# A short introduction to CLASSIFICATION THEORY

Spring 2015

Tapani Hyttinen

## Contents

## 0. Introduction

In the mid 60's, Michael Morley made a number of findings. E.g. he showed that if the theory is $\omega$-stable, then a Cantor-Bendixon rank can be defined for types. This work was continued by Saharon Shelah. During 70's and 80's he created single-handedly a large piece of model theory known as classification theory. The idea behind this work was to determine for which model classes of the form $\{\mathcal{A} | \mathcal{A} \models T\}$, $T$ a complete first-order theory, a structure theorem can be proved. In this paper we try to give a compact introduction to this topic. We concentrate on cases in which $T$ is stable, so a large part of classification theory is left outside the scope of this paper. We also concentrate on ideas and techniques in classification theory, not on results. So our results are not always the best possible.

Unless otherwise stated, all results proved in this paper are from [Sh], but all proofs are not. Some of the proofs are new and also proofs from [HS1], [HS2] and [Hy1] are used. The first version [Hy3] of these notes was written in mid 90's.

To read these notes one needs to know the basic concepts of model theory and how to use them. Also some basic facts from cardinal arithmetics are needed (e.g. $(2^\kappa)^\kappa = 2^\kappa$).

This paper is full of exercises. Usually they are simple but vital parts of the theory, and so they are often used later in the proofs. If an exercise is not needed later in this paper, then it is marked by $^*$. If an exercise is more than just checking definitions, a generous hint is given.

Occasionally we give examples of the concepts we define. The underlying theory in those examples is usually either $T_\omega$ or $T_2$: $T_\alpha = Th((\alpha^\omega, E_n)_{n<\omega})$, where $E_n(\eta, \xi)$ holds if $\eta \restriction (n+1) = \xi \restriction (n+1)$.

Under the name Fact, we give additional information.

Throughout this paper we assume that $T$ is a complete theory in a language $L$ and that $T$ has an infinite model. In order to simplify the notation, we use 'the monster model technique', i.e. we work inside $\mathbf{M}$, where $\mathbf{M} \models T$ is a saturated model of power $\kappa$, and $\kappa$ is assumed to be larger than the cardinality of any object that we come across. So by a model we mean an elementary submodel of $\mathbf{M}$ (of power $< \kappa$). We write $\mathcal{A}$, $\mathcal{B}$ etc. for these. This means e.g. that if $\mathcal{A} \subseteq \mathcal{B}$ then $\mathcal{A} \prec \mathcal{B}$. Similarly by a set we mean a subset of $\mathbf{M}$. We write $A$, $B$ etc. for these. By $a$, $b$ etc. we mean a finite sequence of elements of $\mathbf{M}$. By $a \in A$ we mean $a \in A^{length(a)}$.

If $T$ is stable, then the existence of $\mathbf{M}$ is not a problem (in this paper from Chapter 2 on). Otherwise we have to assume that the inaccessible cardinals form a proper class or use just $\kappa$-saturated strongly $\kappa$-homogeneous monster model. But this assumption is not 'used', it is not hard to see how to modify the definitions and the proofs so that $\mathbf{M}$ is not needed.

Our notation is standard. So e.g. $S^m(A)$ is the set of all complete consistent types over $A$ in $m$ variables (modulo a change of variables). $S(A) = \cup_{m<\omega} S^m(A)$ and by $t(a, A)$ we mean the complete type of $a$ over $A$ (in $\mathbf{M}$). We write $p(x)$

when we want to point out, which are the free variables in the type $p$. $\models \phi$ means $\mathbf{M} \models \phi$ and $\phi(\mathbf{M}, b)$ is the set $\{a \in \mathbf{M} |\ \models \phi(a, b)\}$.

## 1. Stability and ranks

**1.1 Definition.**
*(i) We say that $T$ is $\xi$-stable if for all $A$ of power $\leq \xi$, $|S(A)| \leq \xi$.*
*(ii) We say that $T$ is stable, if for some infinite $\xi$, $T$ is $\xi$-stable.*
*(iii) If $T$ is stable, then by $\lambda(T)$ we mean the least $\lambda$ such that $T$ is $\lambda$-stable.*

**1.2 Exercise.**
*(i)* For all $A$, $|S^1(A)| \geq |A|$.*
*(ii)* Show that the theory of dense linear-orderings without end-points is unstable. (Hint: Choose $\kappa$ so that it is the least cardinal such that $\omega^\kappa > \xi$ and extend the ordering of the tree $\mathbf{Q}^{<\kappa}$ to a linear-order.)*
*(iii)* Show that $T_\omega$ and $T_2$ are stable.*
*(iv) If $T$ is $\xi$-stable and $\xi$ is regular, then for all $A$ of power $\leq \xi$, there exists a saturated model $\mathcal{A}$ of power $\xi$ such that $A \subseteq \mathcal{A}$. (Hint: Choose an increasing continuous sequence $A_i$, $i < \xi$, of sets of power $\xi$ such that every type over $A_i$ is realized in $A_{i+1}$ and $A \subseteq A_0$. Then $\mathcal{A} = \cup_{i<\xi} A_i$ is as wanted.)*

Below, when we write $\phi(x)$, we mean that the free variables of $\phi$ are contained in $x$. When we talk about a formula $\phi$ we assume that $\phi$ is of the form $\phi(x, y)$ and that we always know, which variables belong to the first sequence and which belong to the second. When we talk about $\phi$-types, the variables in $y$ are for parameters, and $x$ remains free. By $\Delta$ we always mean a finite set of formulas and if $\phi(x, y), \psi(x', y') \in \Delta$ then $x = x'$. When we talk about $p \cup \{\phi(x, a)\}$ we of course assume that $x$ is the sequence of free variables of $p$.

