Linear Algebra
Sixth exercise:

1. Prove that if the columns of B are linearly dependent, then so are the columns of AB.
2. Prove that if the rows of A are linearly dependent, then so are the rows of AB.
3. Is the following sentence true? If yes prove it, else justify your answer and give an example.

   "If the columns of AB are linearly dependent, then so are the columns of B."
4. Is the following sentence true? If yes prove it, else justify your answer and give an example.

   "If the columns of B be linearly independent, then so are the columns of AB."

   Note: You can ask the same questions about the rows of A and AB or even go further to rows of A and columns of AB or columns of A and columns/rows of AB.
5. Can you find the requirement (necassary and sufficient condition) for AB to have linearly independent columns? What about rows?

   In exercises 6 to 9 let $A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & -1 & 1 \\ -1 & 6 & 4 \end{bmatrix}$.
6. Compute AB.
7. Write each column of AB as linear combination of columns of B. (hint: Using matrix-column representation will make it easy)
8. Write each row of AB as linear combination of rows of A. (hint: Using row-matrix representation will make it easy)
9. Compute the outer product expansion of AB.
10. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$ and solve the following equations for X.
    - $X - 2A + 3B = O$
    - $2(A - B + X) = 3(X - A)$
11. Write $B = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ as linear combination of $A_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $A_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, if possible.
12. Determine if following matrices are linearly independent.
    - $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$
    - $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$
13. If $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Find conditions on a, b, c, d such that $AB = BA$.
14. Find conditions on a, b, c, d such that $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ commute with both $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.
15. Find conditions on a, b, c, d such that $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ commute with every 2×2 matrix.