William Rowan Hamilton: Mathematician, Poet and Vandal

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William Rowan Hamilton was born in Dublin at the stroke of midnight between the third and fourth of August 1805 and during his lifetime made fundamental contributions to physics and mathematics.

His father was a solicitor who developed grave financial difficulties which led to young William and his four sisters being raised by relatives. So at the age of three he was taken to live with his uncle, the Reverend James Hamilton, who was headmaster of the diocesan school in Trim, a town approximately 50 kilometres north-west of Dublin. Uncle James was an honours graduate of Trinity College Dublin whose interests lay in languages and the classics. He recognised William as a child prodigy from an early age and he set about creating an environment to foster this potential.

William's early genius was displayed through his ability in languages and the performance of lengthy mental computation. At the age of eight he had mastered Greek, Hebrew and Latin and was reportedly proficient in thirteen languages by the age of thirteen, including French, German, Italian, Persian, Arabic, Sanscrit, Syriac and Bengali.

At the age of thirteen William also had a peculiar arithmetical encounter with the *amazing calculating boy* Zerah Colburn. Zerah was an American who could rapidly perform long mental calculations and was taken on a European tour by his father to display his talent. Zerah was a year older than William and a competition was arranged between the boys, from which Zerah emerged the clear winner. This left a lasting impression on William since he had never before been bettered by another child and he later admitted it led to a new appreciation of mathematics.

Around this time the Hamilton family was also struck by tragedy, in 1817 William's mother died and in 1819 his father died leaving the family in an

*Essay submitted for the *Snapshots of the History of Mathematics* course, May 2015.

even more precarious financial situation. This left young William as the head of the family and thereafter he felt responsible for the well-being of his sisters.

In terms of mathematics Hamilton later admitted that before 1821 he had a modest and traditional mathematical background. He was, however, fortunate to arrive during a major revision of the mathematical curriculum at Trinity College Dublin. Mathematical studies in the United Kingdom had not been updated for a long time and were essentially limited to English mathematics, which included Newton's *Arithmetic* and *Optics* with selections from *Principia*, MacLaurin's *Fluxions*, Greek geometry and the first six books of Euclid. At the beginning of the nineteenth century this changed when Bartholomew Lloyd at Trinity and George Peacock, John Herschel and Charles Babbage of Cambridge began translating and teaching the work of continental European mathematicians like Euler, the Bernoullis, Lagrange, Laplace, Fourier, d'Alembert and Monge.

So it was in this environment in 1821 that William was introduced to Bartholomew Lloyd's book *Analytic Geometry* and his eyes opened to a whole new world. He proceed to read books by the French mathematicians from Ecole Polytechnique in Paris as one might read a novel, but he was not merely content to absorb the new ideas, he also speculated and made his own observations. It was thus that he first came to the attention of the Irish scientific community in 1822 when he discovered a previously undetected error in Laplace's *Mécanique Céleste*. He was promptly invited to meet the Astronomer Royal of Ireland John Brinkley, where he further impressed by presenting some of his original mathematical research.

Although William's reputation was growing, his Uncle James was concerned he was neglecting his other studies and how it would impact on the entrance exam to Trinity, so at this time he studied mathematics "at stolen intervals" from reading classics. However Uncle James need not have worried, William finished first in the entrance exam and entered Trinity in 1823 to study classics and mathematics.

Unsurprisingly he was a high achiever at university and during his time he won two Chancellor's Prizes for poetry and was awarded two *optime* grades. An optime was a grade *off the scale* and they were rarely awarded. He received the first optime in classics after an examination on Homer while still a first year and later a second optime in science. It was unheard of for a first year to receive an optime and nobody in living memory had been awarded two optime grades.

Hamilton also studied extra curricular mathematical material, in particular the books written for the Ecole Polytechnique in Paris. It is therefore somewhat surprising that Hamilton did not follow continental mathematical trends in later life, nor did he even travel to the continent. This characteristic as an independent thinker forging his own path is perhaps what allowed him to make such original contributions to science.

