Probabilistic approximations, spring 2015 **Azmoodeh/Gasbarra**

Third Exercise Sheet

Wednesday May 27 at 10-12 in B322

Remark. The exercises should be returned to Dario Gasbarra in his e/mail in/box, B314 or `dario.gasbarra@helsinki.fi`**, before the time of the exercise class.**

1. Use Moment Matrices $\mathbf{M}_2$ and $\mathbf{M}_3$ in the lecture note to deduce the following generalization of the fourth moment theorem: Let $N \sim \mathscr{N}(0,1)$. For a sequence $\{X_n\}_{n\geq 1}$ of the elements in a fixed Wiener chaos of order $p \geq 2$, we have

$$\begin{matrix} \mathbb{E}(X_n^4) \to \mathbb{E}(N^4) = 3 \\ \mathbb{E}(X_n^6) \to \mathbb{E}(N^6) = 15 \end{matrix} \quad \Longleftrightarrow \quad X_n \xrightarrow{\text{law}} \mathscr{N}(0,1).$$

An interested maths-problem solver can also try to show that the even moments couple $(4,6)$ can be also replaced with the even moments couples $(4,8)$ and $(6,8)$! What about the couple $(8,10)$? This is, in fact, still an open problem!

2. Use moment matrix $\mathbf{M}_3$ to prove that if $X = I_p(f)$ is a multiple integral of **odd** order p such that $\mathbb{E}(X^2) = 1$ and $\kappa_4(X) \geq 3$, then $\kappa_6(X) \geq 0$. [Hint: the order p is odd implies that $\mathbb{E}(X^3) = 0$.]
Any idea how to relax the condition $\kappa_4(X) \geq 3$? In fact, it is conjectured that (**known as Γ_2 conjecture**) for any multiple integral $X = I_p(f)$ of any order $p \geq 2$ there exists some constant C_p such that

$$\mathbf{Var}(\Gamma_2(X)) \leq C_p \, \kappa_6(X), \qquad (\star)$$

where $\Gamma_1(X) = \langle DX, -DL^{-1}X\rangle$, and $\Gamma_2(X) = \langle DX, -DL^{-1}\Gamma_1(X)\rangle$. A direct consequence of the Γ_2 conjecture is that always $\kappa_6(X) > 0$! The estimate $(\star)$ has to be compared with the well-known fourth moment estimate

$$\mathbf{Var}(\Gamma_1(X)) \leq C_p \, \kappa_4(X).$$

3. (**Convergence to centered chi-square law**) Fix an integer $\nu \geq 1$, and let $N_1, \cdots, N_\nu$ be independent $\mathscr{N}(0,1)$ random variables. Consider $F_\infty = -\nu + \sum_{i=1}^{\nu} N_i^2$, that is, F_∞ has the centered chi-square law with ν degrees of freedom. Let $p \geq 2$ be an **even** integer, and define

$$c_p := \frac{1}{(p/2)! \binom{p-1}{p/2-1}^2} = \frac{4}{(p/2)! \binom{p}{p/2}^2}. \qquad (0.1)$$

Let $F_n = I_p(f_n)$ be a sequence of multiple integrals such that $\mathbb{E}(F_n^2) = 2\nu$ for all n. Consider the following four assumptions, as $n \to \infty$:

(i) $\mathbb{E}(F_n^4) - 12\mathbb{E}(F_n^3) \longrightarrow 12\nu^2 - 48\nu$.

(ii) $\|f_n \widetilde{\otimes}_{p/2} f_n - c_p f_n\| \to 0$ and $\|f_n \widetilde{\otimes}_r f_n\| \to 0$ for every $r = 1, \cdots, p-1$ such that $r \neq p/2$.

(iii) $\|DF_n\|^2 - 2pF_n \to 2p\nu$ in $L^2(\Omega)$.

(iv) F_n converges in distribution toward F_∞.

