Linear Algebra Part II
Exercise 4

1. (Lemma 4.13 Page 276) Assume A is an $n \times n$ matrix. Prove that Laplace expansion along the first column of A is equal to determinant of A.

2. (Lemma 4.14 Page 277) Assume A is an $n \times n$ matrix and let B be obtained by interchanging any two rows or columns of A. Prove that $\det A = -\det B$.

3. Compute the determinant of the following matrices.

   - $A = \begin{bmatrix} 1 & 0 & 3 \\ 5 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$
   - $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$
   - $C = \begin{bmatrix} a & b & 0 \\ 0 & a & b \\ a & 0 & b \end{bmatrix}$
   - $D = \begin{bmatrix} 4 & 1 & 3 \\ -2 & 0 & -2 \\ 5 & 4 & 1 \end{bmatrix}$
   - $E = \begin{bmatrix} 1 & -1 & 0 & 3 \\ 2 & 5 & 2 & 6 \\ 0 & 1 & 0 & 0 \\ 1 & 4 & 2 & 1 \end{bmatrix}$
   - $F = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & c \\ 0 & d & e & f \\ g & h & i & j \end{bmatrix}$
   - $G = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

4. Prove that the determinant of a triangular matrix is the product of entries on its main diagonal.

5. Find that the determinant of the following matrices assuming that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 4$.

   - $\begin{bmatrix} 2a & 2b & 2c \\ d & e & f \\ g & h & i \end{bmatrix}$
   - $\begin{bmatrix} 3a & -b & 2c \\ 3d & -e & 2f \\ 3g & -h & 2i \end{bmatrix}$
   - $\begin{bmatrix} a & b & c \\ 2d-3g & 2e-3h & 2f-3i \\ g & h & i \end{bmatrix}$

6. For the following matrices, compute (a) the characteristic polynomial, (b) the eigenvalues, and (c) algebraic multiplicity of each eigenvalue, determine (d) a basis for each eigenspace and (e) geometric multiplicity of each eigenvalue, (f) determine whether the given matrix is diagonalizable and if so find the diagonal matrix similar to it and corresponding invertible matrix.

   - $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$
   - $B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 3 \\ 2 & 0 & 1 \end{bmatrix}$
   - $C = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ -1 & 0 & 2 \end{bmatrix}$
   - $D = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ -1 & -1 & 1 \end{bmatrix}$
   - $E = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$
   - $F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 \\ -2 & 1 & 2 & -1 \end{bmatrix}$

7. For matrix B in last exercise and $x = \begin{bmatrix} 12 \\ 27 \\ -6 \end{bmatrix}$ find $B^{20}x$.

8. Let A and B be $n \times n$ matrices with eigenvalues λ and μ respectively.

   - Give an example to show that $\lambda + \mu$ need not to be an eigenvalue of $A + B$.
   - Give an example to show that $\lambda\mu$ need not to be an eigenvalue of AB.
   - Suppose λ and μ correspond to same eigenvector x. Show that, in this case, $\lambda + \mu$ is an eigenvalue of $A + B$ and $\lambda\mu$ is an eigenvalue of AB.

9. Show that A and B are not similar.

   - $A = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
   - $A = \begin{bmatrix} 3 & -1 \\ -5 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ -4 & 6 \end{bmatrix}$
   - $A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 4 & 0 \\ 2 & 3 & 4 \end{bmatrix}$
   - $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$