

UNIVERSITY OF HELSINKI
DEPARTMENT OF MATHEMATICS AND STATISTICS

FINAL ESSAY FOR THE COURSE
SNAPSHOTS OF THE HISTORY OF MATHEMATICS

**"THE ESSENCE OF MATHEMATICS LIES
IN ITS FREEDOM"**

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May 10, 2015

1 Class or set?

Nowadays we know Georg Cantor's name as the founder of "naive set theory". Firstly we could try to define what a set is. Generally the concept of set risks to be confused with the idea of class. Let's consider three sentences:

- a) the rose, that is in the vase, is red;
- b) roses of the vase are red;
- c) roses of the vase are twelve;

In the first sentence we can see a subject, *the rose that is in the vase*, and a noun phrase, *being red*. Then we observe that a *property* is associated to a *substance*. The other two sentences involve the presence of a plurality of elements. We can understand this thanks to the plural form, *roses*, but in two different ways. In the sentence b), being red is not related to the set of roses, because a set has never been red, but to the single elements composing the collection of the elements, to any red rose. We could similarly say that "every rose in the vase is red" or, using set's language, "set of roses in the vase is a subset of the set of red things". In this case using sets is not essential or needed.

In the sentence c), instead, sets are involved: a rose has never been "twelve". Then the verb *having twelve elements* is related to the set of roses in the vase: to their totality. The real subject of sentence c) is the whole set of roses in the vase. The set assumes a singular, individual nature. Before Cantor's researches, the set was not considered a mathematical entity and so the third sentence was not considered "mathematical". Definitely there exist two way to involve a collection of elements in a scientific speech. We could adopt as verb in a sentence a property that refer to the single elements of the collection. In this case we speak about a "class" of objects. Otherwise we can refer to a collection also as a single mathematical entity and so we can also give to it a property that cannot be referred to its own elements. This is the concept of set (a set so is also a class). One of first definitions of set was introduced by Bernard Bolzano in 1847: "A set is a collection of objects, without any consideration about their order."

2 Infinite sets

The idea of infinite is really common in mathematics: we often have to face numeric sets that don't have a limit. First of all, we can define the sets of natural numbers

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

and the one of integer numbers

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

These sets are sets without an ending. This is simple to prove.

Firstly, assume that the set of natural numbers is finite.

Let us consider N , the biggest of these numbers. We can build $N + 1$. It is a new natural number but it is bigger than N . By this contradiction, we can conclude that the initial hypothesis was wrong: the set of natural numbers is infinite.

3 Potential infinite and actual infinite

According to Pythagoras, Elea school and the philosophers Parmenides and Plato, infinite was accepted as a concept, but negatively connoted: it was unreachable. It was impossible to describe through finite

terms and so it was similar to all irrational elements, and it does not have a proper shape because of you cannot add or remove anything from it.

Already Aristotle, focusing on the concept of infinite, had distinguished the idea of potential infinite from the one of actual infinite. We all agree saying that there exist endless sequences of mathematical objects, as the natural numbers, and that, thanks to our human skills, that are limited, we can embrace a large but finite part of them, called potential infinite. The operation of considering its totality and of identifying the objects as a single being is called actual infinite. It is beyond human possibilities. Latins say: "Infinitum actu non datur". Potential infinite is a continuously changing possibility never definitively realized. It is an endless process, without end, and so not finite. It is a gradual approach to a goal that you cannot reach.

The scholastic philosophy moves this discussion in a religious environment, where matters about infinite focused on God: how could the divine perfection fit with the impossibility of actual infinite? Tommaso D'Aquino and many others tried to solve this question saying that God, that is almighty, could do everything that is possible, but neither God could do impossible things, otherwise they would be possible.

According to Tommaso, God is not able to create or to think infinite entities, even if God is infinite itself, otherwise one should accept existence of infinite things different from God.

During the XVII century some mathematics have started to study the potential infinite and its mathematical inverse, the infinitesimal. Newton and Leibnitz have begun to define first hints of differential analysis, that focused on the behaviour of a function that get closer and closer to a point, or that goes to infinite.

4 Zeno's paradoxes

Zeno from Elea, Parmenides disciple, showed for the first time the potentialities of infinite applied to space's and time's concepts. Thanks to his paradoxes, in facts, this philosopher broke two of the most evident certainties in the world: the movement and the passing of time.

Zeno's aim was to display some contradictory phenomena hidden in multiplication and division. Pythagorians had predicted that space and time could be composed by points and instants but, moreover, now we know that these two entities have also a peculiar feature, called *continuity*.

Probably Zeno developed his reasonings in order to contrast Pythagorean concept of point, defined by Aristotle as "unity provided with position", or "unity considered in the space". The most famous are paradoxes of the "dichotomy", of Achilles, of the "arrow" and, last one, paradox of the "stadium". The first one is based on this argument: a moving object, before having covered a given distance, has to cover half of that distance. Before doing that, it has to cover the first quarter of that distance. Before doing that, it has to cover the first eighth, and so on, through an infinite number of subdivisions.

A runner should walk through this infinite number of subdivisions in a finite period of time, but, because of finishing an infinite collection is impossible, to begin the movement is impossible. The second paradox is very similar to the first one except that the infinite subdivision is progressive instead of regressive: Achilles challenges with a turtle which starts from an advanced position. Even if Achilles is faster than the turtle, he will never reach the turtle, and this does not depend on Achilles's speed. In facts, suppose that they start in the same moment. When Achilles will reach the turtle's initial position, the animal will have covered a distance (maybe little but never equal to zero). When Achilles will arrive in this second turtle's position, the turtle will have gone ahead and so on. This process never arrives to an end so the fast Achilles will never reach the slow turtle.

This two paradoxes show that, assuming the infinite subdivisibility of space and time, the movement is impossible. The next two paradoxes, instead, show that the movement seems equally impossible if we assume the opposite hypothesis: that we can divide space and time in indivisible elements. The paradox

of the arrow is built on this reasoning: a flying object will always occupy a space equal to itself. Things that fill always a space equal to themselves are not moving. So the flying arrow rests at every moment, so its movement is only an illusion.

The paradox of the "stadium" is maybe the most difficult to describe: let's A_1, A_2, A_3, A_4 be objects of same dimension, motionless. Let's B_1, B_2, B_3, B_4 be objects of same dimension of the A_i but moving to the right so that every B_i passes every A_i in the *minimum possible interval of time*. Let's C_1, C_2, C_3, C_4 again be objects of same size of A_i (and B_i) and moving uniformly to the left, respect to the A_i so as every C_i passes every A_i in the minimum possible time interval.

Let's consider now that the objects' initial condition is:

$$\begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \\ B_1 & B_2 & B_3 & B_4 \\ C_1 & C_2 & C_3 & C_4 \end{array}$$

After a single instant the objects' disposition will be:

$$\begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \\ B_1 & B_2 & B_3 & B_4 \\ C_1 & C_2 & C_3 & C_4 \end{array}$$

We can observe that C_1 has passed two B_i . This means that the instant cannot be the minimum possible interval of time because we could assume the time C_1 used to pass a B_i as new and smaller unity of time.