Hamilton also enjoyed the move to the city and he was much in demand at Dublin's social occasions where he was described as having "radiance and magnetism". He was expected to graduate from Trinity top of the class with gold medals in classics and science and to progress along the academic career path by becoming a fellow of Trinity.

However in 1926 the position of Professor of Astronomy, which included the title Astronomer Royal, became vacant. Hamilton was persuaded to apply and a week later in June 1927, at the age of 22, he was appointed to the post! Among the other applicants was George Biddell Airy, Senior Wrangler from Cambridge who was later appointed Astronomer Royal at Greenwich. A key consideration for Hamilton in accepting the position was that it came with the directorship of Dunsink Observatory, located about ten kilometres from the city centre, which meant he could accommodate his four sisters.

His predecessor as Astronomer Royal considered the work of an astronomer at the time incompatible with Hamilton's creative energy, since it mainly involved data gathering and performing routine calculations. He also had no previous experience as an astronomer, so to address this he was invited to the Armagh Observatory to learn practical astronomy.

It was in Armagh that he met the Scottish engineer Alexander Nimmo who was performing an ordinance survey of Ireland. He joined Nimmo on a two month tour of Ireland and Britain and it was on this trip that he befriended the English poet William Wordsworth. They struck up a lifelong friendship and Wordsworth deserves credit for steering Hamilton towards mathematics when he considered dedicating himself to poetry.

As it turns out Hamilton did not distinguish himself as an astronomer and as predicted the tedious work involving nightly observations did not suit him. The logbook of the observatory shows his initial enthusiasm for observation soon died away and that his assistant performed most of the work.

We will describe the details of his main mathematical contributions later, but we mention briefly that during his life he received a knighthood at the age of thirty and he was elected President of the Royal Irish Academy in 1837 and served in this position until 1846.

Around this time he also married Helen Bayly, in 1833, and they had three children, born in 1834, 1835, 1840. This was unfortunately not a happy marriage and in later years contributed to the isolation and melancholy that overtook him. His marriage also resulted in the loss of a stabilising force in his life since his sisters moved away from Dunsink.

Historians of mathematics are fortunate that Hamilton's life was well documented and these sources have been preserved. His guardians actively recorded the development of their prodigy for posterity and as an adult Hamilton was encouraged to document and reveal the workings of his mind in the hope of gaining a better understanding of the phenomenon of genius.

After Hamilton's death, on 2 September 1865, his friend Robert Perceval Graves gathered a large body of material including his correspondence, manuscripts and pieces of paper and wrote the *Life of Sir William Rowan Hamilton* in three volumes from 1882 to 1891. This archive is now stored in Trinity College Dublin and according to Hankin, historians owe a great debt of gratitude to Graves for gathering this collection [7]. Indeed Hankin noted that he read 6000 letters, 250 manuscripts and ten boxes of loose paper while writing his biography of Hamilton. The biographies are complemented by *The Mathematical Papers of Sir William Rowan Hamilton* which were published by Cambridge University Press in four volumes between 1931 and 2001 [6].

The primary sources for this essay are the two modern biographies of Hamilton, by Hankins [7] and O'Donnell [10], and unless otherwise stated the facts and information presented here were drawn from these books.

Optics and Mechanics

It is natural to ask how a 22 year old undergraduate could be appointed by Trinity College as the professor of astronomy. The answer lies in his original work in geometrical optics. His approach was to apply analytic geometry to the properties of light, which until this time had only a geometric representation.

He published his work in the article *Theory of Systems of Rays* [4] and its strengths were its great abstraction and generality. Through his approach the properties of any optical system could be completely described in a single equation called the *characteristic function.* As a result the path of any ray of light entering, for instance, a telescope, microscope, or any system of lenses or mirrors, could now be traced. His theory could also accommodate both the particle and wave theories of light.

During his research in 1832 he also predicted the phenomenon called *conical refraction*, which occurs when a ray of light passes through a particular type of biaxial crystal and emerges as a hollow cone, as illustrated in Figure 1. Nobody had ever conceived of such optical behaviour before and a few months later it was verified experimentally. This became a landmark scientific event since it was the first occasion that theory made a prediction that was later confirmed experimentally.

