Linear Algebra
Third exercise:

1. We know that line ℓ goes through $A = (3, 1, 1)$ with direction vector $\mathbf{d} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. Use another point of the line (not A) to determine the distance between the point $B = (1, 0, 2)$ and the line ℓ.

2. Assume that $P = (x_0, y_0)$ is a point and $ax + by = c$ is the general form of line ℓ in $\mathbb{R}^2$. Prove that the distance between P and the line ℓ is

$$d(P, \ell) = \frac{|ax_0 + by_0 - c|}{\sqrt{a^2 + b^2}}$$

3. Assume that $P = (x_0, y_0, z_0)$ is a point and $ax + by + cz = d$ is the general form of plane $\mathcal{P}$ in $\mathbb{R}^3$. Prove that the distance between P and the plane $\mathcal{P}$ is

$$d(P, \mathcal{P}) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

4. Prove that if a linear equation has more than one solution then it has infinitely many solution. Can we conclude the same for systems of linear equations?

In exercises 5 to 10 find the normal form, general form, vector form and parametric form of lines or planes.

5. Line ℓ goes through $P = (1, 2)$ with normal vector $\mathbf{n} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$.

6. Line ℓ goes through $P = (3, 0, -2)$ with direction vector $\mathbf{d} = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$.

7. Line ℓ goes through $P = (0, 1, -1)$ and $Q = (-2, 1, 3)$.

8. Plane $\mathcal{P}$ goes through $P = (0, 1, 0)$ with normal vector $\mathbf{d} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

9. Plane $\mathcal{P}$ goes through $P = (6, -4, -3)$ with direction vectors $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

10. Plane $\mathcal{P}$ goes through $P = (1, 1, 1)$, $Q = (4, 0, 2)$ and $R = (0, 1, -1)$.

11. Suggest a "vector proof" of the fact that, in $\mathbb{R}^2$, two lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$.

12. A cube has vertices at the eight points (x, y, z), where each of x, y and z is either 0 or 1.

a) Find the general equation of the planes that determine the six faces (sides) of the cube.

b) Find the general equation of the plane that contains the diagonal from the origin to $(1, 1, 1)$ and is perpendicular to the xy-plane.

In exercise 13 and 14 find the distance between point Q to the line ℓ and plane $\mathcal{P}$ respectively.

13. $Q = (2, 2)$ and ℓ has equation $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

14. $Q = (2, 2, 2)$ and $\mathbb{P}$ has equation $x + y - z = 0$

*15. Prove that in $\mathbb{R}^2$ the distance between parallel lines with equations $\mathbf{n}.\mathbf{x} = c_1$ and $\mathbf{n}.\mathbf{x} = c_2$ is given by $\frac{|c_1 - c_2|}{\|\mathbf{n}\|}$.

*16. Prove that the distance between parallel planes with equations $\mathbf{n}.\mathbf{x} = d_1$ and $\mathbf{n}.\mathbf{x} = d_2$ is given by $\frac{|d_1 - d_2|}{\|\mathbf{n}\|}$.