Linear Algebra Part II Exercise 5

In exercises 1 and 2, test the sets of matrices for linear dependence in M_{22}. For those that are linearly dependent, express one of the matrices as linear combination of the others.

1. $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \right\}$
2. $\left\{ \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -3 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ -1 & 7 \end{bmatrix} \right\}$

In exercises 3 and 4, test the sets of polynomials for linear dependence in $\mathcal{P}_2$. For those that are linearly dependent, express one of the polynomials as linear combination of the others.

3. $\{1 + x, 1 + x^2, 1 - x + x^2\}$
4. $\{x, 2x - x^2, 3x + 2x^2\}$
5. Find the coordinate vector of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ with respect to the basis $\mathcal{B}$ in M_{22}.
$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$
6. Assume $V = \{A \in M_{22} : A \text{ is upper triangular}\}$. Check that V is a vector space. Give a basis for V.
7. Extend $\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$ to a basis for M_{22}.
8. Find a basis for $span \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right)$ in M_{22}.
9. Extend $\{1 + x, 1 + x + x^2\}$ to a basis for $\mathcal{P}_2$.
10. Find a basis for $span(1 - 2x, 2x - x^2, 1 - x^2, 1 + x^2)$ in $\mathcal{P}_2$.

In the following exercises determine whether T is a linear transformation. If so, find the kernel and the range of T and a basis for each.

11. $T : M_{22} \to M_{22}$ defined by $T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a + b & 0 \\ 0 & c + d \end{bmatrix}$
12. $T : M_{22} \to M_{22}$ defined by $T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$
13. $T : M_{nn} \to \mathbb{R}$ defined by $T(A) = a_{11}a_{22} \cdots a_{nn}$
14. $T : M_{nn} \to \mathbb{R}$ defined by $T(A) = Rank(A)$

In the following exercises, find either the nullity or the rank of T and then use the rank theorem to find the other.

15. $T : M_{22} \to \mathbb{R}^2$ defined by $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a - b \\ c - d \end{bmatrix}$

16. $T : \mathcal{P}_2 \to \mathbb{R}^2$ defined by $T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$

17. $T : M_{22} \to M_{22}$ defined by $T(A) = AB$, where $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

18. $T : M_{22} \to M_{22}$ defined by $T(A) = AB - BA$, where $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

19. For all of the mentioned linear transformations determine whether they are one-to-one and onto.