We will not do, what we said above, in a precise form; We rely on the common sense of the reader.

**1.3 Definition.** *Let $\Delta$ be a finite set of formulas.*
*(i) Let $A \subseteq B$ and $p \in S(B)$. We say that $p$ $\Delta$-splits over $A$ if there are $a, b \in B$ and $\phi \in \Delta$ such that $t(a, A) = t(b, A)$ and $\phi(x, a), \neg\phi(x, b) \in p$. We write $\phi$-splits instead of $\{\phi\}$-splits.*
*(ii) Let $A \subseteq B$ and $p \in S(B)$. We say that $p$ splits over $A$ if it $\phi$-splits over $A$ for some $\phi$.*
*(iii) We say that $\Delta$ is stable, if there are no $A_i$, $i < \omega$, and $a$ such that for all $i < \omega$, $A_i \subseteq A_{i+1}$ and $t(a, A_{i+1})$ $\Delta$-splits over $A_i$. We say that $\phi$ is stable instead of $\{\phi\}$ is stable. (Notice that this definition differs from the one given in [Sh], but as we shall see, they are equivalent.)*
*(iv) We say that $p$ is an $\Delta$-type if it is a set of formulas of the form $\phi(x, a)$ or $\neg\phi(x, b)$, $a, b \in \mathbf{M}$ and $\phi \in \Delta$. By $t_\Delta(a, A)$ we mean the complete $\Delta$-type of $a$ over $A$. We write $S_\Delta(A)$ for the set of all complete $\Delta$-types over $A$. As above, we write $t_\phi(a, A)$, $S_\phi(A)$ and $\phi$-type instead of $t_{\{\phi\}}(a, A)$, $S_{\{\phi\}}(A)$ and $\{\phi\}$-type.*

**1.4 Exercise.**

*(i) If $\phi$ is not stable, then for all $\kappa$, there are $A_i$, $i<\kappa$, and $a$ such that for all $i<j<\kappa$, $A_i \subseteq A_j$ and $t(a,A_{i+1})$ $\phi$-splits over $A_i$. (Hint: Use compactness.)*

*(ii) If every formula is stable, then every finite $\Delta$ is stable.*

**1.5 Lemma.** *If $\phi$ is not stable, then for all infinite $\xi$, there is $A$ of power $\le \xi$ such that $|S_\phi(A)|>\xi$ and so $T$ is not stable.*

**Proof.** Let $\kappa$ be the least cardinal such that $2^\kappa>\xi$. Then $\kappa\le\xi$. By Exercise 1.4, we can find $a$, $a_i$ and $b_i$, $i<\kappa$, such that for all $i<\kappa$, $t(a_i,\cup_{j<i}(a_j\cup b_j)) = t(b_i,\cup_{j<i}(a_j\cup b_j))$ and $\models \phi(a,a_i)\wedge\neg\phi(a,b_i)$.

By induction on $i\le\kappa$ we define automorphisms $f_{\eta\restriction i}$ of $\mathbf{M}$, $\eta\in 2^\kappa$, as follows:

(i) $f_{\eta\restriction 0}=id_{\mathbf{M}}$,

(ii) $f_{\eta\restriction(i+1)}=f_{\eta\restriction i}$ if $\eta(i)=0$ and otherwise $f_{\eta\restriction(i+1)}$ is any automorphism of $\mathbf{M}$ such that $f_{\eta\restriction(i+1)}(a_i)=f_{\eta\restriction i}(b_i)$ (or $f_{\eta\restriction(i+1)}(b_i)=f_{\eta\restriction i}(a_i)$) and for all $j<i$, $f_{\eta\restriction(i+1)}(a_j)=f_{\eta\restriction i}(a_j)$, $f_{\eta\restriction(i+1)}(b_j)=f_{\eta\restriction i}(b_j)$,

(iii) if $i$ is limit, then $f_{\eta\restriction i}$ is any automorphism of $\mathbf{M}$ such that for all $j<i$, $f_{\eta\restriction i}(a_j)=f_{\eta\restriction(j+1)}(a_j)$ and $f_{\eta\restriction i}(b_j)=f_{\eta\restriction(j+1)}(b_j)$.

Let $A=\bigcup_{i<\kappa}\cup\{f_{\eta\restriction i}(\cup_{j<i}(a_j\cup b_j)|\ \eta\in 2^\kappa\}$ and for all $\eta\in 2^\kappa$, we let $p_\eta = t_\phi(f_\eta(a),A)$. Then $|A|=2^{<\kappa}$ and by (ii) above, if $\eta\ne\eta'$, then $p_\eta$ and $p_{\eta'}$ are contradictory. By the choice of $\kappa$, $A$ is as wanted. □

**1.6 Exercise.** *If $T$ is $\xi$-stable and $2^\kappa>\xi$, then there are no $A_i$, $i<\kappa$, and $a$ such that for all $i<j<\kappa$, $A_i\subseteq A_j$ and $t(a,A_{i+1})$ splits over $A_i$. (Hint: The proof of Lemma 1.5 works also here.)*

We say that a type $p$ over $A$ $(\Delta,\phi)$-splits over $B\subseteq A$, if there are $a,b\in A$ such that $t_\Delta(a,B)=t_\Delta(b,B)$, $\phi(x,a)\in p$ and $\neg\phi(x,b)\in p$.