5 Galileo: is the whole greater than a part?

Usually in Pythagorean environments was common to represent quantities using some small stones, called "calculus". During Euclid's age, the reference system had already changed: quantities were associated to segments, and not to stones or numbers anymore.

The numbers' kingdom was again considered discontinuous but the kingdom of continuous quantities was something completely different and had to be investigated in a geometric way.

It seemed geometry led the world, not numbers. The first progress about actual infinite analyzed in a numeric way bears from Galileo Galilei's reasoning, now called Galileo's paradox. It was written in the "Dialogue Concerning the Two Chief World Systems", of 1638.

The first argument is about infinitely small. Firstly, Simplicio admits that, in order to divide a segment, is not necessary to separate the parts but is enough to mark the division points. Salviati then goes ahead saying that, if you use the segment to build a regular triangle, you are basically dividing the segment in three equal parts, if you use it to build a square, one can consider you are dividing the segment in four parts and so on. If you follow this way of thinking, and if you use the segment to build a circle, you can say to have realized in *act* that infinite parts that you could consider contained in the starting segment. This holds because we can consider the circle as a polygon with an infinite number of sides. From geometric infinite, Galileo started to study a property of algebraic infinite, that he considered *curious*. Galileo observed that was possible to create a bijective correspondence between the sequence of natural numbers and the one of perfect squares, even if the second ones seem being rarer on the numeric line than the first ones, and so they seem less than the natural numbers. When you face a finite set, if you have enough time, you can always count its number of elements, that is called *cardinality* of the set.

Counting is equivalent to use the mathematical concept of bijective correspondence with \mathbb{N} . Already in ancient times, in order to count the number of sheep belonging to their pen, sheperds linked every sheep to a little stone. When a sheep went out from the pen, they added a little stone. When it entered the pen, they removed a little stone. In this way they could know if all the sheep were in the pen or not. Instead of counting the sheep, they could count the set of little stones, that has the same number of elements of the first one, but is simpler to count. This procedure can, of course, be applied to all finite sets but also to the infinite ones: given two sets, if I can build a bijective correspondence between them, you can say that they have the same number of elements; if, otherwise, I found at least one elements of the two sets that cannot be insered into the correspondence, I have to conclude that one set has more element than the other one and so that its cardinality is higher than the one of the other set.

More precisely: Given A, B sets, if you can find

$$f : A \longrightarrow B$$

Injective, you can say that :

$$Card(A) \leq Card(B)$$

In order to count square numbers, Galileo assigned to every natural number its square: to 1, 1, to 2,4, to 3,9, and so on. Because of is possible to build a bijective corrispondence, one to one, between the elements of the two sets, they have the same number of elements. The set of square numbers is a proper subset of the set of natural numbers. Even if not all natural numbers are perfect squares, we should say that there are as many natural numbers as many perfect squares. Galileo, without being completely aware of that, had discovered the defining condition of an infinite set: a part of the set is "similar" to the whole set. The scientist, anyway, did not manage to say that the two sets considered, had the same number of elements. He arrived only to say that, when one talk about infinite quantities, speaking about *equal, or more or less* is meaningless . In facts, Galileo have observed that the number of perfect squares is not less than the one of the natural numbers. Adfirming that they are equal, he would have called into question the Euclidean axiom, according to which "the whole is greater than the part".

6 More infinities?

The first attempt in discerning different infinities, is due to Giordano Bruno that proposed an acute mental experiment. If we distance vertically the earth, our horizon widens more and more up to, when we are at infinite distance from the earth, we can see half of it. Considering that from the earth you can see half of the moon, Giordano went on with this reasoning and images that going beyond the infinite, we could start to see the hidden half, until to embrace the whole earth or the whole moon once we have reached infinity for the second time. In 1584, in “De l’infinito universo et mundi”, Bruno distingued between two kinds of infinite in terms of universe and God. If the first one is composed of an infinite number of finite parts, parts of the second one are infinite as their number.

7 Real Numbers

Galileo had considered only the discrete structure of infinite, so infinite that can be counted. Infinite sets of that kind are called countable or denumerable. Against the countable infinite, we find the concept of continous: union of both rational and irrational numbers. Rational numbers are defined as:

$$\mathbb{Q} = \{p/q \text{ with } p \text{ and } q \text{ coprimes} \mid p, q \in \mathbb{Z} \mid q \neq 0\}$$

Irrational numbers are numbers without finite or periodic decimal expansion. In their decimal expansion you cannot find any regularity. Irrational numbers are divided into algebraic numbers, when they

are root of an equation with rational coefficients, or transcendental numbers, if they cannot be described as roots of any equation. Between them, the most famous are π and e .

The expression *real number* was introduced by many mathematicians at the end of XIX century for example by Cantor and Haine in 1872 . They defined every real number as the limit of a Cauchy sequence. In \mathbb{Q} there exist Cauchy sequences that do not admit rational limit. As an example:

$$1, \frac{14}{10}, \frac{141}{100}, \frac{1414}{1000}, \frac{14142}{10000}, \dots$$

converges to the square root of 2 that does not belong to \mathbb{Q} .

Let us consider the set of Cauchy sequences in \mathbb{Q} and let us call it C :

Infinite sequences can share the same limit so we can define an equivalence relation. An equivalence class is the set of sequences that share the same limit. The set of real numbers \mathbb{R} is defined, by Cantor and Haine, as the quotient set of C , with respect to that equivalence relation. Every equivalence class represents a real number.

They proved how every real number can be represented through an infinite decimal expansion.

Let's:

$$x = u, a_1 a_2 a_3 a_4 a_5 \dots$$

where:

- $u \in \mathbb{N}$
- $a_i \in \{0, 1, 2, 3, \dots, 9\} \forall i$

Every real number is defined by a Cauchy sequence:

$$\begin{aligned} &u, a_1 \\ &u, a_1, a_2 \\ &u, a_1 a_2 a_3 \\ &\dots \end{aligned}$$

Bernard Bolzano focused first on the concept of infinite sequence. These concepts were not defined in a rigorous way yet. In the book "Paradoxes about infinite", published in 1851, three years after Bolzano's death, Bolzano observed that correspondences between an infinite set and its own subsets are really common. He considered a simple mathematical function that describe the line $y = 2x$ and consider this function applied to any point between 0 and 1. To every $x \in [0, 1]$ the function assigns one and only one $y \in [0, 2]$. Bolzano concluded that there are as many numbers in $[0, 1]$ as many in $[0, 2]$. In other words there is an equal number of points in a segment of length 1 and in a segment of length 2. Extending this reasoning, we can say that, in every given closed interval, there is the same number of points. This number does not depend on the length of the interval.

8 Hilbert's hotel

Hilbert presented a really famous example that shows how an infinite set can be put in a bijective correspondence with some of its own subsets. This example is called the Hilbert's hotel. The Hilbert's hotel is an hotel with an infinite number of rooms. One evening a traveller arrives and asks for a room for the night. The janitor replies that the hotel is already full, every room are occupied. The traveller replies that moving the guest of the room 1 in the room 2, the guest of the room 2 in the room 3, and so on, the room 1 will be free and noone will be left without a room. This conclusion is stricly related to the fact that number of rooms is infinite. If we would like we could obtain also more free rooms: as an example, we could ask to every guest to move to the room with the number equal to the perfect square of the room



Figure 1: Georg Cantor, St Petersburg, 3 March 1845; Halle, 6 January 1918

that he was occupying previously (using Galileo's correspondence), or with the number double of the one occupied previously. In the first case we would have free all the rooms which number is not a perfect square, in the second case, all the rooms marked with an odd number.

