Probabilistic approximations, spring 2015 **Azmoodeh/Gasbarra**
First Exercise Sheet
Friday March 27 at 10-12 in room B321

Remark. The exercises should be returned to Dario Gasbarra to his e/mail in/box, B314 or `dario.gasbarra@helsinki.fi`, before the time of the exercise class.

1. **(a)** Let $f : \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function such that $f' \in L^1(\gamma)$ (here γ stands for the standard Gaussian measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R})))$. Show that the function $x \mapsto x\, f(x) \in L^1(\gamma)$, and moreover

$$\int_{\mathbb{R}} x f(x) \gamma(dx) = \int_R f'(x) \gamma(dx).$$

[Hint: Use Fubini's Theorem.]
(b) The moment sequence $\{m_n(\gamma)\}_{n\geq 0}$ of standard Gaussian measure γ is defined as

$$m_n(\gamma) := \int_{\mathbb{R}} x^n \gamma(dx), \qquad n \geq 0.$$

Show that the sequence $\{m_n(\gamma)\}_{n\geq 0}$ satisfies in the recursion formula $m_{n+1}(\gamma) = n \times m_{n-1}(\gamma)$. Moreover, show that

$$m_n(\gamma) = \begin{cases} (n-1)!! & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

where the notation double factorial in s defined as $n!! = n(n-2)(n-4)\cdots$. [Hint: Use part (a) and induction.]

2. **(a)** Let $d \geq 1$. Show that the topologies induced by three distances $d_{\mathrm{TV}}, d_{\mathrm{Kol}}$ and d_{W} on the set of probability measures on $\mathbb{R}^d$ is strictly stronger than the topology induced by the convergence in distribution.
(b) Let $d \geq 1$. Show that for any two $\mathbb{R}^d$-valued random variables F and G we have

$$d_{\mathrm{TV}}(F, G) = \frac{1}{2} \sup\{|\mathbb{E}[h(F)] - \mathbb{E}[h(G)]| : h \text{ is Borel-measurable and } \|h\|_\infty \leq 1\}.$$

(c) Let F be a real-valued random variable and $N \sim \mathscr{N}(0,1)$. Show that

$$d_{\mathrm{Kol}}(F, N) \leq 2\sqrt{d_{\mathrm{W}}(F, N)}.$$

[Hint: Choose a "good" parameter α, and for fixed $z \in \mathbb{R}$, define the Lipschitz function

$$h_\alpha(x) = \begin{cases} 1 & \text{if } x \le z, \\ 0 & \text{if } x \ge z + \alpha, \\ \text{linear function} & \text{if } z < x < z + \alpha.] \end{cases}$$

3. **Stein's bound for the Kolmogorov distance (a)** For every $z \in \mathbb{R}$, write $f_z = f_{\mathbf{1}_{(-\infty,z]}}$, that is, f_z is the solution of the Stein's equation associated to the indicator function $h = \mathbf{1}_{(-\infty,z]}$. Also, Φ stands for the cumulative distribution function of a $\mathscr{N}(0,1)$ random variable. Show that

$$f_z(x) = \begin{cases} \sqrt{2\pi} e^{\frac{x^2}{2}} \Phi(x)[1 - \Phi(z)] & \text{if } x \le z, \\ \sqrt{2\pi} e^{\frac{x^2}{2}} \Phi(z)[1 - \Phi(x)] & \text{if } x \ge z. \end{cases}$$

(b) Prove that, for every $x \in \mathbb{R}$, $f_z(x) = f_{-z}(-x)$ (this implies that, in the estimates below, one can assume that $z \ge 0$ without loss of generality).
(c) Compute the derivative $\frac{d}{dx}[x f_z(x)]$, and deduce that the mapping $x \mapsto x f_z(x)$ is increasing.
(d) Show that $\lim_{x\to-\infty} x f_z(x) = \Phi(z) - 1$ and also that $\lim_{x\to+\infty} x f_z(x) = \Phi(z)$.
(e) Use part (a) to prove that

$$f'_z(x) = \begin{cases} [\sqrt{2\pi} x e^{\frac{x^2}{2}} \Phi(x) + 1][1 - \Phi(z)] & \text{if } x < z, \\ [\sqrt{2\pi} x e^{\frac{x^2}{2}} (1 - \Phi(x)) - 1]\Phi(z) & \text{if } x > z. \end{cases}$$