**1.7 Lemma.** *If $\phi$ is stable, then for all infinite $A$, $|S_\phi(A)|\le|A|$.*

**Proof.** Let $c$, $c_i$ and $d_i$, $i<\omega$, be sequences of new constants and $C_i = \cup_{j<i}(c_j\cup d_j)$. Since $\phi$ is stable, there are finite $\Delta$ and $n$ such that the following set is not consistent

$$\{\phi(c,c_i)\wedge\neg\phi(c,d_i)|\ i<n\}\cup\{\psi(c_i,d)\leftrightarrow\psi(d_i,d)|\ i<n,\ d\in C_i,\ \psi\in\Delta\}.$$

But then for all $A$ and $p\in S_\phi(A)$, we can find a finite $B\subseteq A$ such that $p$ does not $(\Delta,\phi)$-split over $B$. Since $B$ and $\Delta$ are finite, $S_\Delta(B)$ is finite and so also

$$\{q\in S_\phi(A)|\ p\restriction B\subseteq q,\ q \text{ does not } (\Delta,\phi)\text{-split over } B\}$$

is finite. Because the number of finite subsets of $A$ is $\le|A|$, the claim follows. □

**1.8 Definition.** *For every finite set $\Delta$ of formulas and cardinal $\xi$ (not necessarily infinite), we define $R_\Delta(p,\xi)$, for all types $p$, in the following way:*

*(i) $R_\Delta(p,\xi)\ge 0$ if $p$ is consistent.*

*(ii)* $R_\Delta(p,\xi) \geq \alpha+1$ *if for all finite* $q \subseteq p$ *and* $\gamma < \xi$ *there are* $\Delta$*-types* $q_i$, $i \leq \gamma$, *such that*

*(a) for all* $i < j \leq \gamma$ *there are* $\phi(x,y) \in \Delta$ *and* $a$ *such that* $\phi(x,a) \in q_i$ *and* $\neg\phi(x,a) \in q_j$ *or vice versa (in this case we say that* $q_i$ *and* $q_j$ *are* $\Delta$*-contradictory),*

*(b) for all* $i \leq \gamma$, $R_\Delta(q \cup q_i, \xi) \geq \alpha$.

*(iii) If* $\alpha$ *is limit, then* $R_\Delta(p,\xi) \geq \alpha$ *if* $R_\Delta(p,\xi) \geq \beta$ *for all* $\beta < \alpha$.

*We say that* $R_\Delta(p,\xi) = \alpha$ *if* $\alpha$ *is the least ordinal such that* $R_\Delta(p,\xi) \not\geq \alpha+1$. *If such* $\alpha$ *does not exist, then we write* $R_\Delta(p,\xi) = \infty$. *We write* $R_\Delta(p,\xi) = -1$ *if* $p$ *is not consistent and* $R_\phi$ *for* $R_{\{\phi\}}$.

**1.9 Exercise.**

*(i) If* $R_\Delta(p,\xi) = \infty$, *then* $R_\Delta(p,\xi) \geq \alpha$, *for all ordinals* $\alpha$.

*(ii) If* $p \vdash q$, *then* $R_\Delta(p,\xi) \leq R_\Delta(q,\xi)$.

*(iii) If* $R_\Delta(p,\xi) \geq \alpha$ *and* $\beta < \alpha$, *then* $R_\Delta(p,\xi) \geq \beta$.

*(iv) If* $\xi \geq \xi'$ *then* $R_\Delta(p,\xi) \leq R_\Delta(p,\xi')$.

*(v)* $R_\Delta(p,\xi) = min\{R_\Delta(q,\xi) |\ q \subseteq p \text{ finite}\}$.

*(vi) If* $p$ *is algebraic, then* $R_\Delta(p,\omega) = 0$.

*(vii) If* $x = y \in \Delta$ *and* $R_\Delta(p,\omega) = 0$, *then* $p$ *is algebraic.*

**1.10 Lemma.** *Let* $\xi > 1$ *be a cardinal and* $\Delta$ *a finite set of formulas.*

*(i) There is* $\alpha$ *such that for all finite* $p$, $R_\Delta(p,\xi) \geq \alpha$ *implies* $R_\Delta(p,\xi) = \infty$.

*(ii) If* $R_\Delta(p,\xi) = \infty$ *and* $p$ *is finite then there are finite* $p_1$ *and* $p_2$ *such that* $p \subseteq p_1 \cap p_2$, *for some* $d$ *and* $\phi \in \Delta$, $\phi(x,d) \in p_1$, $\neg\phi(x,d) \in p_2$ *and* $R_\Delta(p_1,\xi) = R_\Delta(p_2,\xi) = \infty$.

*(iii) If for all infinite* $A$, $|S_\Delta(A)| \leq |A|$, *then for all* $p$, $R_\Delta(p,\xi) < \infty$.

**Proof**. (i) follows immediately from the fact that the number of $t(A,\emptyset)$ for finite $A$, and the number of finite $p$ over a finite $A$ are restricted.

(ii) Immediate by (i) and the definition of $R_\Delta$.

(iii) By Exercise 1.9 (v), it is enough to prove this for finite $p$. But this follows immediately from (ii). □

**1.11 Exercise.** *Let* $\Delta$ *be a finite set of formulas.*

*(i) For all finite types* $p$, $a \in \mathbf{M}$ *and* $\phi \in \Delta$, *if* $R_\Delta(p,2) < \infty$, *then either* $R_\Delta(p \cup \{\phi(x,a)\}, 2) < R_\Delta(p,2)$ *or* $R_\Delta(p \cup \{\neg\phi(x,a)\}, 2) < R_\Delta(p,2)$.

*(ii) Assume* $p \subseteq q \cap r$, $q, r \in S_\Delta(A)$ *and* $A$ *is finite. If* $R_\Delta(q,2) = R_\Delta(r,2) = R_\Delta(p,2) < \infty$, *then* $q = r$.

We write $|T|$ for the number of $L$-formulas modulo the equivalence $T \vdash \forall x(\phi(x) \leftrightarrow \psi(x))$.