Early on he realised that this approach to optics could be applied to

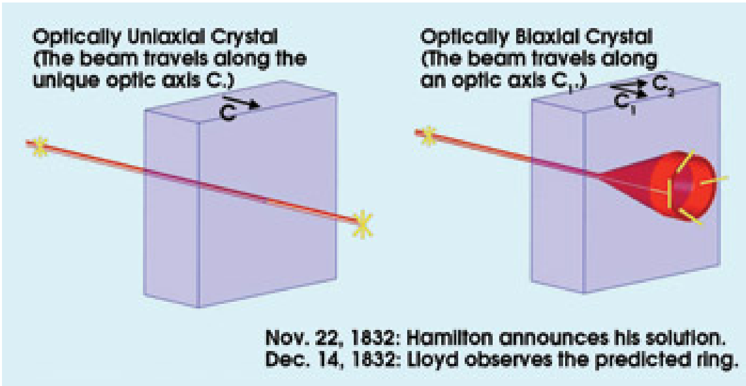

Figure 1: Conical Refraction (Image from `www.photonics.com`)

mechanics and this was published in his 1834 paper *General Methods in Dynamics*. In it he basically identified an analogy to the characteristic function in the case of particle motion in mechanics.

Although his optical and mechanical theories offered powerful and comprehensive models, it was not always easy to apply them and it was often easier to solve a problem in a less elegant but more practical way. So at this time he was respected by his peers while his ideas were somewhat unappreciated.

Heinrich Hertz, for instance, did not recognise that Hamilton had united optics and mechanics, but that his ideas were entirely formal. This was, however, before quantum theory and at the outset of quantum mechanics Arnold Sommerfield stated

> It seems almost as if Hamilton's methods were expressly created for treating the most important problems of physical [quantum] mechanics.

Hamilton's work also suited the quantum theory of wave mechanics when it arrived. Erwin Schrödinger said

> The Hamiltonian principle has become the cornerstone of modern physics... His famous analogy between mechanics and optics virtually anticipated wave mechanics, which did not have to add much to his ideas, [but] only had to take them seriously – a little more seriously than he was able to take them, with the experimental knowledge of a century ago.

From a physicists perspective Hamilton's work on optics and dynamics was his greatest contribution to science and it is a complement of sorts that he is considered by them a twentieth century theoretical physicist who lived in the nineteenth century.

Quaternions

Hamilton's first contribution to algebra came in 1835 in the paper *Theory of Conjugate Functions, or Algebraic Couples; With a Preliminary and Elementary Essay on Algebra as the Science of Pure Time* [5]. His ideas were heavily influenced by metaphysics and in particular by the philosophies of Immanuel Kant and Samuel Taylor Coleridge. His position was if geometry corresponds to our three dimensional world, then algebra corresponds to the metaphysical notion of *pure time*.

In this paper Hamilton expressed his dissatisfaction with the way complex numbers were presented as a sum of elements that cannot in fact be added, that is $a + bi$ where a, b are real numbers and $i^2 = -1$. Casper Wessel had already shown in 1797 that complex numbers could be represented geometrically by a pair of axes in what is now referred to as the complex plane.

Hamilton took this idea further and was the first to represent complex numbers as an ordered *couple* of real numbers, so $a + bi$ becomes (a, b) with addition and multiplication defined as

$$\begin{aligned}(a, b) + (c, d) &= (a + c, b + d)\\ (a, b)(c, d) &= (ac - bd, bc + ad).\end{aligned}$$

His paper is notable for taking algebra beyond the limitations of the ordinary algebra of real numbers and his algebraic treatment of complex numbers is considered by some historians as Hamilton's greatest achievement in algebra [7, p.264]. (Which considering his later contributions is a rather harsh judgment!)

