Probabilistic approximations, spring 2015 **Azmoodeh/Gasbarra**

Second Exercise Sheet

Thursday April 30 at 12-14 in B321

Remark. The exercises should be returned to Dario Gasbarra in his e/mail in/box, B314 or `dario.gasbarra@helsinki.fi`**, before the time of the exercise class.**

1. **(a)** Let $F_n = I_p(f_n)$ for some fixed $p \geq 2$ such that $\mathbb{E}(F_n^2) = 1$ for all n. Denote by $\phi_n(\lambda) = \mathbb{E}(e^{i\lambda F_n})$ the characteristic function of F_n. Show that, for every real λ,

$$\phi_n'(\lambda) = \frac{i\lambda}{p}\mathbb{E}(e^{i\lambda F_n}\|DF_n\|^2).$$

(b) By using part (a), as well as the fact that the mapping $t \mapsto \mathbb{E}(e^{itN})$, $N \sim \mathcal{N}(0,1)$, is the unique solution to the differential equation $\phi'(t) + t\phi(t) = 0$, $\phi(0) = 0$ prove that the following implication:

$$\mathbf{Var}\|DF_n\|^2 \to 0 \Longrightarrow F_n \overset{\text{law}}{\to} \mathcal{N}(0,1). \qquad (\star)$$

2. **(a)** For any two multiple integrals $F = I_p(f)$ and $G = I_q(g)$, where $p, q \geq 1$ show that

$$\mathbb{E}(\langle DF, DG\rangle^2_{L^2(\mu)}) = \sum_{r=1}^{p\wedge q} \frac{(p!\,q!\,)^2}{((p-1)!\,(q-1)!\,(r-1)!\,)^2}\|f\tilde{\otimes}_r g\|^2.$$

(b) In addition, assume that $\mathbb{E}(F^2) = 1$. Show that $\mathbb{E}(\|DF\|^4) = p^2$ if, and only if, $\mathbf{Var}\|DF\|^2 = 0$.

(c) Use parts (a) and (b) to give a proof of implication $(\star)$ for a sequence $F_n = I_p(f_n)$ of multiple integrals with fixed order such that $\lim_{n\to\infty} \mathbb{E}(F_n^2) = 1$.

3. **(a)** Let $p, q \geq 1$. For two symmetric kernels $f \in L^2(\mu^p)$ and $g \in L^2(\mu^q)$ show that

$$\|f\tilde{\otimes}g\|^2 = \frac{p!\,q!}{(p+q)!}\sum_{r=0}^{p\wedge q}\binom{p}{r}\binom{q}{r}\|f \otimes_r g\|^2.$$

(b) For any two multiple integrals $F = I_p(f)$ and $G = I_q(g)$, where $p, q \geq 1$, use multiplication formula to show that

$$\mathbb{E}(F^2G^2) = \sum_{r=0}^{p\wedge q} r!^2\binom{p}{r}^2\binom{q}{r}^2(p+q-2r)!\,\|f\tilde{\otimes}_r g\|^2.$$

(c) Using $\mathbb{E}(F^2)\mathbb{E}(G^2) = p!\,q!\,\|f\|^2\|g\|^2$, parts (a) and (b) to deduce that

$$\mathbf{Cov}(F^2, G^2) \geq p!\,q!\,pq\|f \otimes_1 g\|^2.$$

As a result, always $\mathbf{Cov}(F^2, G^2) \geq 0$!.

(One can also show that $\mathbf{Cov}(F^2, G^2) \geq \max_{r=1,\cdots,p\wedge q}\|f \otimes_r g\|^2$.)
(d) Prove that $f \otimes_1 g = 0$ whenever F and G are independent. [In fact, the condition $f \otimes_1 g = 0$ is sufficient for independence of F and G. This is know as **Zakai-Üstünel** criterion.]

4. Let $F = (F_1, F_2) = (I_2(f_1), I_2(f_2))$ be a two dimensional random vector with components of multiple integrals of order 2. Denote the Malliavin matrix $\Gamma(F) := (\langle DF_i, DF_j\rangle)_{1\leq i,j\leq 2}$, and the covariance function of F by C. Show that

$$\mathbb{E}\det(\Gamma(F)) \geq 4\det C$$

and using the following result to deduce that the law of random vector F has a density if, and only if, $\det C > 0$.

Theorem 0.1. *Let $F = (F_1, \cdots, F_d)$ where each component F_i lives in a finite sum of Wiener chaoses for all $1 \leq i \leq d$. Then, the law of random vector F is not absolutely continuous with respect to Lebesgue measure if, and only if det$(\Gamma(F)) = 0$.*

5. Let $F_n = I_p(f_n)$ and $G_n = I_q(g_n)$, be two sequences of **independent** multiple integrals, with $p, q \geq 1$. Assume that $\mathbb{E}(F_n^2) \to \alpha^2 > 0$, $\mathbb{E}(G_n^2) \to \beta^2 > 0$, and $F_n + G_n \overset{\text{law}}{\to} \mathcal{N}(0, \alpha^2 + \beta^2)$ as $n \to \infty$. Prove that both

$$d_{\text{TV}}(F_n, \mathcal{N}(0, \alpha^2)) \to 0, \quad \text{and} \quad d_{\text{TV}}(G_n, \mathcal{N}(0, \beta^2)) \to 0.$$