Inverse problems course Exercise 6 (Last exercise, February 24–26, 2015)

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Related book sections (Mueller & Siltanen 2012): 5.5 and 6.

Theoretical exercises:

T1. Compute the derivative $f'(t)$ when the function $f : \mathbb{R} \to \mathbb{R}$ is given by

$$\text{(a) } f(t) = \sqrt{t^2 + \beta}, \qquad \text{(b) } f(t) = \frac{1}{\beta}\log(\cosh(\beta t)),$$

where $\beta > 0$.

T2. Let $x \in \mathbb{R}^4$ and $\beta > 0$. Compute the gradient of the approximate total variation penalty functional

$$\Phi_{\text{TV}}(x) = \sum_{k-\ell=1} |x_k - x_\ell|_\beta,$$

where $|t|_\beta = f(t)$ as in exercise T1(a).

T3. Read the Wikipedia article on steepest descent minimization and explain in your own words how it works.

Matlab exercises:

M1. Resolution-based choice of total variation regularization parameter. Download the routine `deconv9_TVreg_comp.m`. Choose some value for the regularization parameter α and compute reconstructions with the same alpha but with three resolutions: $n = 128, 256, 512$. Calculate the TV norm of each of the three reconstructions using the formula

$$|f_2 - f_1| + |f_3 - f_2| + \ldots + |f_n - f_{n-1}| + |f_1 - f_n|.$$

Do you get approximately the same norms? If not, make α a bit bigger and try again. If you get approximately the same results, make α a bit smaller and try again. This way you should be able to find the smallest $\alpha > 0$ that still gives roughly the same TV norm for the reconstruction with different resolutions. Finally, show the reconstruction with the optimal α so determined.

M2. Resolution-based choice of smoothness-penalty regularization parameter. Download the routine `deconv7_genTikhonov_comp.m`. Choose some value for the regularization parameter α and compute reconstructions with the same alpha but with three resolutions: $n = 128, 256, 512$. Calculate the "smoothness norm squared" of each of the three reconstructions using the formula

$$n\left(|f_2 - f_1|^2 + |f_3 - f_2|^2 + \ldots + |f_n - f_{n-1}|^2 + |f_1 - f_n|^2\right).$$

Do you get approximately the same numbers? Repeat the procedure of Problem M1 and see if this works as a regularization parameter choice rule.

To my best knowledge this is the first time ever for anyone to try this. It is quite interesting to see if it works.

M3. Go to this page and download the Matlab routines under the headline *Compute reconstruction using matrix-free iterative (approximate) total variation regularization.* Compute TV regularized one-dimensional deconvolutions by simplifying those files from the 2D tomography case to the 1D deconvolution case. You should use the smooth penalty of Problem T2.

How do the results compare to the reconstructions computed with the quadratic programming approach (`deconv9_TVreg_comp.m`)?