He ended the paper with the intention to publish "many other applications of this view; especially to Equations and Integrals, and to a Theory of Triplets and Sets of Moments, Steps and Numbers, which includes this Theory of Couples". True to his word for the next eight years he intermittently sought an algebra composed of *triplets*. His motivation was that such an algebra would be a natural extension of couples and would provide an elegant way of describing our three-dimensional world.

During his investigation he initially considered triplets containing one real and two complex parts, that is

$$a + bi + cj$$

where a, b, c are reals and $i^2 = j^2 = -1$. The geometrical interpretation of this is the extension of the complex plane by adding a third axis, perpendicular to the plane, which by symmetry also gives the property that $j^2 = -1$.

Taking the square of this triplet he got

$$(a + bi + cj)^2 = a^2 - b^2 - c^2 + 2abi + 2acj + 2bcij \tag{1}$$

but for the algebra to be closed the product of two triplets should give a triplet and hence the ij term must vanish. The obvious option was to set $ij = 0$ however this seemed "odd and uncomfortable" to Hamilton, so instead he tried setting $ij = -ji$.

He again considered (1) but this time preserved the order of multiplication, so the last term became

$$bcij + bcji = bcij - bcij$$

and thus his assumption that $ij = -ji$ gave the necessary cancellation. This observation occurred around 1842 so Hamilton had for a while considered dispensing with the property of commutativity.

He also wanted the triplets to preserve the law of the modulus. We recall for the complex number $a + bi$ the modulus is defined as

$$|a + bi| = \sqrt{a^2 + b^2}$$

and the law of the modulus states that the product of the moduli of two complex numbers equals the modulus of the product, that is

$$|a + bi||c + di| = |(a + bi)(c + di)|.$$

Hamilton calculated the product of the moduli of the triplet $a + bi + cj$ as

$$\begin{aligned} |a + bi + cj||a + bi + cj| &= \sqrt{a^2 + b^2 + c^2}\sqrt{a^2 + b^2 + c^2} \\ &= \sqrt{a^4 + b^4 + c^4 + 2a^2b^2 + 2a^2c^2 + 2b^2c^2} \\ &= \sqrt{(a^2 - b^2 - c^2)^2 + 4(ab)^2 + 4(ac)^2}. \end{aligned} \tag{2}$$

While the modulus of the square of $a + bi + cj$ turns out to be

$$|(a + bi + cj)^2| = |a^2 - b^2 - c^2 + 2abi + 2acj + 2bcij|. \tag{3}$$

So for (2) and (3) to agree the ij term must again disappear and thus Hamilton had another compelling reason to set $ij = -ji$.

However difficulties arose with the product of two arbitrary triplets. If we consider the following product

$$(a + bi + cj)(d + ei + fj) = ad - be - cf + (ae + bd)i + (af + cd)j + (bf - ce)ij$$

where a, b, c, d, e, f are real numbers, this time the ij term does not cancel to give a triplet. Among the approaches to overcome this difficulty Hamilton tried to set ij equal to some unknown k and hope for some cancellation.

We now know Hamilton's difficulty in constructing a closed algebra of triplets under his assumptions is impossible. Hankel showed in 1867 that no algebra with dimension greater than two can satisfy all the laws of arithmetic and later Frobenius [3] demonstrated that there are exactly three such algebras over the real numbers: the real numbers themselves, complex numbers and the quaternions (which have dimension four). In 1958 Milnor [9] showed that by dropping the associativity assumption the Cayley numbers gave an algebra of dimension eight.

As a brief aside we will present a well known argument taken from [8] to see why triplets cannot form an algebra. We make the counter assumption that there exists an algebra composed of triplets with one real and two complex parts i and j. Since it is closed under multiplication the product of i and j is another triplet

$$ij = a + bi + cj \tag{4}$$

where a, b, c are real numbers. Multiplying (4) by i and then substituting (4) back in we get

$$\begin{aligned} i^2 j = -j &= ai - b + cij \\ &= ai - b + c(a + bi + cj) \end{aligned}$$

which gives

$$0 = ac - b + (a + bc)i + (c^2 + 1)j$$

from which it follows that $c^2+1 = 0$. Since c was assumed to be a real number we arrive at a contradiction and therefore there cannot exist an algebra based on triplets.

