we can conclude that no solution exist to the case $n = 4$.

One would notice that we have not used any method of Wiles' proof of the

Fermat Last Theorem because at the moment not so many mathematicians understand the proof Wiles provided, though this does not 'water down' the genius work of the proof.

Chapter 4

Conclusion

Conclusively, by the time Wiles submitted his proof for the Fermat Last Theorem in 1995, the prize of the 100,000 marks which was worth 2 million dollars was staring at him, but due to the condition of the Wolfskehl prize which says that the solution should be scrutinized for 2 years following publication of the proof, Wiles was able to collect his reward in 1997, but by this time the hyper-inflation of the Reichsmark has reduced it value to 50,000 dollars. For Wiles, the monetary value was insignificant, his proof is the realization of his boyhood dream which started when a 10 years old boy with passion for Mathematics borrowed a book from his local library in Cambridge, England with the title **'The Last Problem'** by Eric Temple Bells and the determination of a decade of serious efforts. Though today many mathematicians believe that they can still achieve fame and glory by discovering Fermat original proof, but as far as Wiles is concerned, the battle to prove the Fermat Last Theorem is over, 'There is no other problem that will mean same to me' he was quoted. He continued, 'This was my childhood passion, there is nothing to replace that. I had the very rare priviledge of being able to pursue in my adult life what had been my childhood dream. I know it is a rare priviledge, but if you can tackle something in adult life that means that much to you, then it's more rewarding than anything imaginable'.

So, the next time you hear or read about Fermat Last Theorem or its proof, then know that what you are reading spans over 400 years, has made someone commit sucide, has saved someone from committing sucide and has been so many people life-time research.

Thus, thanks to **Sir Andrew Wiles** and other mathematicians, the seemingly most difficult mathematical problem now has a solution(even though not so many people understand it). But don't get to happy, there are still lots of other mathematical problems which has no solution and one of them is also pointing in the direction of the French mathematician **Pierre de Fermat**. Are all Fermat's number square free? Fermat number is a positive integer of the form:

$$F_n = 2^{2n} + 1, \text{ where } n \text{ is a non-negative integer.}$$

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