Highlights from the History of Graph Theory

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1 Definitions

- Graph: An ordered pair $G = (V, E)$ made up of a set of vertices V and a set of edges E acting as relations on the set V.
- Subgraph: A subgraph of a graph $G = (V, E)$ is a graph $G' = (V', E')$ where $V' \subset V$ and $E' \subset E$.
- Degree: The number of edges connected to a vertex.
- Loop: An edge that begins and ends on the same vertex, i.e. an edge $e = (x, x)$.
- Path: A sequence of edges which connect a sequence of vertices.
- Cycle: A path that begins and ends at the same vertex.
- Connected graph: A graph in which there exists a path between any two vertices.
- Simple graph: A graph which contains no loops.
- Planar graph: A graph which can be embedded into the Euclidean plane.
- Triangulation: The process of adding edges onto a planar graph until all faces have 3 edges.
- Subdivision: The process of removing from a graph some edge e with endpoints $\{u, v\}$, and then adding a vertex w along with two new edges $\{u, w\}$ and $\{w, v\}$.
- Subdivision of a graph: A graph G' formed from a graph G through one or more iterations of subdivision.

2 Introduction

Graph theory is a subbranch of topology which deals with deceptively simple looking mathematical objects, named graphs. It has its roots in 18th century and arose as one of the earliest forms of topology. The story begins with an awkward problem, whose solution could not be categorized into any known branch of mathematics of the day. Slowly in the 19th century more mathematical curiosities began to be lumped into the same category of graphs. By the 20th century, graph theory had become a rigorous field with countless applications in not only mathematics itself, but many scientific disciplines. This paper will survey some of the highlights in the history of graph theory, beginning from its inception and continuing in a linear fashion. Along the way discussions of different problems arising in graph theory are given, including the Königsberg Bridges Problem, enumeration of trees, the Icosian game, the Four Color conjecture, the Kuratowski Theorem, and the Traveling Salesman problem. Together these snapshots and stories of the people involved will give a basic understanding of the evolution of the field of graph theory.

3 The Birth of Graph Theory

The birth year of graph theory is often considered to be 1736. The 29 year old Leonhard Euler had already established his reputation as a mathematician two years earlier with his solution to the Basel problem. He was working at the Imperial Academy of Sciences in St Petersburg at the time, when he published a paper in *Commentarii Academiae Scientiarum Imperialis Petropolitanae* named *Solution of a problem relating to the geometry of position.* In the paper he solved a problem known as the Königsberg Bridges Problem. The paper is now widely considered the first paper of both topology and graph theory.

3.1 The Königsberg Bridges Problem

Königsberg, known as Kaliningrad today, was the capital of East Prussia. Through the city flows the river Pregel (Pregelya today), which divides the city into four regions, which were connected to each other with seven bridges. It is reported that the inhabitants of Königsberg enjoyed taking walks over the bridges. They wondered, however, whether it was possible to take a walk that crossed each bridge only once. None had succeeded in taking such a walk, nor proven that it was impossible.

It is not clear where Euler first heard of the Königsberg bridges problem, but after solving it, he did have correspondence with several people about it in 1736. One such person was Carl Leonhard Gottlieb Ehler, the mayor of Danzig, Prussia, who was aware of Euler's knowledge of the problem. He had inquired Euler for its solution on behalf of a mathematics professor in Danzig. Euler replied:

> Thus you see, most noble Sir, how this problem bears little relationship with mathematics, and I do not understand why you expect a mathematician to produce it, rather than anyone else, for the solution is based on reason alone, and

> its discovery does not depend on any mathematical principle. Because of this, I do not know why even questions which bear so little relationship to mathematics are solved more quickly by mathematicians than by others.

Mathematics had not dealt with problems of this kind before, and it seems that Euler, despite having solved it, did not quite think of the problem as a mathematical one. Regardless, he seems to understand that there is some mathematical essence in the problem, because as he admits the solution comes more readily to mathematicians.

