Linear Algebra
Part II, First exercise

1. Find inverse of following matrices (if it exists). For 2×2 matrices use formula and for the rest use Gauss-Jordan method.

   - $\begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix}$
   - $\begin{bmatrix} 4 & -2 \\ 2 & 0 \end{bmatrix}$
   - $\begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$
   - $\begin{bmatrix} 2 & 3 & 0 \\ 1 & -2 & -1 \\ 2 & 0 & -1 \end{bmatrix}$
   - $\begin{bmatrix} 0 & a & 0 \\ b & 0 & c \\ 0 & d & 0 \end{bmatrix}$
   - $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & d \end{bmatrix}$
   - $\begin{bmatrix} 0 & -1 & 1 & 0 \\ 2 & 1 & 0 & 2 \\ 1 & -1 & 3 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

2. Prove that for invertible matrix A we have $(A^T)^{-1} = (A^{-1})^T$.

3. Solve the given matrix equation for X.

   - $XA^2 = A^{-1}$
   - $AXB = (BA)^2$
   - $ABXA^{-1}B^{-1} = I + A$

4. Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -1 & 3 \\ 2 & 1 & -1 \end{bmatrix}$

   Find matrix E that satisfies the given equation. Determine if E is elementary or not. If not write it as product of elementary matrices (if possible)? Also find the inverse of any matrices that you will find.

   - $EA = B$
   - $EB = A$
   - $EA = C$

- $EC = A$
- $EC = D$
- $ED = C$
- $EA = D$
- $ED = A$

5. (a) Prove that if A is invertible and $AB = O$, then $B = O$.
   (b) Give a counterexample to show that the result in previous part may fail if A is not invertible.
6. (a) Prove that if A is invertible and $BA = CA$, then $B = C$.
   (b) Give a counterexample to show that the result in previous part may fail if A is not invertible.
7. A square matrix A is called **idempotent** if $A^2 = A$.
   (a) Find three idempotent 2×2 matrices.
   (b) Prove that the only invertible idempotent $n \times n$ matrix is identity matrix.
8. Prove that if a symmetric matix is invertible, then its inverse is symmetric also.
9. Prove that if A and B are square matices and AB is invertible, then both A and B are invertible.

   Partitioning large matrices can sometimes make their inverses easier to compute, particularly if the blocks have a nice form. In next exercises verify by block multiplication that the inverse of a matrix, if partitioned as shown, is as claimed. (Assume that all inverses exist as needed.)

10. $\begin{bmatrix} A & B \\ O & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}BD^{-1} \\ O & D^{-1} \end{bmatrix}$
11. $\begin{bmatrix} O & B \\ C & I \end{bmatrix}^{-1} = \begin{bmatrix} -(BC)^{-1} & (BC)^{-1}B \\ C(BC)^{-1} & I - C(BC)^{-1}B \end{bmatrix}$
12. $\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$, where $P = (A - BD^{-1}C)^{-1}$, $Q = -PBD^{-1}$, $R = -D^{-1}CP$ and $S = D^{-1} + D^{-1}CPDB^{-1}$

Monday 16 March we go thorugh almost all material form section 3.3 (The inverse of a Matrix) just a couple of examples left. We will jump to section 3.5 (Subspaces, Basis, etc.) on Friday after covering some examples from Monday.