In the autumn of 1843 Hamilton returned to the problem of triplets and the breakthrough came on 16 October 1843 as he and his wife walked along the Royal Canal on their way to a council meeting of the Royal Irish Academy in Dublin. He realised if the product of triplets produced four terms then the solution might lie in taking the product of expressions with four terms. So he considered k to be a third imaginary part and again set $ij = k$. So his assumptions now stood at $i^2 = j^2 = -1$ and $ij = -ji = k$.

Using his powers for mental computation he saw that the square of such a four-tuple was

$$\begin{aligned} (a + bi + cj + dk)^2 = a^2 - b^2 - c^2 + d^2k^2 + 2abi + 2acj + 2adk \\ + bc(ij + ji) + bd(ik + ki) + cd(jk + kj) \end{aligned}$$

and by assumption $ij = -ji$ so one term already cancels. The product of the moduli gives

$$\begin{aligned}|a + bi + cj + dk||a + bi + cj + dk| \\ &= \sqrt{a^2 + b^2 + c^2 + d^2}\sqrt{a^2 + b^2 + c^2 + d^2} \\ &= \sqrt{(a^2 - b^2 - c^2 - d^2)^2 + 4(ab)^2 + 4(ac)^2 + 4(ad)^2}\end{aligned}$$

and he noticed that the law of the modulus was satisfied by taking $k^2 = -1$, $ik = -ki$ and $kj = -jk$.

Next he needed to check that general products respect the law of the modulus so he needed values for jk and ik. Having already assumed that $ij = k$ he calculated that the desired result came from setting $ik = i(ij) = -j$ and $kj = (ij)j = -i$. The excitement of this *eureka* moment led to a harmless piece of vandalism when he famously scratched the following multiplication formula into Broombridge over the Royal Canal

$$i^2 = j^2 = k^2 = -1$$
$$ij = -ji = k, \quad jk = -\,kj = i, \quad ki = -ik = j.$$

He called these new objects *quaternions* and confirmed that they satisfied every operation of arithmetic bar commutativity.

Discarding commutativity instigated a revolution in algebra as it challenged the conventional Principle of the Permanence of Forms, which was defined by the English mathematician George Peacock in his 1830 *Treatise on Algebra*. Peacock stated that the operations of arithmetical algebra should also apply to symbolical algebra, or as he put it himself

> Whatever form is algebraically equivalent to another when expressed in general symbols, must continue to be equivalent whatever those symbols denote. Whatever equivalent form is discoverable in arithmetical algebra considered as the science of suggestion, when the symbols are general in their form, though specific in their value, will continue to be an equivalent form when the symbols are general in their nature as well as in their form.

Non-commutative algebras came as a surprise to many in the mathematical community as did the realisation that algebras could just be constructed in this way. Many significant non-commutative algebras followed and modern algebra was born. For this reason Hamilton is sometimes known as *The Liberator of Algebra*.

Hamilton's geometrical interpretation of quaternions was that the i, j, k terms represent space and the real term represents time. This idea was an amazing precursor to Einstein's special theory of relativity which describes the relationship between space and time, so once again Hamilton's intuition found its value in the twentieth century. However the level of sophistication of experimental physics at the time meant that predicting relativity theory would have been an impossible mental leap.

Hamilton also quickly recognised that the multiplication of quaternions corresponds to rotations in three dimensional space and his idea was to create a system free of coordinates to reveal the essential properties of space. The unknown French banker Benjamin Olinde Rodrigues also discovered this method of representing rotations in space and published it in 1840 [11]. Unfortunately for Olinde his paper went unnoticed for many years and credit for the discovery of quaternions went to Hamilton [1].

To understand how these rotations work we first recall the simple case in the complex plane, which was first observed by Argand in 1806. If we begin with position $(1, 0)$, then multiplying it by i gives position $(0, 1)$ and multiplying again by i gives position $(-1, 0)$. So through the operation of multiplication we have rotated the point $(1, 0)$ twice through an angle of $90°$, as illustrated in Figure 2.