The aim of this exercise is to show that these four assumptions are equivalent. **(a)**: Use Hypercontractivity on Wiener chaoses to show that $\sup_{n\geq 1} \mathbb{E}|F_n|^q < +\infty$ for all $q \geq 2$. deduce that (iv) implies (i). **(b)**: Use product formula and isometry to prove that

$$\mathbb{E}(F_n^3) = p!\,(p/2)! \binom{p}{p/2}^2 \langle f_n, f_n \widetilde{\otimes}_{p/2} f_n \rangle. \tag{0.2}$$

(c): Use the expression in terms of the norms of the contractions obtained in the lecture note for $\mathbb{E}(F_n^4)$ with (0.2) to deduce that

$$\begin{aligned}
\mathbb{E}(F_n^4) - 12\mathbb{E}(F_n^3) = 12\nu^2 + 3/p \sum_{r=1,\cdots,p-1\,\&\, r\neq p/2} p^2 (r-1)! \binom{p-1}{r-1}^2 p! \binom{p}{r}^2 \\
\times (2p-2r)!\, \|f_n \widetilde{\otimes}_r f_n\|^2 \\
+ 3p(p/2-1)! \binom{p-1}{p/2-1}^2 (p/2)! \binom{p}{p/2}^2 p!\, \|f_n \widetilde{\otimes}_{p/2} f_n\|^2 \\
- 12p!\,(p/2)! \binom{p}{p/2}^2 \langle f_n, f_n \widetilde{\otimes}_{p/2} f_n \rangle.
\end{aligned} \tag{0.3}$$

(d): Use elementary simplifications, check that

$$\begin{aligned}
3/2 \frac{(p!)^5}{[(p/2)!]^6} \|f_n \widetilde{\otimes}_{p/2} f_n - c_p \times f_n\|^2 \\
= -12p!\,(p/2)! \binom{p}{p/2}^2 \langle f_n, f_n \widetilde{\otimes}_{p/2} f_n \rangle + 24p!\, \|f_n\|^2 \\
+ 3p(p/2-1)! \binom{p-1}{p/2-1}^2 (p/2)! \binom{p}{p/2}^2 p!\, \|f_n \widetilde{\otimes}_{p/2} f_n\|^2.
\end{aligned} \tag{0.4}$$

Deduce that (i) and (ii) are equivalent.
(e): Show that

$$\mathbb{E}[F_n\|DF_n\|^2] = p^2(p/2-1)!\binom{p-1}{p/2-1}^2 p!\,\langle f_n, f_n\widetilde{\otimes}_{p/2}f_n\rangle. \qquad (0.5)$$

(f): Use (0.5) and the expression in terms of the norm of the contractions for $\mathbf{Var}(1/p\|DF_n\|^2)$ in the lecture note to show that, if (ii) holds, then $\mathbb{E}[(\|DF_n\|^2-2pF_n - 2p\nu)^2] \to 0$, as $n \to \infty$, that is, (ii) implies (iii).
(g): Show that (iv) holds iff any subsequence $\{F_{n_k}\}$ converging in distribution to some random variable G is necessarily such that $G \overset{\text{law}}{=} F_\infty$. (Hint: Use Prokhorov's theorem.)
(h): Assume that F_n converges in distribution to some G, and let $\varphi_n(\lambda) := \mathbb{E}(e^{i\lambda F_n})$. Prove that $\varphi_n'(\lambda) \longrightarrow \mathbb{E}(e^{i\lambda G})$.
(i): Show that $\varphi_n'(\lambda) = -\lambda/p\mathbb{E}[e^{i\lambda F_n}\|DF_n\|^2]$.
(j): Assume that (iii) holds and F_n converges in distribution to some G. Let $\varphi_\infty(\lambda) = \mathbb{E}(e^{i\lambda G})$. Prove that

$$(1-2i\lambda)\varphi_\infty'(\lambda) + 2\lambda\nu\varphi_\infty(\lambda) = 0.$$

Deduce that $\varphi_\infty(\lambda) = \left(\frac{e^{-i\lambda}}{\sqrt{1-2i\lambda}}\right)^\nu$, and then $G \overset{\text{law}}{=} F_\infty$.
(k): Show that (iii) implies (iv).