In another letter written just few weeks earlier Euler writes to Giovanni Marioni, an Italian engineer and mathematician about the Königsberg bridges problem. He states:

> This problem is so banal, but seemed to me worthy of attention in that geometry, nor algebra, nor even the art of counting was sufficient to solve it. In view of this, it occurred to me to wonder whether it belonged to the geometry of position [*geometriam Situs*], which Leibniz had once so much longed for. And so after some deliberation, I obtained a simple, yet completely established, rule with whose help one can immediately decide for all examples of this kind, with any number of bridges in any arrangement, whether such a round trip is possible, or not.

Here Euler clearly entertains the thought of this problem belonging to a new field. The reference to the geometry of position was to a notion that Gottfried Leibniz had introduced in 1679 in a letter to Christiaan Huygens. Though Leibniz had not defined exactly what the *analysis situs* was, he had expressed a need for such a field which "deals directly with position, as algebra deals with magnitudes".[1] The ideas of the *analysis situs* or *geometria situs* is what we today interpret as topology. Euler was aware of this idea in 1736, but there was a high level of ambiguity to what such a field would contain. The Königsberg bridges problem became the first official contender to the field, as Euler named his paper *Solutio problematis ad geometriam situs pertinentis*, or Solution of a problem relating to the geometry of position. He begins the paper crediting Leibniz with the idea of *geometriam situs.*[2]

3.2 Euler's solution

The solution given by Euler is not what the modern graph theorist would give. He makes no mention of words such as graph, edge, or vertex. The logical structure of the argument, however, is somewhat similar, although by modern standards not fully rigorous. Euler's first step was to abstract the four land masses as single points A,B,C,D, and the seven bridges as letters a,b,c,d,e,f and g. The following image is Euler's sketch of the scenario from his paper.

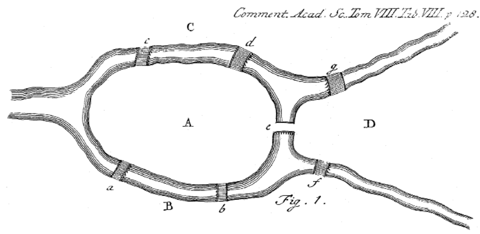

Figure 1. Euler's diagram of the Königsberg bridge problem.

A walk could then be represented as a sequence of letters. For example, ABD would represent crossing a bridge from land mass A to land mass B and then to land mass D. It did not specify which bridge was used if multiple bridges were available between any two land masses. Euler first examined the case of only one land mass and noted that if a land mass M has an odd number n bridges connecting to it, each to be crossed once, then M must appear in the final sequence $\frac{(n+1)}{2}$ times. Thus, in the Königsberg problem, since each land mass has an odd number of bridges connecting to it, we can apply the idea above to count how many times each land mass is to appear in the complete walk. Since A has five bridges connecting to it, a walk crossing all of them must contain the letter A $\frac{5+1}{2} = 3$ times. Similarly the letter B, C and D must each appear in the walk twice. Thus the entire walk must contain $3 + 2 + 2 + 2 = 9$ letters. However, if one is to cross seven bridges once, the resulting sequence must contain only eight letters. A contradiction is found, making such a walk impossible.

With similar argumentation Euler proceeds to deal with the general case of any number of land masses and bridges. He summarizes his conclusions with the following three facts:

1. If there are more than two areas to which and odd number of bridges lead, then such a journey is impossible.
2. If, however, the number of bridges is odd for exactly two areas, then the journey is possible if it starts in either of these two areas.
3. If, finally, there are no areas to which an odd number of bridges lead, then the required journey can be accomplished starting from any area.

3.3 In modern terms

Today the Königsberg bridges problem is abstracted one step further from Euler's picture and formed into what today is known as a *graph*. The the land masses are represented by points known as *vertices* and the bridges are represented by lines known as *edges*. Thus the problem can be represented as follows:

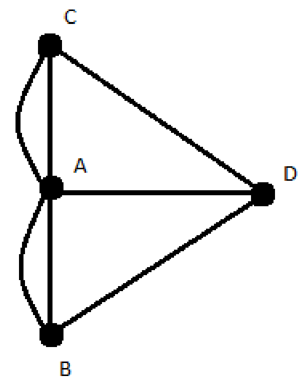

Figure 2. Graph of the Königsberg problem.