9 Georg Cantor

Georg Cantor is the mathematician who gave the turning point in the development of set theory. He was born in Saint Petersburg in 1845. Origins of his family are not well known yet. His mother was Russian and catholic, his father German and lutheran. He was the first of six children. His family moved to Frankfurt in 1856 and in 1862 Cantor obtained the permission to study math by his father. Cantor's father in fact had a great influence on the son and encouraged him to reach success in the academic world. Some historians think that his father's influence has contributed to develop a first nucleus of that form of mental instability for which Cantor has suffered during his lifetime.

He began to study in Zurich Polytechnic but stayed here a few time before moving to the the University of Berlin where he graduated and studied for a PhD. Hereafter he accepted the role of Privatdozent in the University of Halle, where he remained for the rest of his life. In Berlin, Cantor met as theachers Kummer, Kronecker and Weierstrass. During one of Weierstrass' seminar Cantor got the idea that finding a bijective correspondence between two sets is equivalent to say that they have the same cardinality. This concept lead the mathematician to the first proof about infinite cardinalities of sets.

10 Are natural number less than fractions?

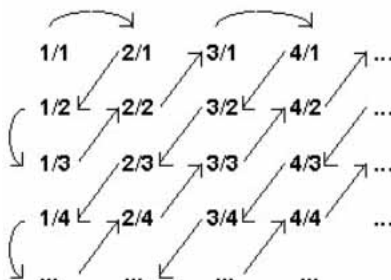
Taking a rapid and superficial look, we could think that \mathbb{Q} has more elements than \mathbb{N} because the first ones are denser on the numeric line. They are also dense according to the mathematical concept: given two elements of the set, is always possible to find a number belonging to the set that lies between the two.

Cantor's first step was proving that there exists a bijective correspondence between \mathbb{Q} and \mathbb{N} . He proved in this way that these two sets share the same level of infinite.

First of all, he arranged rational numbers in that order:

| | | | | | | |
|-----|-----|-----|-----|-----|-----|-----|
| 1/1 | 2/1 | 3/1 | 4/1 | 5/1 | ... | ... |
| 1/2 | 2/2 | 3/2 | 4/2 | 5/2 | ... | ... |
| 1/3 | 2/3 | 3/3 | 4/3 | 5/3 | ... | ... |
| 1/4 | 2/4 | 3/4 | 4/4 | 5/4 | ... | ... |
| 1/5 | 2/5 | 3/5 | 4/5 | 5/5 | ... | ... |
| ... | ... | ... | ... | ... | ... | ... |
| ... | ... | ... | ... | ... | ... | ... |

Now we can associate to every natural number a rational number simply counting the rational numbers following this path:



Basically we are building this bijective correspondence:

| <i>Natural Number</i> | \leftrightarrow | <i>Rational Number</i> |
|-----------------------|-------------------|------------------------|
| 1 | \leftrightarrow | 1/1 |
| 2 | \leftrightarrow | 2/1 |
| 3 | \leftrightarrow | 1/2 |
| 4 | \leftrightarrow | 1/3 |
| 5 | \leftrightarrow | 3/1 |
| 6 | \leftrightarrow | 4/1 |
| 7 | \leftrightarrow | 3/2 |
| 8 | \leftrightarrow | 2/3 |
| 9 | \leftrightarrow | 1/4 |
| ... | \leftrightarrow | ... |

We can now confirm that the set of natural numbers has the same number of elements of the set of rational numbers: the first impression was wrong. Cantor proceeded proving that \mathbb{Z} 's cardinality is equal to \mathbb{N} 's cardinality. It is enough to order \mathbb{Z} 's elements as:

$$\mathbb{Z} = \{0, -1, 1, -2, 2, -3, 3, \dots\}$$

. Now we can build the bijective correspondence between \mathbb{Z} and \mathbb{N} :

| <i>Natural Number</i> | \leftrightarrow | <i>Integer Number</i> |
|-----------------------|-------------------|-----------------------|
| 1 | \leftrightarrow | 0 |
| 2 | \leftrightarrow | -1 |
| 3 | \leftrightarrow | 1 |
| 4 | \leftrightarrow | -2 |
| 5 | \leftrightarrow | 2 |
| 6 | \leftrightarrow | -3 |
| 7 | \leftrightarrow | 3 |
| 8 | \leftrightarrow | -4 |
| 9 | \leftrightarrow | 4 |
| ... | \leftrightarrow | ... |

When two sets have the same cardinality, we call them *equipotent*.

So we have proved that \mathbb{N} is equipotent to \mathbb{Q} , and that \mathbb{N} is equipotent to \mathbb{Z} . Using the transitive property, we can say that also \mathbb{Z} is equipotent to \mathbb{Q} . So even if:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$$

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are equipotent.

11 Trigonometric Series

In 1869 Cantor moved to the University of Halle where he met Heine. This mathematicians was working on the theory of trigonometric series. Cantor started to research about the same topic and focused in particular on a problem: the *problem of uniqueness for trigonometric series*.

For which sets E in $[0, 2\pi)$ is it true that:

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} = 0 \quad \forall x \in [0, 2\pi) \setminus E \quad \Rightarrow \quad c_n = 0 \quad \forall n \in \mathbb{Z} ?$$

A set of that kind is called a *uniqueness set* or U-set. A set that does not satisfy this property is called set of multiplicity, and is denoted as M-set.

In 1870, Cantor reached his first result:

Theorem (Cantor, 1870):

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} = 0 \quad \forall x \in [0, 2\pi) \quad \Rightarrow \quad c_n = 0 \quad \forall n \in \mathbb{Z}$$

In other words, \emptyset is a U-set.

He managed to generalize this result a year later:

Theorem (Cantor, 1871):

If $F \subset [0, 2\pi)$:

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} = 0 \quad \forall x \in [0, 2\pi) \setminus F \quad \Rightarrow \quad c_n = 0 \quad \forall n \in \mathbb{Z}$$

In other words, finite sets are U-sets.

Young later proved that every countable sets are U-sets.

Again a year later Cantor found that:

Theorem (Cantor, 1872):

If

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} = 0 \quad \forall x \in P$$

where P is a set such that $\exists n \in \mathbb{N}$ such that $P^{(n)} = \emptyset$, then

$$c_n = 0 \quad \forall n \in \mathbb{Z}$$

The set P^n is related to the concept of derived set: given a set S , the derived set of S , denoted with S' , is the set of all the limits points of S . Cantor had proved that these exceptional sets P , for which there exists a natural number n such that P^n is the empty set, are set of uniqueness. Cantor called these sets *sets of the first species*.

At this point Cantor understood that, in order to go on with his researches, he needed to know more about the structure of real numbers.