**1.12 Theorem.** *The following are equivalent:*

*(i)* $T$ *is stable.*

*(ii) Every formula is stable.*

*(iii) Every finite* $\Delta$ *is stable.*

*(iv) For every* $\phi$ *and infinite* $A$, $|S_\phi(A)| \leq |A|$.

*(v) For every finite $\Delta$ and infinite $A$, $|S_\Delta(A)| \le |A|$.*
*(vi) For every finite $\Delta$, cardinal $\xi > 1$ and type $p$, $R_\Delta(p,\xi) < \infty$*
*(vii) $T$ is $\xi$-stable for all $\xi$ such that $\xi^{|T|} = \xi$.*

**Proof**. (i)$\Rightarrow$(ii): This is Lemma 1.5.
(ii)$\Rightarrow$(iii): This is Exercise 1.4 (ii).
(iii)$\Rightarrow$(iv): This follows from Lemma 1.7.
(iv)$\Rightarrow$(v): Every type $p \in S_\Delta(A)$ is determined by the sequence $(p \restriction \phi)_{\phi\in\Delta}$, from which the claim follows.
(v)$\Rightarrow$(vi): This is Lemma 1.10 (iii).
(vi)$\Rightarrow$(v): Let $p \in S_\Delta(A)$. By Exercise 1.9 (v), choose finite $B \subseteq A$ such that

$$(*) \qquad R_\Delta(p \restriction B, 2) = R_\Delta(p, 2).$$

By Exercise 1.11 (ii), $p$ is determined by $p \restriction B$ and (*). Since for finite $B$, $S_\Delta(B)$ is finite and the number of finite subsets of $A$ is $|A|$, $|S_\Delta(A)| \le \omega \times |A| = |A|$.
(v)$\Rightarrow$(vii): Assume $|A| = \xi$ and $\xi^{|T|} = \xi$. Every type $p \in S(A)$ is determined by the sequence $(p \restriction \phi)_{\phi\in L}$. So $|S(A)| \le |\prod_{\phi\in L} S_\phi(A)| = |A|^{|T|} = \xi$.
(vii)$\Rightarrow$(i): Trivial. □

**1.13 Exercise.** *If $T$ is stable, then for every cardinal $\xi > 1$, finite $\Delta$ and type $p$, $R_\Delta(p,\xi) < \omega$. (Hint: By Exercise 1.9 (iv), it is enough to prove the claim for $\xi = 2$. For a contradiction, assume that the claim does not hold for $\xi = 2$ and use compactness to show that the following set of formulas is consistent ($c_\eta$ and $d_i$ are sequences of new constants):*

$$\{\neg \bigwedge_{\phi\in\Delta,\ d\subseteq d_i} (\phi(c_\eta, d) \leftrightarrow \phi(c_{\eta'}, d)) |\ \eta,\ \eta' \in 2^\omega,\ \eta \restriction i = \eta' \restriction i,\ \eta(i) \ne \eta'(i)\}.)$$

**1.14 Fact.** *([Sh]) If $T$ is not stable, then there is $\phi(x,y)$ such that for all linear-orderings $\eta$ there are $a_i \in \mathbf{M}$, $i \in \eta$, such that $\models \phi(a_i, a_j)$ iff $i < j$. (Notice that by the proof of Exercise 1.2 (ii), this $\phi$ is not stable.)*

We say that $p$ and $q$ are $\Delta$-contradictory if there are $\phi \in \Delta$ and $a$ such that $\phi(x,a) \in p$ and $\neg\phi(x,a) \in q$ or vice versa.

**1.15 Theorem.** *Assume $T$ is stable. Then*

$$R_\Delta(p \cup \{\vee_{i<n}\phi_i\}, \omega) = max_{i<n} R_\Delta(p \cup \{\phi_i\}, \omega).$$

**Proof**. By Exercise 1.9 (ii), it is enough to show that for all $p$, $R_\Delta(p\cup\{\vee_{i<n}\phi_i\}, \omega) \ge \alpha$ implies $max_{i<n} R_\Delta(p \cup \{\phi_i\}, \omega) \ge \alpha$. We prove this by induction on $\alpha$. The cases $\alpha = 0$ and $\alpha$ is limit, are trivial.

We prove the case $\alpha = \beta + 1$: For a contradiction, assume that for all $i < n$, there are a finite $p_i \subseteq p$ and $n_i < \omega$, which satisfy the following: there are no pairwise $\Delta$-contradictory $q^i_j$, $j < n_i$, such that $p_i \subseteq q^i_j$ and $R_\Delta(q^i_j \cup \{\phi_i\}, \omega) \ge \beta$.

Let $p^* = \cup_{i<n} p_i$ and $n^* = n \cdot (max_{i<n} n_i)$. Then $p^* \cup \{\vee_{i<n}\phi_i\} \subseteq p \cup \{\vee_{i<n}\phi_i\}$ is finite and there are no pairwise $\Delta$-contradictory $q_i$, $i < n^*$ such that for all $i < n^*$, $p^* \subseteq q_i$ and for all $i < n^*$, there exists $j < n$, such that $R_\Delta(q_i \cup \{\phi_j\}, \omega) \geq \beta$ (i.e. if $q_i$, $i < n^*$, are $\Delta$-contradictory and $p^* \subseteq q_i$, then for some $i < n^*$, $max_{j<n} R_\Delta(q_i \cup \{\phi_j\}, \omega) < \beta$). By the induction assumption there are no pairwise $\Delta$-contradictory $q_i$, $i < n^*$ such that $p^* \subseteq q_i$ and $R_\Delta(q_i \cup \{\vee_{j<n}\phi_j\}, \omega) \geq \beta$. So $R_\Delta(p \cup \{\vee_{i<n}\phi_i\}, \omega) \not\geq \alpha$, a contradiction. □