A walk is represented as a sequence of edges, and it is known as an *Euler path* if it contains each edge only once. A graph that contains an euler path is said to be *Eulerian.* If an Eulerian path exists that begins and ends at the same vertex, it is known as an*Euler circuit.*

In today's language Euler's results would be stated as follows:

1. If a graph contains more than two vertices of odd degree, it is not eulerian.
2. If a graph contains exactly two vertices of odd degree, an euler path exists and must begin at either vertex of odd degree.
3. If a graph contains no vertices of odd degree, an euler path exists beginning at any vertex.

Hence since the graph in Figure 2, representing the Königsberg bridges, contains four vertices of odd degree, it is not Eulerian.

Today bridges in Kaliningrad are no longer the seven that existed in Euler's time. All seven bridges were destroyed in World War II.[4] Since then eight bridges have been built, as seen in the following Google Maps image:

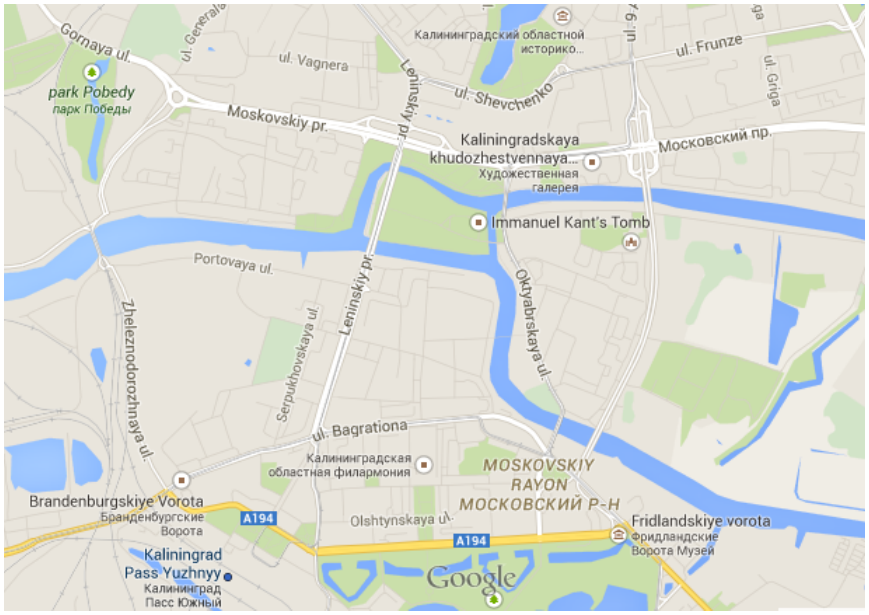

Figure 3. Map of Kaliningrad 2015.

A graph of the current bridges looks as follows.

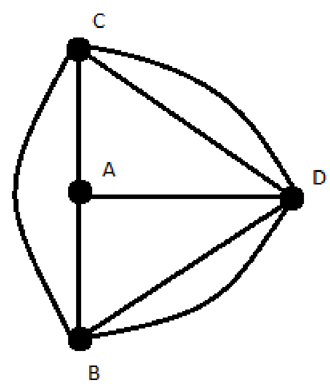

Figure 4. Graph of Kaliningrad bridges today.

Since the graph contains exactly two vertices of odd degree, namely vertices A and D, an euler path exists! It is therefore possible to take a walk in Kaliningrad today crossing each bridge exactly once. However, since vertices A and D represent two islands, and the euler path must begin at either one of the islands, such a walk is impractical in reality.

