Extra Exercises

Snapshots of the History of Mathematics
spring 2015, Prof. Eero Saksman

Paola Elefante

April 9, 2015

1 Instructions

In order to pass the final exam, you need to attend 80% of lectures (that is, all but 5). If you missed more than 5 lectures, you can cover the missing hours by writing solutions to some of the following exercises. One completed (and solved) problem covers one missed lecture hour (unless differently stated).
Please write your solutions in LaTeXor in a easy-to-read format. Do not forget to include your name and your student number. Solutions can be sent **before 15.05.2015** to `paola.elefante@helsinki.fi`.

2 Exercises

Exercise 1
Prove the following proposition.
Let $a, b \in \mathbb{R}$ and $f, g \in C([a,b], \mathbb{R})$. Assume $\int_a^b f(t)\varphi(t)dt = 0$ whenever $\int_a^b g(t)\varphi(t)dt = 0$ and $\varphi \in C_0([a,b], \mathbb{R})$. Then there exists $\lambda \in \mathbb{R}$ such that $f = \lambda g$.
Exercise 2
Show that if $3|n$, n being a natural number, then $\frac{2}{n}$ can be expressed ad a sum of two different Egyptian fractions[1].
Exercise 3
Show that if $5|n$, n being a natural number, then $\frac{2}{n}$ can be expressed ad a sum of two different Egyptian fractions.
Exercise 4
Prove the following theorem.
An even natural number n is perfect if and only if $n = \frac{q(q+1)}{2}$, where q is a Mersenne prime.
Exercise 5
Use the Euclidean algorithm to show that $\sqrt{2}$ is not rational.

[1] An Egyptian fraction is the sum of distinct unit fractions, that is fractions having 1 as numerator. For example, $\frac{5}{8} = \frac{1}{2} + \frac{1}{8}$

EXERCISE 6 (=only 30 minutes)
Consider Fibonacci's result:

$$\sum_{j=1}^{n}(2j-1) = 1 + 3 + 5 + ... + (2n-1) = n^2$$

Assume $(2n-1) = q^2$ for some $q \in \mathbb{N}$. How does this help you to create Pythagorean triples?

EXERCISE 7 (=only 30 minutes)
Show that the complex number $\sqrt{x+iy}, x, y \in \mathbb{R}$ can be expressed in terms of real roots of the real and imaginary parts x, y.

EXERCISE 8
Show that the statement of exercise 7 does not hold for the cubic root $\sqrt[3]{x+iy}, x, y \in \mathbb{R}$.

EXERCISE 9 Show that the cubic equation $(x-1)(x-2)(x-3) = 0$ leads to the case of casus irreducibilis.

EXERCISE 10
Find all roots of $x^3 = 16x + 4$ by Cardano formulas.

EXERCISE 11
Use Ferrari's formula to find all solutions of $x^4 - 8x + 6 = 0$.

EXERCISE 12
Express $x_1^3 + ... + x_n^3$ in terms of the elementary symmetric polynomials

$$e_j(x_1, ..., x_n) = \sum_{1 \leq i_1 < i_2 < ... < i_j \leq n} x_{i_1} ... x_{i_j}$$

EXERCISE 13
Show by Cauchy-Schwartz that $y(x) = x$ is the unique C^1 function that minimizes the integral

$$\int_0^1 (y'(x))^2 dx$$

with boundary conditions $y(0) = 0, y(1) = 1$.