**1.16 Exercise.** *Assume $T$ is stable. If $p$ is over $A$ and $R_\Delta(p, \omega) = \alpha$, then there is $q \in S(A)$ such that $p \subseteq q$ and $R_\Delta(q, \omega) = \alpha$. (Hint: By Theorem 1.15, show that*

$$\{\neg\phi(x, a) |\ a \in A,\ R_\Delta(p \cup \{\phi(x, a)\}, \omega) < \alpha\}$$

*is consistent.)*

## PART I: INDEPENDENCE

Forking was invented by S. Shelah in the mid 70's. Since then, the use of this concept has dominated research in model theory. In this part we prove the basic properties of forking in a compact style. We follow the approach of [Sh], so we do not try to find the simplest way to see the basic properties of forking. The reason for this is that the author of this paper believes, that it is important to know the relations between indiscernible sets and ranks, forking, and finite equivalence relations. In details we do not necessarily follow [Sh], e.g. our definition of forking differs from the one given in [Sh]. For other approaches to forking, see [Ba], [Bu], [La] and/or [Pi].

### 2. Forking

From now on in this paper we assume that $T$ is stable.

**2.1 Definition.**

*(i) We say that a consistent formula $\phi(x,m)$, $m \in \mathbf{M}$, forks over $A$ if for all $p = p(x) \in S(A)$ the following holds: If $p \cup \{\phi(x,m)\}$ is consistent, then there is a finite $\Delta$ such that for all finite $\Delta' \supseteq \Delta$, $R_{\Delta'}(p \cup \{\phi(x,m)\}, \omega) < R_{\Delta'}(p,\omega)$. (Notice that this definition differs from the one given in [Sh], but, as we shall see, they are equivalent.)*

*(ii) We say that $p$ forks over $A$ if there is a finite $q \subseteq p$ such that $\wedge q$ forks over $A$.*

*(iii) We write $a \downarrow_A B$ if $t(a, A \cup B)$ does not fork over $A$.*

Below we give examples of forking. We delay, until Exercise 5.12, the proof that the claims in the example are actually true. (The reader may try to prove this straight from the definition. It is of course possible, but needs a bit work.)

**2.2 Example.**

*(i) Assume $T = T_\omega$. Let $a$ be a singleton. Then $t(a,B)$ forks over $A \subseteq B$ iff $a \in B - A$ or there are $n < \omega$ and $b \in B$ such that $\models E_n(a,b)$ but for all $c \in A$ $\models \neg E_n(a,c)$.*

*(ii) Assume $T = T_2$. Let $a$ be a singleton. Then $t(a,B)$ forks over $A \subseteq B$ iff $a \in B - A$.*

**2.3 Exercise.**

*(i) If $p$ is a consistent type over $A$ then $p$ does not fork over $A$.*

*(ii) If $p \in S(B)$ forks over $A \subseteq B$, then there is $\phi(x,b) \in p$ such that $\phi$ forks over $A$, especially if $a \not\downarrow_A B$ then there is finite $B' \subseteq B$ such that $a \not\downarrow_A B'$.*

*(iii) If $t(a,A)$ is algebraic, then $a \downarrow_A B$ for all $B$. (Hint: Use Exercise 1.9 (vi).)*

**2.4 Lemma.** *Assume $A \subseteq B$, $t(a,B)$ is algebraic but $t(a,A)$ is not algebraic. Then $a \not\downarrow_A B$.*

**Proof**. Choose $\phi(x,b) \in t(a,B)$ such that $\phi(x,b)$ is algebraic. Since $t(a,A)$ is not algebraic and $\phi(\mathbf{M},b)$ is finite, there is $\psi(x,c) \in t(a,A)$ such that for all $a'$, if $\models \phi(a',b) \wedge \psi(a',c)$, then $t(a',A)$ is not algebraic. By Exercise 1.9 (vi) and (vii), $\phi(x,b) \wedge \psi(x,c)$ forks over $A$. □

**2.5 Lemma.** *If $\phi_i$, $i < n$, fork over $A$ and $p \vdash \vee_{i<n}\phi_i$, then $p$ forks over $A$.*

**Proof**. Clearly we may assume that $p$ is finite. Let $q \in S(A)$ be such that $q \cup p$ is consistent. Let $I \subseteq n$ be such that $I \neq \emptyset$, $q \cup p \vdash \vee_{i\in I}\phi_i$ and for all $i \in I$, $q \cup p \cup \{\phi_i\}$ is consistent (as an exercise, prove the existence of $I$). Then for all $i \in I$ there is a finite $\Delta_i$ such that for all finite $\Delta' \supseteq \Delta_i$, $R_{\Delta'}(q \cup \{\phi_i\}, \omega) < R_{\Delta'}(q,\omega)$. Let $\Delta = \cup_{i\in I}\Delta_i$. Then for all $i \in I$ and finite $\Delta' \supseteq \Delta$, $R_{\Delta'}(q \cup p \cup \{\phi_i\}, \omega) < R_{\Delta'}(q,\omega)$. By Theorem 1.15, $R_{\Delta'}(q \cup p \cup \{\vee_{i\in I}\phi_i\}, \omega) < R_{\Delta'}(q,\omega)$. Since $q \cup p \vdash \vee_{i\in I}\phi_i(x,m_i)$, Exercise 1.9 (ii) implies that $R_{\Delta'}(q \cup p, \omega) < R_{\Delta'}(q,\omega)$. □

Notice that from Lemma 2.5 it follows that if $q \vdash p$ and $p$ forks over $A$, then $q$ forks over $A$.