4 The 19th Century

It was not until over a century after Euler had worked on the Königsberg bridges problem that any significant work was done in graph theory. Around the 1850's different graph theoretical problems began to arise, and in somewhat unexpected places, including differential calculus and chemistry. The slowly growing prevelence of graphs in mathematics promoted further study into graphs themselves. The term "graph" was coined in 1878 by the British mathematician James Joseph Sylvester in a paper published in Nature regarding connections between algebra and chemistry.[3]

4.1 Arthur Cayley and the Trees

Some of the most important work in graph theory in the 19th century was done by British mathematician Arthur Cayley. He was studying analytical forms arising from differential calculus, when he noted that they can represented by a specific type of graph, which he coined as *trees*.[5] Trees are graphs that contain no cycles.

Cayley's studies formed the beginning of enumerative graph theory, which focuses on calculating the number of certain types of graphs. In his case, the types of graphs were the trees. He extended a formula originally found by Carl Wilhelm Borchardt in 1860 for counting trees on a given number of vertices.[5] As is not uncommon in the history of mathematics the formula was misnamed. It became known as Cayley's formula and it states that given the number of vertices n, the number of trees on n labeled vertices is n^{n-2}. The following is a simple example.

Example 1. *Let $n = 4$. By simple calculation we see that Cayley's formula gives $4^{4-2} = 16$ possible labeled trees. We find the possible labeled trees in the following picture.*

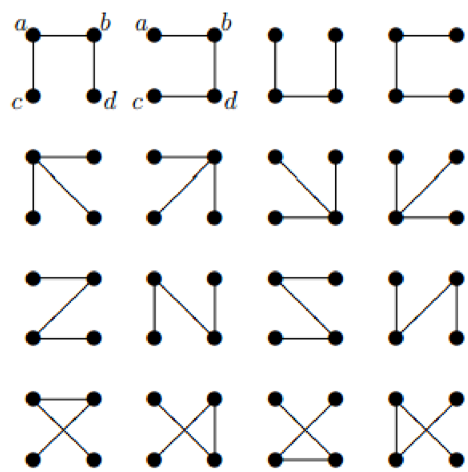

Figure 5. The 16 possible labeled trees with four vertices.[7]

Cayley linked his studies on trees with the theoretical chemistry. He was interested in enumerating the isomers of different possible hydrocarbons. In chemistry there is not always a bijective correspondence between the chemical formula and the set of chemicals. For example, although butane and isobutane have the same chemical formula of C_4H_{10}, they are chemically and graphically very different. Cayley considered what he called "diagrams", made of points labeled by atoms and connected by links of atomic bonds.[6] In Figure 6 the two possible graphs of C_4H_{10} are shown. The bold lines represent carbon bonds, while the non-bold lines represent hydrogen bonds.

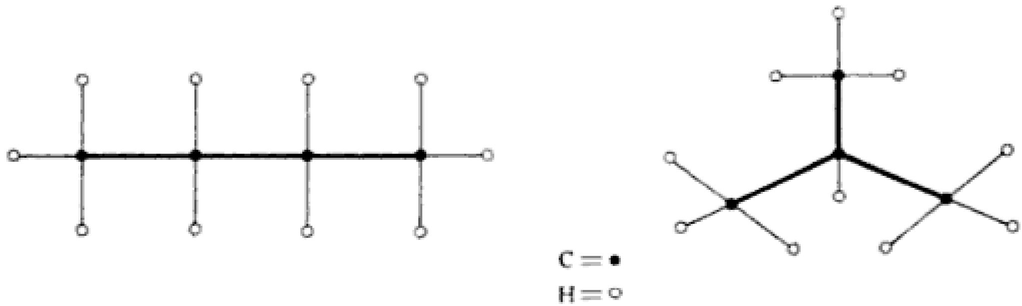

Figure 6. Cayley's representation of the tree structure of butane (left) and isobutane (right).[6]

4.2 The Icosian Game

Another 19th century mathematician who dabbled in graph theory was Sir William Rowan Hamilton. In 1857 he invented the graph theoretical puzzle called the icosian game. The game stemmed from an algebra of Hamiltons invension which he called *icosians*. He sold the game for 25 pounds, and it was distributed commercially in Europe. It turned out to be a complete flop because it was too easy.[16] The game was to find a specific kind of path on the graph of a dodecahedron projected onto a plane (see Figure 7). The path had to be one such that each point was visited only once.