12 Meeting Dedekind

Therefore Cantor met Richard Dedekind, in 1872, during an holiday in Switzerland. They remained good friends for the whole life. Dedekind too was working on the structure of real numbers. Dedekind had introduced in the mathematics the concept of *cut*. The cut is an entity that divides all numbers greater than one number from the ones smaller than that number. The cut itself can be rational or not: as an example we could divide all numbers bigger than 2 from the ones smaller than 2. In this case the entity is rational (is 2!). In other cases the cut is not a rational number. Every real number can be thought as a cut. Dedekind was also the first one to define an infinite set exactly through the condition observed by Galileo and Bolzano: "A set S is called infinite if it is similar to a part of itself; otherwise S is called a finite set." Using more modern terms, a set S is infinite when its element can be put in a bijective correspondence with the elements of one of its proper subsets. Dedekind, as Cantor, had recognized the necessary (and sufficient) condition to consider a set infinite. Maybe he also grasped that there exist more than one level of infinite.

13 Are the fractions less than the real number?

\mathbb{Q} is dense in \mathbb{R} so, given two rational numbers, even if they are really close one to the other, you can always find a rational number between them. After first Cantor's results one may hypotize that all numeric sets are equipotent. Cantor proved, on the 7th December 1873, in a definitive way, that this is not true. As an example \mathbb{R} 's cardinality is bigger than \mathbb{Q} 's one.

Let's assume that all real numbers between 0 and 1 are countable. This means that there exists a bijective correspondence between real numbers and rational ones or, more simply, because of \mathbb{Z} and \mathbb{Q} are equipotent, we can find a bijective correspondence between \mathbb{R} and \mathbb{Z} . There exist many functions that put in a bijective correspondence the interval $[0, 1]$ with the whole \mathbb{R} (the whole numeric line) so we can reduce our research to numbers in the unit interval. Let's assume also that every real number is expressed as a number with limitless decimal expansion, and that real number (a_i) are ordered in a countable order (because we have assumed the existence of the bijective correspondence).

$$a_1 = 0, a_{11}a_{12}a_{13}, \dots$$

$$a_2 = 0, a_{21}a_{22}a_{23}, \dots$$

$$a_3 = 0, a_{31}a_{32}a_{33}, \dots$$

.....

where $a_{ij} \in \{0, 1, 2, \dots, 9\}$.

If we consider the bijective correspondence:

$$\text{Natural Number} \leftrightarrow \text{Real Number in } [0, 1]$$

| | | |
|-----|---|-------|
| 1 | ↔ | a_1 |
| 2 | ↔ | a_2 |
| 3 | ↔ | a_3 |
| ... | ↔ | ... |

If this would be a bijective correspondence, every real number would appear in the column to the right, linked to an integer situated to the left.

Cantor described how to create a limitless decimal expansion that cannot be at any point of the column to the right. To have a number of that kind, let's consider:

$$b = 0, b_1b_2b_3\dots$$

where $b_k = 9$ if $a_{kk} = 1$ and $b_k = 1$ if $a_{kk} \neq 1$.

This real number will belong to $[0, 1]$ but it will be different from all numbers listed in that arrangement that, by hypothesis, should have contained all real numbers included in $[0, 1]$.

More simply we can see that this number created by Cantor is obtained considering the digits that are on the diagonal of the infinite square that the real number written with their decimal expansion one under the other, shape. This digits are taken and modified one by one (using any kind of operation). This sequence of digits will establish the decimal expansion of a new real number that will never be associated to an integer number, according to this correspondence. This contradiction leads us to say that a correspondence between \mathbb{N} and \mathbb{R} is not possible. This means that there exist more real numbers than integer ones. This kind of proof by Cantor, used also later in his works, is called the "diagonal argument". At the end of 1873, Cantor showed his proof to Weierstrass that understood the great importance of that research and tried to persuade Cantor to publish his work.

14 "God has created the integer numbers, everything else is work of man"

Cantor had showed that there exist different levels of infinite. At least two: the one of the rational numbers and an other level for real numbers, belonging to the numeric line. Cantor followed Weierstrass hint. He was scared because he thought that the scientific community of Berlin could reject his ideas but, moreover, he dreaded charges by Leopold Kronecker. Kronecker had been Cantor's teacher in Berlin and was one of the editorial of the Crelle's journal. He used to repeat "God has created the integer numbers, everything else is work of man". He thought that math should involve only discrete quantities of algebra. He pretended Weierstrass's results and many other analytic results not to exist, only because analysis involved continuous entities. For example, in 1870, he had already rejected the Bolzano-Weierstrass theorem. Cantor was also afraid of publishing because the mathematical world was not used to speak about the concept of actual infinite. Until the end of XIX century, analysis had focused on

studying continuous functions. At the end of the 1800s, mathematicians tried to apply their theory also to discontinuous functions. In these years they discovered that these functions can be approximated using particular simple functions. This observation was really important to develop the concept of integral. We can use for example step functions to approximate a function. The approximation through step functions is perfect when their number reaches the infinite. Gauss himself believed only in the potential infinite, an ideal concept that can never be realized. When we use many steps, total area is near to the area of the curve we are interested in. The approximation could be as good as one wants at any finite level. Gauss and his contemporary, but also Newton and Leibniz, that invented the calculus, were satisfied by the idea of potential infinite, not reachable. Noone was gone beyond. Because of this complex historical and scientific behaviour, in which Cantor's ideas sound as extraordinary, we can understand why he published his results, in 1874, in a sort of hidden way. Article's title was "On a property of the collection of All Real algebraic numbers". All the work was mainly about numbers that remain in the numeric line once removed numbers that belong to countable sets, and so also algebraic numbers. Main goal of the work is that on the line effectively some number remains. They are called transcendental irrational numbers, and they cannot be counted because their level of infinite is bigger than the rational numbers' one or algebraic numbers' one. The trick worked and the article was published on Crelle's journal. After the publication Kronecker started to define Cantor a "renegade", a "scientific charlatan" and a "youth's corruptor". Poincarè instead define Cantor's set theory, a "disease".

15 “I see it but I don't believe it at all”

“Je le vois, mais je ne le crois pas” (I see it but I don't believe it at all): this is the sentence that Cantor sent to Dedekind, in 1878, to inform him about the new exciting feature regarding the new infinite discovered. This feature regards the concept of dimension: Euclides had defined the point as a line without length, the line as a plane without width, the plane as a space without profundity. Theoretically we could use the same mental procedure also to define the spaces with more than three dimensions, even if our imagination cannot conceive them.

Cantor asked to himself: which level of infinite do object with different dimension have? The perspective used has its origin in the quantification of every point of the plane, one of most important Cartesio's results. Thanks to the Cartesian idea, leading to Cartesian plane's building, was possible to merge arithmetical, algebraic and, later, analytical results. Cantor, using the Cartesian coordinate's system, would like to understand how does an entity's dimension affect the number of points by which it is composed. Easily speaking he was interested in understanding if on a line are there less points than on a plane or in a space. Denying this statement seemed something really absurd!

