Linear Algebra Part II Exercise 3

1. Prove that the following transfomations are linear. (Use the defenition or the following remark)
   - $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$
   - $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-y+z \\ 2x+y-3z \end{bmatrix}$

2. Give a counterexample to show that the following transformations are not linear.
   - $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x^2 \end{bmatrix}$
   - $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} |x| \\ |y| \end{bmatrix}$
   - $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+1 \\ y-1 \end{bmatrix}$

3. Prove that the following transformations are linear by showing that they are matrix transformations. Find the standard matrix of each transformation. (all from $\mathbb{R}^2$)
   - R rotates a vector 45 degree counterclockwise about the origin.
   - P projects a vector onto the line $x = y$.
   - C converts the cordinate of a vector with respect basis $B_1 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ to the cordinate of same vector with respect to $B_2 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$.

4. Show that $[S]o[T] = [S][T]$ for following transformations.
   - $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$, $S\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2y_1 \\ -y_2 \end{bmatrix}$
   - $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 - x_3 \\ 2x_1 - x_2 + x_3 \end{bmatrix}$, $S\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 - y_2 \\ y_1 + y_2 \\ -y_1 + y_2 \end{bmatrix}$

5. Prove that the range of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is the column space of it's matrix $[T]$.

6. Show that v is an eigenvector of A and find the corresponding eigenvalue.
   - $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $v = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$
   - $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

7. Show that λ is an eigenvalue of A and find a corresponding eigenvector.

- $A = \begin{bmatrix} 0 & 4 \\ -1 & 5 \end{bmatrix}, \lambda = 1$
- $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}, \lambda = -1$

8. Show that the eigenvalues of the upper tirangular matrix $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ are $\lambda = a$ and $\lambda = d$. Find the corresponding eigenspaces.

9. Find all eigenvalues of the matrix A over indicated $\mathbb{Z}_p$ and $\mathbb{R}$.

- $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ over $\mathbb{Z}_3$
- $A = \begin{bmatrix} 1 & 4 \\ 4 & 0 \end{bmatrix}$ over $\mathbb{Z}_5$