If we now consider the three-dimensional axes in the i, j, k directions, as illustrated in Figure 3, then notice if we wish to rotate the *north* line, or j-direction, about the *east* line, or i-direction, to the left through an angle of $90°$ we of course end up on the k-line.

Using the quaternions this geometric rotation can be succinctly described as multiplying j by i, or $ij = k$. If we continue the rotation by another $90°$ then we return to the j-line but this time pointing in the opposite direction,

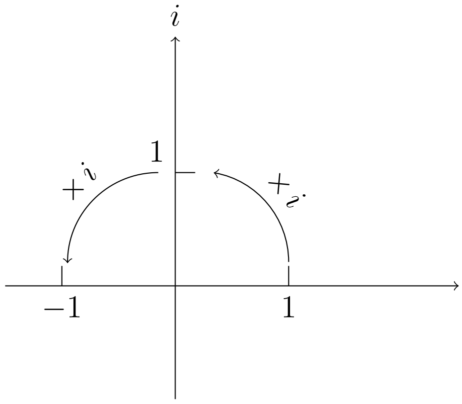

Figure 2: Rotations in the complex plane

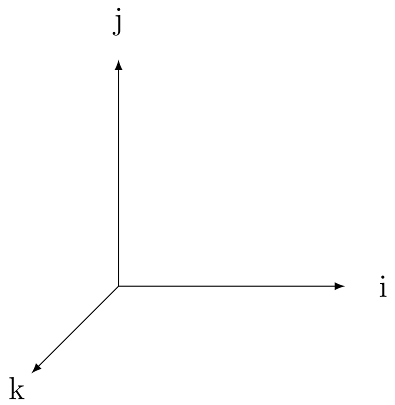

Figure 3: The i, j, k components of a quaternion

which can again be described by multiplying by i,

$$i(ij) = ik = -j.$$

Through his investigation of quaternions Hamilton also initiated the modern usage of the terms scalar to indicate the real part of a quaternion and vector to refer to the complex component. Early in his research he noticed that multiplying the vector parts of two quaternions gives a quaternion containing both scalar and vector parts. That is

$$\begin{aligned}&(ai + bj + ck)(di + ej + fk)\\&\quad = -(ad + be + cf) + (bf - ce)i + (cd - af)j + (ae - bd)k.\end{aligned}$$

When he took the product of two quaternions, say α and α', he denoted the scalar part as $S.\alpha\alpha'$ and the vector part as $V.\alpha\alpha'$. The scalar and vector parts of the product are what are today known in vector analysis as (the negative of) the dot product and the cross product, respectively. The *del* operator also appears in his work and in 1846 he was using expressions like

$$\triangleleft = i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}$$

and

$$-\triangleleft^2 = \left(\frac{d}{dx}\right)^2 + \left(\frac{d}{dy}\right)^2 + \left(\frac{d}{dz}\right)^2$$

Hamilton considered the quaternions the most important algebra after the complex numbers and dedicated the rest of his life to investigating them.

He produced a series of papers including eighteen instalments of *On Quaternions: Or on a New System of Imaginaries in Algebra* which appeared in the Philosophical Magazine between 1844 and 1850, and ten papers in the series *On Symbolical Geometry* in the Cambridge and Dublin Mathematical Journal from 1845 to 1849. In 1848 he gave a series of lectures on quaternions in Trinity College Dublin on which he based the book *Lectures on Quaternions* which was published in 1853.

Unfortunately Hamilton's publications suffered from his characteristic impenetrable writing style, so he was convinced to write a short elementary manual on the topic which he began in 1858. However this project dragged on and ballooned into *Elements of Quaternions* which in the end was longer than *Lectures* when it was posthumously published in 1866.

Many historians consider Hamilton's dedication to quaternions as a waste of his genius and Bell [2] considered the rest of his life an "Irish tragedy", similar to Einstein's later life when he pursed a unified field theory. His isolation in Dunsink Observatory from mathematical circles deprived him of regular critical comment on his research and his decline was compounded by an unhappy domestic life and excessive consumption of alcohol.