Cantor managed to prove that the number of points of an object does not depend on its dimension. First of all, Cantor restricted the line to the closed interval $[0, 1]$. Thanks to Bolzano's proof this is not a consistent restriction. Cantor drew alongside of the interval $[0, 1]$, a square of side length one. Every point of the square can be identified thanks to two numbers. The first of them is the coordinate x , the second one is the coordinate y . Every point of $[0, 1]$ is a coordinate x . Cantor would like to build a bijective correspondence between square's and segment's points.

A point p of the square is denoted by its coordinates:

$$p = (0, a_1 a_2 a_3 a_4 a_5 \dots, 0, b_1 b_2 b_3 b_4 b_5 \dots)$$

Cantor defined the transformation of the square in the segment in this way:

$$p \longrightarrow 0, a_1 b_1 a_2 b_2 a_3 b_3 \dots = q$$

q is in the interval $[0, 1]$. This means that every point of the square is linked to one and only one point on the considered segment.

This means that we can say that there are as many points on the plane as on a line. We can extend this result: the number of points on a line is equal to the one of any bidimensional, tridimensional or with four or more dimensions space. Any continuous space, any space with n dimensions has as many points as the points of the continuous are.

In the same year, 1878, Cantor developed an hypothesis:

Claim: Every infinite set of reals either is countable or has the power of the continuum.

This is the first kernel of a larger question Cantor would have faced later.

As soon as Cantor tried to publish the article in which he supported that, speaking about infinite, dimension is not important because all spaces, independently from their dimension, have the same level of infinite, Kronecker worked hard to prevent or at least delay its publication. Kronecker gave dignity only to integer numbers. Cantor instead had started to believe that infinities were a present from God. He thought that infinite was the kingdom of God and he discovered (in 1891) that it was composed of infinite different levels, the transfinite numbers, also called transfinite cardinals. Integer's, rational's or algebraic's infinite was only the lowest level of infinite. Beyond transfinite numbers instead there was another level of infinite, last one and not reachable, the Absolute, that was the God itself.

Advancing through studies about infinite, the personal war between Cantor and Kronecker became more and more violent to such an extent that, in 1884, Cantor faced his first depressive crisis, that lasted more than one month.

Theoretically speaking about knowledge, Cantor thought that everyone should be free to investigate reality without prejudices: mathematics and philosophy had to go ahead without any conditioning and everyone should be left free to undertake his experiment, everywhere they would have lead to. This way of thinking is the reason that explains why in Halle's city center, there is a bust representing Georg Cantor. Under the bust are placed these words: "The essence of mathematics lies in its freedom." This sentence is contained in Cantor's "Grundlagen einer Allgemeinen Mannigfaltigkeitslehre", a book published in 1880, in which he explained his set theory but also his general philosophical conception about knowledge and research.

Kronecker's followers read in the infinite's mathematics an attempt to damage solid basis of their mathematical conception, as strengthened as blind, because these intellectuals, supporting conservative ideas, would have excluded from mathematics everything not concerning integer numbers and finite quantities.

16 God's infinite

Giordano Bruno's intuition was confirmed: there existed at least two levels of infinite. In 1891 Cantor managed to prove that there exist infinite levels of infinite, and so that, given an infinite set, is always possible to build another infinite set which cardinality is an higher level of infinite. This proof determined a spoiling of relationship between new mathematical results and the Roman Curia that was used to identify infinite as God. This perspective of existence of more levels of infinite could be interpreted as a discussion about Christian monotheism, in spite of evident affinities between Cantorian infinite and God's image.

Agostino in "City of God" wrote: "Every number is known by the one which ability of understand cannot be subject to numeration. Even if the infinite numbers' series cannot be counted, this infinite is not out of his comprehension. From this has to follow that every infinite entity is, in a way that we cannot express, done finite for God". Church's worries could not be undertaken: Cantor in fact was Christian. He visited the cardinal Frenzelin in the Vatican and explained to him that the infinities of his theory, called transfinite numbers, were relative. In fact he discerned transfinite numbers from the Absolute infinite, bigger than all other infinities and not existing in reality, otherwise, thanks to Cantor method, one could have built an infinite even bigger. From Cantor's own words we can understand how he discerned between three kinds of infinite: the Absolute, space-time's infinite and mathematical infinite:

"The actual infinite arises in three contexts: first when it is realized in the most complete form, in a fully independent otherworldly being, in Deo, where I call it the Absolute Infinite or simply Absolute; second when it occurs in the contingent, created world; third when the mind grasps it in abstracto as a mathematical magnitude, number or order type."

Surely one of Cantor's aims was to smooth reasons of contrast with high ecclesiastic hierarchies but he understood the need of a new language to express and describe his infinities.

17 Transfinite numbers: the Alef

ω is the symbol given to state the cardinality of countable infinite sets. It is bigger than every finite numbers.

This symbol has been changed in 1893, year from which transfinite cardinals are defined using the letter \aleph , the letter *alef* of the Hebrew alphabet. Probably this symbol emerged among the others because it seems from Cabala's studies that Jewish with that letter denoted the *En Sof*, so God's infinity.

Cantor's name, according to recent researches, would descend from the verb "to sing" in a peculiar interpretation that would refer to a synagogue's singer. Probably Cantor himself knew the mystic and religious implications that go with the letter \aleph . In front of the scientific community, the mathematician declared he was really satisfied with his choice: \aleph was the first letter of Jewish alphabet and he thought that there was not a symbol that could symbolize in a better way the starting point of a new mathematics. Cantor supposed the existence of a sequence of \aleph s. The smallest one should be the symbol for the integer or natural or rational numbers' infinite. This transfinite was denoted with the symbol \aleph_0 , from the 1895. Cantor defined then with $\aleph_1, \aleph_2, \aleph_3, \dots$ bigger infinities. They all together were denoted as a set with the letter Ω . Cantor thought about the existence of an order in Ω because he believe to the truth of a principle called the *Well-ordering principle*: it is always possible to bring any well-defined set into the form of a well-orderer set. For any subset of a well-ordered set you can find the minimum element.

Cantor knew that the real numbers' infinite was bigger than \aleph_0 , but he did not know if this level of infinite was immediately consecutive to \aleph_0 . As a consequence, he did not know if the infinite of real numbers could be denoted with \aleph_1 . This reason moved Cantor to assign a different symbol, c to the cardinality of \mathbb{R} .

Cantor tried also to fix some roles, like a relations' system among transfinite numbers. As an example:

- $\aleph_0 + 1 = \aleph_0$;
- $\aleph_i + \aleph_i = \aleph_i$, equivalently, if you add to a set of cardinality \aleph_i another set of the same cardinality, we obtain a set that has the same cardinality of the starting sets. We do not obtain a set of higher cardinality.

18 Infinite infinities

Cantor succeeded in proving the existence of many levels of infinite in 1891. He started from the concept of power set. The power set is the set of all subsets of a given set. As an example the power set of :

$$A = \{a, b, c\}$$

is

$$P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$$

. Let's indicate with $Card(A)$ the cardinality of A .

There exists a relation between cardinalities of the starting set and his power set:

if $Card(A) = n \Rightarrow Card(P(A)) = 2^n$. As an example:

$$Card(\{a, b, c\}) = 3 \text{ and } Card(P(\{a, b, c\})) = 8 = 2^3$$

Every subset of a given set is defined by the fact that it includes or not a particular element. This means that for every element we have at least two kinds of subset: ones that contain the element and the other ones that do not contain it. This implies that, if we have two elements, we will have $2 * 2$ subsets, if we start with 3 elements, we will have $2 * 2 * 2$ subsets; if we start with n elements, we will have 2^n subsets.