(f) Use part (e) in order to prove that

$$0 < f'_z(x) \le z f_z(x) + 1 - \Phi(z) < 1, \quad \text{if}\, x < z,$$

and

$$-1 < z f_z(x) - \Phi(z) \le f'_z(x) < 0, \quad \text{if}\, x > z$$

to deduce that $\|f'_z\|_\infty \le 1$.
(g) Use part (f) to show that $x \mapsto f_z(x)$ attains its maximum in $x = z$. Compute $f_z(z)$ and prove that $f_z(z) \le \frac{\sqrt{2\pi}}{4}$ for every $z \in \mathbb{R}$, to complete a proof of the following theorem.

Theorem 0.1. *Let $z \in \mathbb{R}$. Then the function f_z is such that $\|f_z\|_\infty \leq \frac{\sqrt{2\pi}}{4}$ and $\|f_z'\|_\infty \leq 1$. Therefore, for $N \sim \mathscr{N}(0,1)$, and for any integrable random variable F,*

$$d_{Kol}(F, N) \leq \sup_{f \in \mathscr{F}_{Kol}} |\mathbb{E}[f'(F)] - \mathbb{E}[Ff(F)]|,$$

where $\mathscr{F}_{Kol} = \{f : \|f\|_\infty \leq \frac{\sqrt{2\pi}}{4}, \|f'\|_\infty \leq 1\}$.

4. Define the **divergence operator** δ on the space $\mathrm{Dom}(\delta) \subset L^2(\mathbb{R}, \gamma)$ as $\delta\varphi(x) := -\varphi'(x) + x\varphi = -e^{\frac{x^2}{2}} \frac{d}{dx}(e^{-\frac{x^2}{2}} \varphi(x))$. Let $p \geq 0$ be an integer. We define the pth **Hermite Polynomial** as $H_0 = 1$ and $H_p = \delta^p 1$, where here $\delta^p = \delta \circ \cdots \circ \delta$, p times.
(a) Show that $d\delta - \delta d = \text{Identity}$, where $d = \frac{d}{dx}$, and moreover $\delta H_p = H_{p+1}$, $dH_p = pH_{p-1}$ and $(\delta + d)H_p = xH_p$.
(b) for any $p, q \geq 0$ show that

$$\int_R H_p(x) H_q(x) \gamma(dx) = \delta_{p,q} p!\,,$$

where here $\delta_{p,q}$ stands for the Kronecker delta.
(c) Show that the family $\{\frac{1}{\sqrt{p!}} H_p : p \geq 0\}$ is an orthonormal basis of $L^2(\mathbb{R}, \gamma)$.
(d) Define the **Ornstein-Uhlenbeck generator** $L\varphi(x) = -x\varphi'(x) + \varphi''(x)$. Show that $LH_p = -pH_p$.

5. **(a)** Define the **'carré-du-champ'** operator Γ_1 as

$$\Gamma_1(F, G) := \frac{1}{2}\{L(FG) - FLG - GLF\}.$$

Show that if $F, G \in \mathbb{D}^{1,2}$ then $\Gamma_1(F, G) = \langle DF, DG \rangle$.
(b) Use part (a) to deduce that for $F = I_p(f)$ we have

$$\langle DF, -DL^{-1}F \rangle = \frac{1}{p}\Gamma_1(F, F) = \frac{1}{p}\|DF\|^2.$$

(c) Define the operator

$$\Gamma_2(F, G) = \frac{1}{2}\{L(\Gamma_1(F, G)) - \Gamma_1(F, LG) - \Gamma_1(G, LF)\}.$$

Show that if $F = I_p(f)$ then $\mathbb{E}[\Gamma_2(F, F)] = p^2 \mathbb{E}[F^2]$.