**2.6 Lemma.** *If $p$ is over $B$ and does not fork over $A \subseteq B$, then there is $q \in S(B)$ such that $p \subseteq q$ and $q$ does not fork over $A$.*

**Proof**. By Exercise 2.3 (ii), it is enough to show that the type $p \cup q$ is consistent, where $q = \{\neg\phi(x,b) |\ b \in B,\ \phi(x,b)$ *forks over* $A\}$. If $p \cup q$ is not consistent then there are $\neg\phi_i(x,b_i) \in q$, $i < n$, such that $p \vdash \vee_{i<n}\phi_i(x,b_i)$. By Lemma 2.5, this implies that $p$ forks over $A$, a contradiction. □

Before we can prove further properties of forking, we have to study indiscernible sets and finite equivalence relations.

## 3. Indiscernible sets

The following fact may help understanding this section. (As an exercise, prove this fact after reading this Part I.) Assume $\models \phi(a,b)$ and $t(b,A)$ is not algebraic. If we want to test whether $\phi(x,b)$ forks over $A$ or not, then we can do the following: Choose $I = \{b_i |\ i < \omega\}$, so that $\{b\} \cup I$ is indiscernible over $A$ (see the definition below) and for all $i < \omega$, $b_i \downarrow_A b \cup \bigcup_{j<i} b_j$. If $|\{c \in \{b\} \cup I |\ \models \phi(a,c)\}| = \omega$ (i.e. $\phi(a,y) \in Av(I, A \cup a)$), then $\phi(x,b)$ does not fork over $A$.

**3.1 Definition.** *Assume $I$ is a set of finite sequences. We say that $I$ is indiscernible over $A$ if for all $a_k, b_k \in I$, $k < n$, $a \in A$ and $\phi(x_0,...,x_{n-1},y)$ the following holds: If for all $k < k' < n$, $a_k \neq a_{k'}$ and $b_k \neq b_{k'}$, then*

$$\models \phi(a_0,...,a_{n-1},a) \leftrightarrow \phi(b_0,...,b_{n-1},a).$$

*We say that $I$ is indiscernible if it is indiscernible over $\emptyset$.*

**3.2 Exercise.**

*(i) If $I$ is infinite indiscernible over $A$ then for all $\xi$ there is $J$ such that $|J| = \xi$ and $I \cup J$ is indiscernible over $A$.*

*(ii) Let $I = (I, <)$ be a linearly ordered set. We say that $\{b_i |\ i \in I\}$ is order indiscernible over $A$ if for all $i_k, j_k \in I$, $k < n$, $a \in A$ and $\phi(x_0, ..., x_{n-1}, y)$ the following holds: If for all $k < k' < n$, $i_k < i_{k'}$ and $j_k < j_{k'}$, then*

$$\models \phi(b_{i_0}, ..., b_{i_{n-1}}, a) \leftrightarrow \phi(b_{j_0}, ..., b_{j_{n-1}}, a).$$

*Show that if $I$ is infinite and $\{b_i |\ i \in I\}$ is order indiscernible over $A$, then it is indiscernible over $A$. (Hint: Clearly we may assume that if $i, j \in I$ and $i \neq j$ then $b_m \neq b_n$ (otherwise $\{b_i |\ i \in I\}$ is a singleton) and that $A$ is finite. For a contradiction assume that the claim does not hold. Show that we may assume that $I = (\mathbf{R}, <)$ and find $\phi$, $a \in A$, $n, m$ and $k$ such that for all $i_0 < ... < i_n$ from $\mathbf{R}$, $\models \phi(b_{i_0}, ..., b_{i_n}, a)$ but if $i_k < i^* < i_{k+1}$, then*

$$\models \neg\phi(b_{i_0}, ..., b_{i_{m-1}}, b_{i^*}, b_{i_{m+1}}, ..., b_{i_n}, a).$$

*Let $B = A \cup \{b_i |\ i \in \mathbf{Q}\}$ and for every irrational $r$, let $p_r = t_\phi(b_r, B)$. Finally show that if $r \neq r'$, then $p_r \neq p_{r'}$.)*

*(iii) Assume that $\{b_i |\ i < \omega\}$ and $A$ are such that for all $j < i < \omega$ $t(b_i, A \cup \bigcup_{k<j} b_k) = t(b_j, A \cup \bigcup_{k<j} b_k)$ and $t(b_i, A \cup \bigcup_{j<i} b_j)$ does not split over $A$. Then $\{b_i |\ i < \omega\}$ is indiscernible over $A$.*

**3.3 Theorem.** *If $T$ is $\xi$-stable, $|A| \leq \xi$ and $I$ has power $> \xi$, then there is $J \subseteq I$ of power $> \xi$ such that $J$ is indiscernible over $A$.*

**Proof**. We show first:

**Claim.** There are $B$, $C$ and $p \in S(C)$ such that

(i) $A \subseteq B \subseteq C$ and $|C| \leq \xi$,

(ii) for all $C' \supseteq C$ of power $\xi$, there is $b \in I$ such that $t(b, C') \supseteq p$, $b \notin C'$ and $t(b, C')$ does not split over $B$,

(iii) for all $c$ there is $c' \in C$ such that $t(c', B) = t(c, B)$.

**Proof**. Assume not. Then by induction on $i \leq \xi$, we define $B_i$ of power $\leq \xi$ the following way: $B_0 = A$ and for limit $i$, $B_i = \cup_{j<i} B_i$. Assume $B_i$ is defined. Let $C_i \supseteq B_i$ be such that for all $c$ there is $c' \in C_i$ such that $t(c', B_i) = t(c, B_i)$ and $|C_i| \leq \xi$. Let $p \in S(C_i)$. Since (ii) above does not hold for $B_i$, $C_i$ and $p$, there is $C_p \supseteq C_i$ of power $\xi$ such that

(*) for every $b \in I$, if $b \notin C_p$ and $t(b, C_p) \supseteq p$, then $t(b, C_p)$ splits over $B$.