Next step is proving that the cardinality of the power set is always bigger than the cardinality of the set. Considering only finite numbers, it seems obvious: for any non negative integer n,

$$Card(set) = n < 2^n = Card(P(set))$$

We have to ask ourselves if this property holds also for infinite sets.

We want to show that :

$$Card(A) < Card(P(A))$$

Because of $f : A \rightarrow P(A), f(a) = \{a\}$ is injective, we have only two possibilities:

- $Card(A) = Card(P(A))$;
- $Card(A) < Card(P(A))$;

If we manage to prove that the first case is impossible, we will conclude that the second one must be true.

Let us assume, by contradiction, that there exists a bijective correspondence between A and P(A).

Let us follow the proof with an example:

$$A = \{c, *, a, ?, !, q, t, \dots\}$$

where ... means that the set A is infinite.

Our hypothetical correspondence would assign any element of A to a element of P(A).

Let us remember that the elements of P(A) are subsets of A. In our example our correspondence could be like that:

$$\begin{aligned} A &\rightarrow P(A) \\ c &\rightarrow \{?, a, q\} \\ * &\rightarrow \{c, a, ?, !, *\} \\ a &\rightarrow \{\} \\ &\dots \end{aligned}$$

Element c is associated to the subset $\{?, a, q\}$. c does not appear in the set to which it is linked to. On the contrary * appears in the column of A, being associated to $\{c, a, ?, !, *\}$. Thanks to this observation we can define:

- *alone* elements that do not belong to the set they are linked to;
- *happy* elements belonging to the set they are linked to;

Let us consider L the set of all alone elements of the starting set. L is subset of A so $L \in P(A)$. Because of that L , according to our bijective correspondence, must be associated with an element of A . Let's say that L is linked to the element $t \in A$. Let us think now about t :

- suppose t is happy: t must be associated with a set that contains t . t is associated with L . this means that $t \in L$ but L is defined as the set of all alone elements. t is happy so t cannot belong to L . By this contradiction we can infer that t cannot be happy.
- suppose t is alone: this means that t belongs to L because L is the set of all alone elements. But, if $t \in L$ then t is happy because happiness's condition means for an element to be linked with a set that contains it.

Because t cannot be neither happy neither alone, we have found a contradiction. This means that our hypothesis was wrong: a bijective correspondence between A and $P(A)$ does not exist. This is equivalent to say that $Card(A) \neq Card(P(A))$. Only possibility left is that $Card(P(A)) > Card(A)$.

Let us consider now the set \mathbb{N} ; thanks to $P(\mathbb{N})$ we obtain an other level of infinite. $P(\mathbb{N})$ is itself an infinite set so its power set, $P(P(\mathbb{N}))$, has cardinality bigger than $Card(P(\mathbb{N}))$. We do not have reasons to stop this procedure and so we can find always infinites new levels of infinite.

19 Continuum Hypothesis

Cantor want to know the order of his transfinite cardinals. In particular in 1873 he started to focus on a question that appeared as harmless: there does exist any other transfinite cardinal, an other \aleph , between \aleph_0 and the continuum's cardinality?

If we could say that the answer is no, we could affirm that:

$$c = \aleph_1$$

We now still do not know the answer to this question and so we are not able to order transfinite numbers: we do not know if c is the immediate consecutive number of \aleph_0 .

The same question can be written in that words: there exist others cardinal numbers between \aleph_1 and c ? Is possible to show that

$$c = \aleph_1?$$

?

This matter is known as a form of the continuum hypothesis.

Cantor never managed neither to definitively prove neither to definitively disprove this hypothesis, even if he devoted to it large part of his life.

Sometime he thought to have found the proof that was searching with all his efforts, sometime he thought to have found a key to prove that it is wrong.

At some point of his researches he even believed to have managed to prove the existence of infinite levels of infinite between \aleph_0 and c .

Nowadays we know that Cantor's failure was not due to any mathematician's lack. The continuum hypothesis is an impossible matter. It has no solution within our mathematical system: both the continuum hypothesis and its negation are true, and both the continuum hypothesis and its negation are false: it is an undecidable sentence within our axiomatical system.

20 I ask only for some coherence!

David Hilbert inserted the continuum hypothesis within the 23 problems that he proposed to the International Congress in Paris in 1900. These problems should have been the main research's topics for mathematicians during the successive century.

Hilbert himself opened a new perspective in mathematics. Hilbert thought that reasoning's cornerstone is not hidden in the consideration of mathematical objects but in the formal coherence of the system, in other words the center of reasoning lies in the absence of contradictions and paradoxes among axioms and reasoning's roles that control that system.

In 1900, Hilbert proposed an axiomatic system for real numbers and, in the second position of his list, he put the problem of coherence of Arithmetics that is the possibility to prove that within Arithmetics there are no contradictions. After a long period of silence, in 1917, with the work "Axiomatisches Denken", Hilbert explained more clearly his plan's goals. All mathematical developed branches have to be equipped with a proper axiomatic system and deduction roles, that allow to create finite sentences, propositions, proofs, that are millstones for theorems. These formal systems should have to satisfy a main requirement: the intrinsic coherence.

Formalized mathematical theories are collections of propositions, lacking an intuitive meaning. This is why the only discriminating criterion to decide if a system is valid or not is the absence of contradiction. This is the only requisite Hilbert was asking for. It remained open to establish who would have to testify the coherence.

According to Hilbert, the coherence should be an intrinsic property, that the system should attest itself as a proper theorem.

21 Maniacal eccentricities

Cantor entered a period of "maniacal depression". Probably this crisis were related to the impossibility to give a definitive answer to the continuum hypothesis. Some psychologists prefer to read the origin of the disease in a conflicted relationship between the mathematician and his father. Cantor would like to reach his father's approval thanks to a total compliance.

The most qualified theory, and surely the most suggestive, links Cantor's mental weakness to the frustration due to failing in solving the continuum problem and in defending his theory from intensive Kronecker's attacks. It seems that Cantor, in order to escape the more and more frequent depressive crisis, had tried to lead his interests toward English literature. He tried to prove that Shakespeare's works should have been recognized as written by the scientist-philosopher Bacon. Cantor has been focused on this topics for many years.

22 The well-ordering principle and the axiom of choice

At the same time Cantor carried on his studies about set theory: he understood that, in order to prove the continuum hypothesis, he had to elaborate a method to compare transfinite cardinals.

Every transfinite cardinal must have his own place in the Ω system. Cantor believed that you can always compare two transfinite cardinals because thought the well-ordering principle holds. In 1904 König (16 December 1849 – 8 April 1913), during an International Congress in Heidelberg, seemed to have proved that c is not one of the \aleph 's.