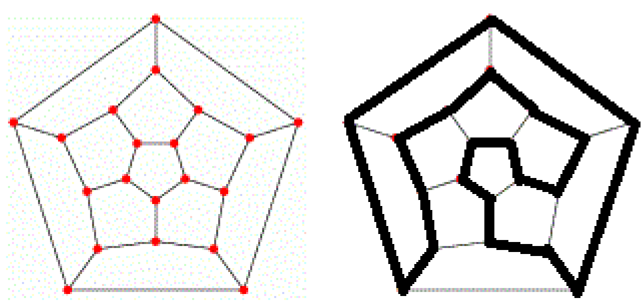

Figure 7. The icosian game and its solution.

Although the game itself has not had great mathematical significance, from a historical point of view it is more interesting. The terms *Hamiltonian path* and *Hamiltonian cycle* originate from this problem, since they are paths and cycles which visit each vertex exactly once. Thomas Kirkman had studied such paths a year before Hamilton's Icosian game was published, but Hamilton's name stuck because of the Icosian game. The game also serves as one of the earliest formulations of the Traveling Salesman Problem, which will discussed in more detail in Section 6.2.[16]

4.3 The Four Color Conjecture

Another famous problem in graph theory that arose in the 19th century was the Four Color Conjecture. As many of the early questions in graph theory, the Four Color Problem was also puzzle-like in its nature. Informally the problem is to prove that four colors are sufficient to color any map such that no two bordering regions share the same color. Any map that can be colored in such a way is known as a 4-colorable map. It may not be immediately obvious that the problem is a graph theoretical one, however, it turns out that any map can be represented in the form of a graph. The regions in the map may be represented with vertices, and edges connecting vertices which border each other on the map.[8] An example of this can be seen in Figure 8.

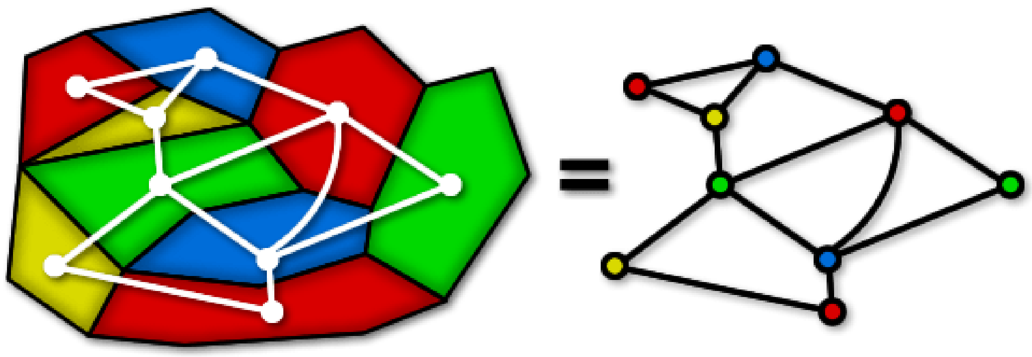

Figure 8. A four colored map and its corresponding graph.[9]

The conjecture that any map could be 4-colored was first proposed on October 23rd 1852 by Francis Guthrie, a South African mathematician and botanist. He had studied under Augustus De Morgan at the University College London, but moved on to study law. Francis' brother Frederik Guthrie was studying under De Morgan, so Francis showed him some results on map colorings which he was unable to prove. He asked his brother to inquire about the solution from De Morgan.[8]

After hearing about the problem, De Morgan wrote to Hamilton the same day saying:

> A student of mine asked me today to give him a reason for a fact which I did not know was a fact - and do not yet. He says that if a figure be anyhow divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured - four colours may be wanted, but not more - the following is the case in which four colours are wanted. Query cannot a necessity for five or more be invented. ...... If you retort with some very simple case which makes me out a stupid animal, I think I must do as the Sphynx did....

