Inverse problems course — Exercise 4 (February 10–12, 2015)

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Related book sections (Mueller & Siltanen 2012): 2.3.6, 3.4, 4.2, 5.1.

Theoretical exercises:

T1. Show that $A^TA + \delta I$ is always an invertible matrix for $\delta > 0$.
Hint: note that A^TA is symmetric and study the eigenvalues of $A^TA + \delta I$.

T2. For any $\alpha > 0$ define the truncated SVD (TSVD) by $A_\alpha^+ = VD_\alpha^+U^T$ where

$$D_\alpha^+ = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 & & \cdots & 0 \\ 0 & 1/d_2 & & & & & \vdots \\ \vdots & & \ddots & & & & \\ & & & 1/d_{r_\alpha} & & & \\ & & & & 0 & & \\ \vdots & & & & & \ddots & \vdots \\ 0 & \cdots & & & & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n\times k}$$

and

$$r_\alpha = \min\Big\{r,\ \max\{j \mid 1 \le j \le \min(k,n),\ d_j > \alpha\}\Big\}. \tag{1}$$

We can then define a reconstruction function $\mathcal{L}_\alpha$ by the formula

$$\mathcal{L}_\alpha(m) = VD_\alpha^+U^Tm. \tag{2}$$

Show that $\mathcal{L}_\alpha$ is a regularization strategy in the sense of Definition 3.4.1 with the choice $\alpha(\delta) = \delta$.

(a) Prove that $\lim_{\alpha\to 0} \mathcal{L}_\alpha(Af) = f$ for any fixed $f \in \mathbb{R}^n$.

(b) Assume given a noisy measurement $m = Ax + \varepsilon$, where we know that $\|\varepsilon\| \le \delta$. Show that for any fixed $f \in \mathbb{R}^n$ we have the worst-case reconstruction error tending to zero in the zero-noise limit:

$$\sup_{m\in\mathbb{R}^k} \{\|\mathcal{L}_{\alpha(\delta)}(m) - f\| : \|Af - m\| \le \delta\} \to 0 \text{ as } \delta \to 0.$$

Matlab exercises:

M1. Use the routine `XR01_buildA.m` to construct a tomographic measurement matrix for resolution $N = 32$ and number of angles $T = 15$. Furthermore, use the routine `XR04_NoCrimeData_comp.m` to create simulated data with no inverse crime.

(a) Implement Tikhonov regularization using the SVD-based definition. You can do this by modifying the file `XR03_truncSVD.m`. Try your method with regularization parameter $\alpha = 1$ and calculate the relative L^2 error in the reconstruction.

(b) Loop over several values of $\alpha > 0$ and compute Tikhonov regularized reconstruction as in (a). Find the value of α giving the smallest error. Hint: it's a good idea to create the parameter range logarithmically for instance like this: `alphavec = 10.^linspace(-3,3,20)`.

(c) Repeat (a) and (b) using $T = 50$ angles. Can you reach a smaller relative error in the reconstruction than in (b)? Why?