The day after the reading of the article Ernst Zermelo, a German mathematician, proved that König had used a lemma in a not allowed way. In these years Zermelo started to focus on set theory and in particular he would like to build an axiomatic system for that theory. Zermelo's axioms are seven. One

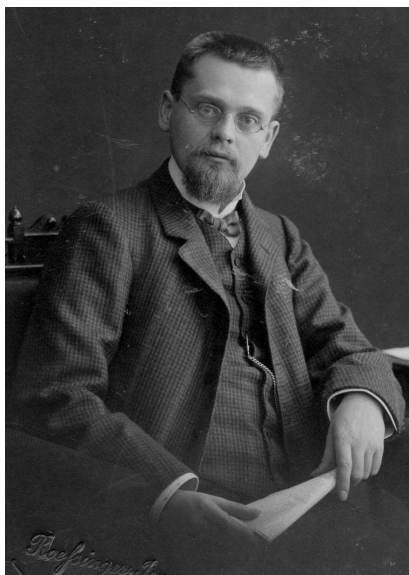


Figure 2: Ernst Zermelo, Berlin, 27 July 1871 – Freiburg, 21 May 1953

of them is called the *axiom of choice*. This principle says that you can always choose a single element from any subset of a given set. This seems to be true if your starting set is finite. Is not evident in the same way that you can do the same for an infinite set. Even if, as an example, we consider only subsets of cardinality 2 of an infinite set, who can guarantee that is possible to carry out the choice's operation an infinite number of times? For mathematicians saying that a choice can be done is not enough. You need to give a role that shows how the choice has to be done. Because of that, the axiom of choice was considered suspicious by the scientific community.

This axiom is radically different from all other axioms. An axiom has to be evident. Starting from axioms is possible to build rigorous proof for problems and theorems. The axiom of choice is not constructive. It does not give a procedure in order to make the choice.

Zermelo managed also to prove the *well-ordering theorem*: every set can be well ordered. It was the same of Cantor's principle but it had become a theorem in the new axiomatic theory. From that moment, Zermelo decided to focus all his researches in finding a rigorous proof for the axiom of choice, top of that axiomatic process of formalization of set theory. In the same time Cantor started to distance the exact, rigorous mathematics. The importance of mathematical proofs became less and less important to him. He thought to be God's assistant that should have protected the continuum hypothesis, by now considered as a dogma, like the Word of God, from possible mugger's assaults. Cantor started to think he had the assignment to put in writing the truth that God was revealing to the world.

23 Paradoxes!

In 1897 the mathematician Cesare Burali-Forti discovered the existence of one of the first paradoxes about sets. The starting point is a simple argument: if there exists the set of all ordinal numbers, extension of natural numbers including also infinite sequences, also the biggest ordinal number should belong to it. Meanwhile we have already observed that such a number does not exist, because if you have a given number, you can always find out one bigger than the first one. If we similarly think about transfinite numbers, we cannot consider the set of all transfinite numbers because the maximum of that set

does not exist. Given an infinite set of cardinality \aleph_i is always possible to get a set with bigger transfinite cardinality, building the power set of the starting one.

Zermelo's axiomatic theory simply solved this paradox assuming the non-existence of the set of all ordinals. Other solutions were offered. Hilbert supported a theory called "Constructivism", within only finite objects and rigid deduction rules are allowed. Brouwer instead supported a theory named "Intuitionism". According to that, a mathematical object does not exist as long as it is not built.

In 1901 Russell, followed one year later by Zermelo that found it independently, discovered an other dangerous paradox about set theory. Let's assume that a set could be element of an other set. Then some sets contain themselves as element. Let's define:

$$I = \{A \text{ set} | A \text{ does not belong to } A\}$$

now let us consider that question:

$$I \in I?$$

To answer that question is not possible:

- if I belongs to itself, it belongs to the set of sets that do not belong to themselves, so it cannot belong to itself;
- if I does not belong to itself, it has to belong to itself!

Russel's paradox was a terrible blow for the new axiomatic system for set theory created by Frege, a German logic. He was going to publish his results but his system included among his basics the "abstraction axiom" according to which, given any property, there always exists a set which elements are exactly the ones that satisfy that property. Russell's paradox showed a property that cannot be satisfied by any set. This paradox implies also the non-existence of the "Universal set", a set that contains everything. This set had been always considered existing by mathematicians.

Russel showed also that in order to define a set is not enough to be sure that it exists. We cannot forget to prove its elements' existence.

Scientific community started to doubt more and more seriously about the axiom of choice's truth. In this contest a third paradox was found: the Banach-Tarsky paradox. Its final form is of 1924 but the first kernel had been already found by Hausdorff twenty years before. It shows how you can break up a sphere of radius r in some pieces that you can rearrange, using the axiom of choice, to create two spheres of radius r, using the same pieces.

Cantor entered a desperate permanent condition because of he understood that probabilities to strengthen set theory's efficiency were decreasing. He was more frequently hospitalized in a clinic. He saved only one faith: God, the Absolute, had used him to disclose the truth to the world.

24 Difficult questions

Kurt Gödel, Czechoslovakian mathematician, graduated at University of Vienna, continued Cantor's researches. More exactly, Gödel, in addition to his mathematics' interests, was deeply involved in physics and philosophy, since when he was young.

He proposed a new approach to set theory. His first questions were:

- what is a proof?
- are the truth of a sentence and the proof of it equivalent?
- can a true sentence always be proved?



Figure 3: Kurt Gödel, Brno, 28 April 1906 – Princeton, 14 January 1978

- within the limits of a system, can we prove something which is "external" to the system itself?

Investigating on that urgent matters, Gödel proved in 1931 some theorems. Today they are well known as *Gödel's incompleteness theorems*.

25 Gödel's incompleteness theorems...

Let's consider a formal system S such that:

- S has a decidable set of axioms;
- S can treat Arithmetics and Number Theory;
- S contains no contradiction;

First incompleteness theorem says that in every S of that kind is possible to build sentences about which the system cannot decide. According to the system's axioms and deduction's rules, they cannot be neither proved, neither disproved.

Sentences of that kind are similar to this one:

$$P \Leftrightarrow P \text{ cannot be proved in } S$$

We can prove P only if P cannot be proved in S. P is true but P cannot be proved in S.

S is not complete. Completeness here means the ability of the system to decide about every matter inside itself and so to decide (and prove) about truth or untruth of every sentences about itself.

Hilbert asked not for completeness but for coherence. However Hilbert believed that a well-founded and well-organized mathematics could really answer any possible question. During an International Congress in 1928, he says "There does not exist any limit to mathematical knowledge, in mathematics there does not exist "ignorabimus". Gödel's first theorem negates this certainty.

Gödel found a second result that destroyed also coherence's main feature, causing the definitive collapse of Hilbert's plane. Gödel's second incompleteness theorem says that none formal system S including Elementary Number Theory, free from contradictions, will not be able to prove itself (so using its own axioms and deduction's rules), his coherence.

In every case of that kind, and in every other situations starting with the same hypothesis, the system S will need an external confirm to guarantee its absence of contradictions and so, in other words, its own license to treat coherently what it is treating.