To which Hamilton replied on 26 October 1852:

> I am not likely to attempt your quaternion of colour very soon.

De Morgan kept actively asking if other mathematicians could give him the solution, and several mathematicians did work on it. Cayley also learnt of the problem from De Morgan. He showed interest toward the problem and sent a paper named *On the colouring of maps* to the Royal Geographical Society, which was published in 1879. The paper explains where the difficulties lie in attempting to prove the Conjecture.[8]

A student of Cayley's named Alfred Bray Kempe announced on July 17th 1879 that he had a proof of the Four Color Conjecture. Under Cayley's guidance Kempe submitted the proof in the American Journal of mathematics, and it was published the same year. Kempe received much acclaim for his proof and was subsequently elected a Fellow in the Royal Society. Unfortunately, a decade later in 1890, a mistake was noticed in the proof by Percy John Heawood. Heawood showed that Kempe's argumentation was valid only to prove the 5-coloring theorem, but left a gap when it came to the 4-coloring theorem. Kempe was unable to mend the gap in his proof, and he reported the error to the London Mathematical Society.[8]

Heawood continued working on the Four Color Conjecture, and in he made some progress. For example, in 1898 he proved that if the number of edges around each region on the map is divisible by 3 then the map is 4-colorable.[8]

Work on the Four Color Conjecture would continue on well into the 20th century, and many false proofs and disproofs would come and go. It was not until 1976 that the Four Color Conjecture was considered proven.[8] More details will be discussed in the section *The Computer Age: 1950 and after*.

5 The Early 1900's

Much of the foundations to modern graph theory was done in the early 1900's. Graphs were no longer considered as curiosities along the outskirts of mathematics, but became a field of study of their own. This is evident from the fact that the first books on the subject began to crop up. The first official textbook on graph theory was written in 1936 by the Jewish Hungarian mathematician Dénes König. It was published in German, and titled *Theorie der endlichen und unendlichen Graphen.*[11] Another Hungarian mathematician named George Polya developed further some of the enumerative techniques of Cayley, applying them to a broad specturm of problems. These applications followed partly in Cayley's footsteps, with further developments in theoretical chemistry and the enumeration of trees. He proved a major results in enumerative graph theory which became known as Polya's Enumeration Theorem. With rigorous foundations developing for graph theory, many of its important theorems were proven in the early 1900's. The number of fascinating theorems and proofs are too numerous to mention in this short overview, so instead of listing them, the author has chosen in this section to present little more closely one interesting result: The Kuratowki Theorem.

5.1 The Kuratowski Theorem

In 1930 the Polish mathematician Kazimierz Kuratowski published a surprising theorem about planar graphs which was later named after him.

Theorem 1. *(Kuratowski) A finite graph is planar if and only if it does not contain a subgraph that is a subdivision of K_5 or of $K_{3,3}$.*

In other words, a graph can only be drawn on a plane such that no edges cross each other if and only if the graph does not contain either of the following graphs:

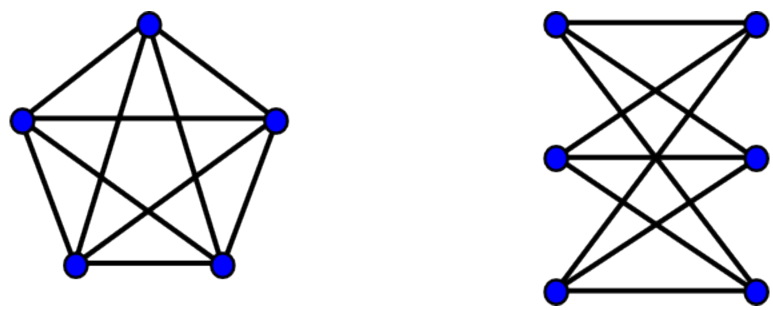

Figure 10. Left: Graph K_5. Right: Graph $K_{3,3}$.[12]

This is a somewhat unexpected result, since there are infinitely many possible configurations of graphs. Only graphs that are subdivisions of two graphs ruin planarity.