Let $B_{i+1} = \bigcup_{p \in S(B_i)} C_p$.

Choose $b \in I$ so that $b \notin C_\xi$. Then by (*), $t(b, B_{i+1})$ splits over $B_i$ for all $i < \xi$ (choose $p = t(b, C_i)$). This contradicts Exercise 1.6. □ Claim.

Let $B$, $C$ and $p$ be as in the claim. For $i < \xi^+$ we define $J_i$ as follows: $J_0 = \emptyset$ and for limit $i$, $J_i = \cup_{j<i} J_j$. Assume $J_i$ is defined. Then by (ii) in the claim, we can find $b \in I$ such that $b \notin C \cup J_i$ and $t(b, C \cup J_i) \supseteq p$ does not split over $B$. Let $J_{i+1} = J_i \cup \{b\}$. By (iii) in the claim and Exercise 3.2 (ii) and (iii), it is easy to see that $J = \cup_{i<\xi^+} J_i$ is as wanted (exercise). □

**3.4 Exercise$^*$.** *Prove so called $\Delta$-lemma: If $A_i$, $i \in I$, are finite sets and $\{A_i | \ i \in I\}$ is uncountable then there are uncountable $J \subseteq I$ and $B$ such that for all $i, j \in J$, if $i \neq j$ then $A_i \cap A_j = B$. (Hint: the theory of an infinite set is $\omega$-stable.)*

**3.5 Exercise.** *For all $\phi(x, y)$ there is $n < \omega$ such that for all indiscernible $I$ and $a$ either*

$$|\{b \in I| \ \models \phi(b, a)\}| < n$$

*or*

$$|\{b \in I| \ \models \neg\phi(b, a)\}| < n.$$

*(Hint: If not, then by compactness find indiscernible $I$ and $a$ such that $|\{b \in I| \ \models \phi(b, a)\}| = |\{b \in I| \ \models \neg\phi(b, a)\}| = \omega$, and show that this implies that for every infinite $\xi$ there is $B$ such that $|B| = \xi$ and $|S_\phi(B)| = 2^\xi$.)*

**3.6 Definition.** *Let $I$ be an infinite indiscernible set. We define $Av(I, A)$, the average type of $I$ over $A$, to be the set*

$$\{\phi(x, a)| \ a \in A, \ \phi \in L, \ |\{b \in I| \ \models \phi(b, a)\}| \geq \omega\}.$$

**3.7 Exercise.**

*(i) If $I$ is an infinite indiscernible set, then $Av(I, A)$ is consistent for all $A$.*

*(ii) Assume $I$ is an infinite indiscernible set over $A$ and $a \notin I$. Then $I \cup \{a\}$ is indiscernible over $A$ iff $t(a, I \cup A) = Av(I, I \cup A)$.*

*(iii) Assume $I$ and $J$ are infinite and $I \cup J$ is indiscernible. Then for all $A$, $Av(I, A) = Av(J, A)$.*

**3.8 Definition.** *Let $I$ be an infinite indiscernible set over $A$. We say that $I$ is based on $A$, if for all $B \supseteq A$, $Av(I, B)$ does not fork over $A$.*

The fact in the beginning of this section, may clarify the idea behind Definition 3.8, see also the proof of Theorem 3.9.

**3.9 Theorem.** *Assume $A \subseteq B$ and $p \in S(B)$ is non-algebraic and does not fork over $A$. Then there is an infinite indiscernible set $I$ based on $A$ such that for all $b \in I$, $t(b, B) = p$.*

**Proof**. Let $\xi > |B| + \omega$ such that $\xi^{|T|} = \xi$. Then by Theorem 1.12, $T$ is $\xi$-stable and $\xi^+$-stable. Let $\mathcal{A} \supseteq B$ be a saturated model of power $\xi^+$. Let $A_i$, $i < \xi^+$, be an increasing continuous sequence of sets of power $\xi$, such that $B \subseteq A_0$ and $\cup_{i<\xi^+} A_i = \mathcal{A}$. For all $i < \xi^+$, choose $a_i \in \mathcal{A}$ so that $t(a_i, B) = p$ and $t(a_i, A_i \cup \bigcup_{j<i} a_j)$ does not fork over $A$. By Lemma 2.4, if $i \neq j$, then $a_i \neq a_j$. So by Theorem 3.3, we may assume that $\{a_i | \ i < \xi^+\}$ is indiscernible over $A$.

We show that $I = \{a_i | \ i < \omega\}$ is as wanted. By Lemma 2.4, $I$ is infinite. So it is enough to show that it is based on $A$. For this let $C \supseteq A$. Clearly we may assume that $C - A$ is finite and so we may assume also that for some $i^* < \xi^+$, $C \subseteq A_{i^*}$. By Theorem 3.3, choose $i_n > i^*$, $n \leq \omega$, such that $\{a_{i_n} | \ n \leq \omega\}$ is indiscernible over $C$. Let $J = \{a_{i_n} | \ n < \omega\}$. Then $a_{i_\omega} \downarrow_A C$ and by Exercise 3.7,

$$t(a_{i_\omega}, C) = Av(J, C) = Av(I, C).$$

□

**3.10 Definition.** *Assume* $A \subseteq B$ *and* $p \in S(B)$. *We say that* $p$ *strongly splits over* $A$, *if there are* $b_i \in B$, $i < \omega$, *such that* $\{b_i| \ i < \omega\}$ *is an infinite indiscernible set over* $A$ *and for some* $\phi$, $\phi(x, b_0), \neg\phi(x, b_1) \in p$.

**3.11 Lemma.** *Assume* $A \subseteq B$ *and* $p \in S(B)$. *If* $p$ *strongly splits over* $A$, *then* $p$ *forks over* $A$.