26 ...proof of some proof's indemonstrability

Gödel's theorems proved then that, given any system, there will always be sentences that cannot be proved from inside that system. Even if a sentence is true, could be mathematically impossible managing to prove it. Gödel's theorem is strictly related to Cantor's proof about the non-existence of a set that contains every cardinal numbers because, given the largest cardinal, is always possible to find out a bigger infinite's level thanks to its power set. Then, given any system, because of we cannot reach the highest level, there will always be sentences that will not be analyzed, or understood, or proved, or about which you can say anything remaining inside that system. In order to know something about that sentences, we should move in a bigger system but, while moving, we will meet even bigger systems and entities living in the new enlarged system.

Every system the limited human mind would adopt, there always will be some properties that escape comprehension within that system.

The biggest system is unintelligible and unreachable for men. This is the reason that lead Cantor to consider it as God's existence. Gödel's proved that some properties cannot be proved. Up to that moment, mathematics had as main aim to build theorem and proofs starting from basic principles, called axioms. Even if mathematicians tried to build a clear and complete axiomatic system, within that system there will always exist some undecidable, unprovable sentences, independently from their truth or not-truth. Gödel's started suffering for psychiatric diseases after having faced the "forbidden" concepts of \aleph and actual infinite. He convinced himself that someone was persecuting him and, over time, he asked to Adele, his wife, to taste the food before him. This persecution's mania, of manic-depressive origin, became worse and worse, up to when Gödel's left himself dying for starvation in 1978.

27 Persecution's maniae: madness

Gödel managed to prove that the axiom of choice was compatible with other set theory's axioms to the top of his depressive attack. Meanwhile in Europe the Nazi Germany was growing. In 1938 it would have invaded Austria to build the first nucleus of the Great Germany about which Hitler wrote in his "Mein Kampf" in 1923. In the same years Gödel moved to the continuum hypothesis.

Probably the mathematician was aware that being in contact with set theory worsened his mental disease but, as Cantor, he was irresistibly attracted by infinite's charme.

During the night between 14th end 15th June in 1937, Gödel wrote in a corner of his notes, that he had managed to prove that the continuum hypothesis is compatible with set theory's axiomatic system. This proof implies that is possible to assume the continuum hypothesis as true, valid, because this does not cause any contradiction or conflict with the other axioms of the system.

The same proof holds also for the axiom of choice. This proof does not imply the continuum hypothesis or axiom of choice are true. It simply says that the continuum hypothesis is, at least, half-independent from the rest of mathematics. They are not false. That hypothesis could work within the set theory's axiomatic system. Initially Gödel decided not to divulge his discovery.

In 1938 taught at University of Notre Dame. Once returned in Austria started inevitably to become aware of the surrounding historical background: in 1939 was stated fitting to Reich's military service. The Nazi call to arms definitely persuaded Gödel to leave Austria. In 1939, really delayed, decided to accept one of the many work's offer that many prestigious American Universities, as Princeton's one, had sent to him during previous years.

He got with some difficulties a visa to the Usa, where arrived in 1940 with his wife.

As Cantor, Gödel was not able to stand actual infinite's contemplation for long time, above all when it was in the form of the continuum hypothesis: Gödel was slipping to madness.

He had proved that the continuum hypothesis' truth does not create any contradiction in set theory's axiomatic system. He had also proved that the controversial axiom of choice can be used without threat-



Figure 4: Gödel and Einstein

ening the whole system's stability. Then he focused to prove the same compatibility with axiomatic system for continuum hypothesis and axiom of choice's negation. In that way he could affirm that the continuum hypothesis and the axiom of choice were totally independent from all the rest of math. The great paranoia that was afflicting him, rapidly worsened. Beyond every persecutive maniae, that became stronger and stronger, he tried to prove that Leibniz had not developed his own theories. Luckily, next to that imaginary certainty, Gödel knew Einstein, with who he began a deep relationship. This friendship, if we can call it a friendship, was supported by two main factors: Einstein preferred to keep on speaking German, Gödel's mother language, and both Einstein and Gödel, even if they had opposite natures, shared physical and philosophical interests.

Thanks to Einstein, Gödel got in contact with the relativity's theory. Gödel developed and solved one of Einstein's equation. Therefore he managed to prove that the universe rotates around itself. It should be non expanding and homogeneous, properties thanks to which travelling through time should be possible. Nowadays physical and astronomical studies seems to prove that the universe does not rotate and that it is expanding. The mistake lies in the fact that Gödel was working in a model different from our. Gödel's solution is not worthy for our universe. While he was becoming Einstein's friend, Gödel asked for obtaining the American citizenship even if, according to American's law, psychiatric patients could not become American citizens, moreover if they had been interned. In order to overcome the decisive hearing, Gödel had to study USA's history and culture, and to know the US Constitution. The mathematician faced this study remaining faithful to his logic: he searched for contradictions and paradoxes hidden among Constitution's lines. Gödel found a certain number of them and submitted his results to his examiner, even if Einstein had tried to dissuade him. Gödel obtained the citizenship thanks to ironic and smart-alek examiner's nature and to Einstein's participation, as guarantor, in the courtroom.



Figure 5: Paul Cohen, April 2, 1934 – March 23, 2007

28 Paul Cohen: a Field medal

In 1963, Paul Cohen, researcher at University of Stanford, managed to prove axiom of choice's independence from the other axioms of set theory and proved that continuum hypothesis is independent from all other axioms, included the axiom of choice. His proof is based on a method called *forcing*: Logics at University of Stanford, "forced" a system of postulates to take one or two truth's values. He started from a collection of sets and logic rules applied to that sets, he gradually extended the system applying rules to wider and wider classes. Within that wider logic system, changing some postulates, Cohen managed to prove that the continuum hypothesis is completely independent from the axioms of set theory. This means that, within actual axiomatic system, the continuum hypothesis can be considered both true and false because neither of that choice produces any contradictions in the system. Proving or disproving continuum hypothesis would need a new axiomatic system but logics do not build it. They have not discovered, up to now, any contradiction in the Zermel-Freankel's system and this system has been a faithful servant for set theory until now.

There is no possibility for that system, to decide if the continuum htpothesis is true or false. Cohen got the Field Medal in 1966 for his result. Near to the end of his research also Cohen, as Gödel, was more and more persuaded about continuum hypothesis' falsity, then also Cohen doubted about what Cantor had assumed as absolute truth. Cohen thought that the continuum number, c , is so rich that it cannot correspond to any of lowest cardinal numbers. It could at most, coincide to one of highest cardinals or, anyway, larger than \aleph_1 . Obviously there does not exist any proof about Gödel's or Cohen's hypothesis.

29 Even forward: large cardinals

Already Gödel had searched for a development on set theory supposing the existence of other transfinite cardinal numbers, larger than the Cantor's ones. Next to that hypothesis we find the theory of large cardinals.

The first cardinal, \aleph_0 , cannot be reached starting from finite numbers through any mathematical calculation. Similarly there could exist other transfinite cardinals that cannot be reached starting from

smaller infinite cardinals. Large cardinals, so, should be so "large" that they cannot be reached using neither exponentialization's operation, neither any other mathematical calculations. The research about these gigantic numbers, with respect to which transfinite cardinals could be considered negligible, allowed to discover some important properties of smaller cardinals. Obviously, because of we are again within Zermelo-Fraenkel's axiomatic system, we could not, and will never managed, in any way, to continue in the study about the continuum hypothesis.

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