Note: Due to the long and technical nature of the complete proof, here only the sufficient condition is proven. But first we begin with a lemma.

Lemma 1. *A bipartite graph contains no cycles of odd degree.*

Proof. Let G=(V,E) be a bipartite graph. By the definition of a bipartite graph $\exists$ subsets A and B of G such that all edges $e \in E$ are of the form $\{a, b\}$, where $a \in A$ and $b \in B$.

Suppose G has at least one odd cycle C.

Now $C = (v_1, v_2, \ldots, v_n, v_1)$, where its length n is odd.

WLOG, let $v_1 \in A$. It follows that $v_2 \in B$ and $v_3 \in A$ and so on.

We see that $\forall\, k \in \{1, 2, ..., n\}$ the following is true:

$$v_k \in \begin{cases} A, & \text{if } k \text{ is odd} \\ B, & \text{if } k \text{ is even} \end{cases}$$

But now if n is odd, we have $v_n \in A$ and $v_1 \in A$, which implies that $v_n v_1 \in C_n$. Hence, C contains an edge $\{v_n, v_1\}$ where both $v_n \in A$ and $v_1 \in A$. This contradicts the assumption that G is bipartite. Hence if G is bipartite, it has no odd cycles.

$\square$

Using Lemma 1 we may now prove the sufficient condition of Kuratowski's Theorem.

Proof. (Kuratowski's Theorem)
Claim: A graph is planar if it does not contain a subgraph that is a subdivision of K_5 or $K_{3,3}$.

First we consider the graph K_5.

> Suppose K_5 is planar. It has 5 vertices and 10 edges, so by Euler's formula $V - E + F = 2$, the number of faces in a planar embedding must be $F = 2 - 5 + 10 = 7$.
>
> Each face is bounded by at least 3 edges, which implies that the maximum number of faces is $F = \frac{1}{3}E$. However, each edge bounds exactly two faces, thus doubling the maximum number of faces to $F = \frac{2}{3}E$. This gives $F = \frac{2}{3}E = \frac{2}{3} \times 10 = 6\frac{2}{3}$. This is a contradiction since $F = 7$. Therefore K_5 is not planar.

Next we consider the graph $K_{3,3}$.

> Suppose $K_{3,3}$ is planar. It has 6 vertices and 9 edges, and again by Euler's formula, the number of faces must be $F = 2 - 6 + 9 = 5$.
>
> Since this graph is bipartite, by Lemma 1 we know that cycles in $K_{3,3}$ cannot have odd degree, which means that each face has at least 4 edges. So the number of faces is at most $F = \frac{1}{4} \times E$. But since each edge bounds two faces, the maximum number of faces increases to $F = \frac{2}{4} \times E = \frac{1}{2} \times E = 4.5$, which is a contradiction. Therefore $K_{3,3}$ is not planar.

Now consider a subdivision of either graph. The number of edges and the number of vertices is each increased by one, resulting in no change to the result of Euler's formula, or the maximum number of faces. The argument above can thus be repeated.

Hence if G contains a subdivision of K_5 or $K_{3,3}$, it is not planar.

Taking the contrapositive gives the claim.

□

6 The Computer Age: 1950 and after

With the development of computers in the mid 20th century, graph theory would no longer be the same. Studying graphs not only became easier, but new applications for graphs were found in computing and networking. During the late 1950's, some major work in graph theory was done by Hungarian mathematicians Paul Erdös and Alfréd Rényi. They began what is considered a probabilistic approach to graph theory. From it was born the entire field which is now known as random graph theory. It studies when certain graph theoretical properties are expected to arise after randomly adding edges to a predetermined number of vertices.[13]

6.1 Proof to the Four Color Theorem

The Four Color Problem finally gained it's status as theorem in 1976, almost a hundred years after Kempe proposed his flawed proof. Kempe's work had not gone to waste, since the final proof relied on developments of his ideas called Kempe chains. The final proof was given by Kenneth Appel and Wolfgang Haken at the University of Illinois.