Quaternions themselves received a mixed reception from the scientific community. Peter Tait became the disciple of Hamilton while he was a professor at Queen's College Belfast (as it was known at the time) and went on to write an introductory book on them [12]. James Clerk Maxwell considered the identification of scalar and vector quantities of great importance to physics and wrote that

> The invention of the calculus of quaternions is a step towards the knowledge of quantities related to space which can only be compared, for its importance, with the invention of triple coordinates by Descartes. The ideas of this calculus, as distinguished from its operations and symbols, are fitted to be of the greatest use in all parts of science.

Maxwell saw the power of quaternions not only as a mathematical technique but the way they represent the physical world.

However many considered quaternions difficult to understand and the unintelligible writing style of Hamilton served only to exacerbated the situation. This led to Josiah Willard Gibbs and Oliver Heaviside creating vector analysis from quaternions. Vector analysis did not utilise the full quaternion algebra but restricted itself to the vector part which proved sufficient for their purpose of describing the theory of electricity and magnetism.

Although vector analysis traces its roots back to quaternions, since both Gibbs and Heaviside were influenced by Maxwell's *Treatise on Electricity and*

Magnetism published in 1873, a battle between the competing ideas began in 1890 when Tait wrote in the preface of the third edition of [12] that

> Even Prof. Willard Gibbs must be ranked as one of the retarders of quaternion progress, in virtue of his pamphlet on vector analysis, a sort of hermaphrodite monster, compounded of the notations of Hamilton.

Gibbs did not respond to this provocation but Heaviside said that quaternions were a "positive evil of no inconsiderable magnitude" and were

> Metaphysical considerations of an abstruse nature, only to be thoroughly understood by consummately profound metaphysicomathematicians, such as Prof. Tait.

Lord Kelvin went on to write in a letter in 1892 that

> Quaternions came from Hamilton after his really good work had been done; and, though beautifully ingenious, have been an unmixed evil to those who have touched them in any way, including Clerk Maxwell.

The hostility of these exchanges seems quite excessive and senseless considering the futility of pitting mathematical theories against each other.

To counteract vector analysis the *International Association for Promoting the Study of Quaternions and Allied Systems of Mathematics* was founded in 1895 and it produced a bulletin between 1900 and 1923. It was represented in fifteen countries and its membership peaked at sixty in 1900. That such a society existed hints that quaternions were finding it difficult to be accepted on their own merit and indeed this was the case as the techniques of vector analysis proceeded to become the standard tool of physicists.

The English mathematician Edmund Whittaker subsequently called for a Hamiltonian revival in 1940, stating that quaternions could be "the most natural expression of the new physics" and it took more than forty years before his idea was realised with the advent of computer graphics in the 1980s. The most efficient method of calculating three-dimensional orientation and rotation is currently by using quaternions and this finds applications not only in computer graphics but in robotics and the mechanics of satellites. So once again the value of Hamilton's notions were found in the twentieth century.

Today quaternions are also celebrated by the annual Hamilton walk which is organised by the Department of Mathematics at Maynooth University on 16 October. It began in 1990 and retraces the route taken by Hamilton from Dunsink Observatory to Broombridge. It attracts an international crowd and among its participants have been Fields Medallists Timothy Gowers and Efim Zelmanov, Nobel laureates Murray Gell-Mann, Steven Weinberg and Frank Wilczek along with Andrew Wiles and Roger Penrose.

The bridge is built of stone so the original inscription is no longer visible but a plaque was unveiled by Taoiseach Éamon de Valera in 1958. Incidentally de Valera himself performed an act of quaternion graffiti while on death-row for his part in the 1916 rebellion against British rule in Ireland. Before becoming a rebel and politician he was a mathematics teacher and was naturally proud of Hamilton's achievements, so in 1916 he etched, as Hamilton had done, the quaternion formula on his prison wall.

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