The outline of the idea of the proof is as follows.

Supposing that the four-color conjecture is false, there would be a smallest map that requires five colors. Using the following technical concepts developed in the early 1900's, they showed that such a minimal counterexample could not exist.

1. An *unavoidable set* is a set of configurations that must appear in every minimal non-4-colorable map.
2. A *reducible configuration* is an arrangement of countries that cannot occur in a minimal counterexample. As suggested by the name, a map containing a reducible configuration can be reduced to a smaller map. This smaller map has the condition that if it is 4-colorable, then so is the original map. This implies that if the original map is not 4-colorable, then the smaller map is not either, and hence the original map is not minimal.

Appel and Hanken used properties of the reducible configurations and found an unavoidable set of reducible configurations in the minimal map. This showed that a minimal counter example could not exist. Using their method, all maps could be reduced to 1936 reducible configurations. Each of these configurations had to be checked by computer which took over 1200 hours (over seven weeks). The final proof was printed in a 400 page volume which included an enormous unavoidable set. This was checked by peer review.[9]

The Four Colour Theorem was the first major theorem with a proof that could not be verified directly by other mathematicians, instead it relied on computer assisted work. There were initial worries about the results for this reason. Since then work has been done to simplify and improve the algorithms, and other independent verifications have also been done.[9]

6.2 The Traveling Salesman Problem

Given a set of cities and the distances between them, the Traveling Salesman Problem asks to find the shortest route that visits each city once, and returns to the starting point. As eluded to earlier in section 4.2, a special case of this problem was asked by Rowan Hamilton with his Icosian game. There the task was merely to find a cycle, with no restriction on length. Although its first official formulation was given in the 1930's, it became popular problem of study in the 1950's and 1960's with the rise of computers. Although it was approached as an optimization problem in computation complexity theory, it can be modeled and easily visualized by a weighted graph as shown by the following example.

Example 2. *Suppose vertices 0,1,2,3,4 represent cities, the following weighted graph shows the distances between each city by the edge traveled.*

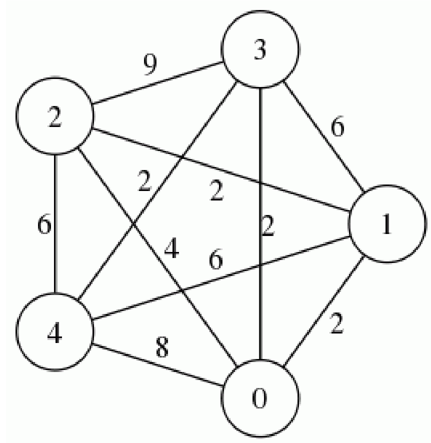

Figure 11. Weighted graph for example 2.

The shortest path (not necessarily unique) that begins and ends at city 0 is the path (0,1,2,0,3,4,3,0), with length 2+2+4+2+2+2+2=16. To prove that no shorter path exists, one must check all other possibilities.

The Traveling Salesman Problem turned out to be computationally challenging, and in 1972 it was proven to be an NP-Hard problem.[15]

7 Conclusion

What began as obscure mathematical curiosities such as the Königsberg bridges problem and the Icosian game, gave rise to a robust field of study with unexpected applications in many sciences. After having developed a rigorous foundation, graph theory thrived with the dawn of computing. The proof of the by then century-old Four Color Conjecture was a milestone not only in graph theory, but in mathematics as a whole, pushing the limits of the notion of proof. With new solution have come new problems. Today graph theory is still an active area of research, with many applications in a wide variety of sciences. Graph theoretical ideas are working behind the scenes in our daily lives as we use GPS, communication networks, and even Facebook. So the next time someone asks you to solve an obscure puzzle, think twice before declining. A whole new field of study could be hidden behind it, at least that is what happened with graph theory.

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