**Proof**. Let $\phi$ and $b_i$, $i < \omega$, be as in the definition of strong splitting. Let $n$ be the number given by Exercise 3.5 for $\phi$ and let

$$\psi(x, y_0, ..., y_n) = \phi(x, y_0) \wedge \bigwedge_{0<i\leq n} \neg\phi(x, y_i).$$

Without loss of generality we may assume $\psi(x, b_0, ..., b_n) \in p$.

We show that $\psi(x, b_0, ..., b_n)$ forks over $A$. For this let $q \in S(A)$ be such that $q \cup \{\psi\}$ is consistent. For a contradiction, assume that there is finite $\Delta$ such that $\phi \in \Delta$ and $R_\Delta(q \cup \{\psi\}, \omega) = R_\Delta(q, \omega) = \alpha$. By Exercise 1.16, for $i < \omega$, there are types $q_i \in S(A \cup \{b_i| \ i < \omega\})$ such that $q \subseteq q_i$, $R_\Delta(q_i, \omega) = \alpha$ and $\psi(x, b_{i\cdot(n+1)}, ..., b_{i\cdot(n+1)+n}) \in q_i$. By the choice of $n$, there is infinite $I \subseteq \omega$ such that $q_i \restriction \phi$, $i \in I$, are pairwise contradictory. But then $R_\Delta(q, \omega) \geq \alpha + 1$, a contradiction. □

**3.12 Lemma.** *Assume* $A \subseteq \mathcal{B} \subseteq C$, $\xi = (|A| + 2)^{|T|}$ *and* $\mathcal{B}$ *is* $\xi^+$*-saturated. If* $a \downarrow_A C$, $b \downarrow_A C$ *and* $t(a, \mathcal{B}) = t(b, \mathcal{B})$, *then* $t(a, C) = t(b, C)$.

**Proof**. Assume not. Choose $\phi(x, c)$, $c \in C$, so that $\models \phi(a, c) \wedge \neg\phi(b, c)$. By Exercise 1.6, choose $A' \supseteq A$ such that $A' \subseteq \mathcal{B}$, $|A'| \leq \xi$ and $t(c, \mathcal{B})$ does not split over $A'$. For all $i < \omega$, choose $c_i \in \mathcal{B}$ so that $t(c_i, A' \cup \bigcup_{j<i} c_j) = t(c, A' \cup \bigcup_{j<i} c_j)$. By Exercise 3.2 (iii), $\{c\} \cup \{c_i| \ i < \omega\}$ is indiscernible over $A'$ and so also over $A$. But then either $t(a, C)$ or $t(b, C)$ splits strongly over $A$. By Lemma 3.11, either $t(a, C)$ or $t(b, C)$ forks over $A$, a contradiction. □

**3.13 Exercise.** *For all* $A \subseteq B$, *the set* $\{t(a, B)| \ a \in \mathbf{M}, \ a \downarrow_A B\}$ *has power* $\leq ((|A| + 2)^{|T|})^+$.

## 4. Finite equivalence relations

We write $Aut(A)$ for the set of all automorphisms of $\mathbf{M}$, which fixes $A$ pointwise.

**4.1 Definition.**

*(i) We say that a relation* $R(x)$ *of* $\mathbf{M}$ *is over* $A$ *if it is definable by some formula* $\phi(x, a)$, $a \in A$.

*(ii) We say that* $\phi(x, b)$ *is almost over* $A$ *if the set* $\{\phi(\mathbf{M}, f(b))| \ f \in Aut(A)\}$ *is finite. We say that* $p$ *is almost over* $A$, *if every formula* $\phi \in p$ *is almost over* $A$.

*(iii) We say that an equivalence relation* $E(x, y)$ *in* $\mathbf{M}$ *is finite, if the number of equivalence classes is finite. We write* $FE(A)$ *for the set of all finite equivalence relation over* $A$.

**4.2 Exercise.**

*(i)$^*$: $R = \phi(\mathbf{M}, b)$ is over $A$ iff $\{\phi(\mathbf{M}, f(b)) |\ f \in Aut(A)\}$ is a singleton. (Hint for $\Leftarrow$: First show that $\models \phi(a, b)$ iff for all $c$ such that $t(c, A) = t(b, A)$, $\models \phi(a, c)$. Then use compactness.)*

*(ii) If $E \in FE(A)$, then for all $a$, $E(x, a)$ is almost over $A$.*

**4.3 Lemma.** *$\phi(x, b)$ is almost over $A$ iff there is $E(x, y) \in FE(A)$ such that*

$$\forall x, y(E(x, y) \to (\phi(x, b) \leftrightarrow \phi(y, b))).$$

*(In this case we say that $\phi(x, b)$ depends on $E$.)*

**Proof**. $\Leftarrow$: Clearly if $t(c, A) = t(b, A)$, then $\phi(x, c)$ depends on $E$. So the cardinality of $\{\phi(\mathbf{M}, f(b)) |\ f \in Aut(A)\}$ is at most $2^n$, where $n$ is the number of equivalence classes of $E$.

$\Rightarrow$: Now there is $n < \omega$, such that the set

$$\{\theta(y_i, a) |\ \theta(y_i, a) \in t(b, A),\ i < n\} \cup \{\neg\forall x(\phi(x, y_i) \leftrightarrow \phi(x, y_j)) |\ i < j < n\}$$

is contradictory. Let $n$ be minimal. Then there is $\theta(y, a) \in t(b, A)$ such that

$$\{\theta(y_i, a) |\ i < n\} \cup \{\neg\forall x(\phi(x, y_i) \leftrightarrow \phi(x, y_j)) |\ i < j < n\}$$

is contradictory.

We define $E(x, y)$ to be

$$\forall z(\theta(z, a) \to (\phi(x, z) \leftrightarrow \phi(y, z))).$$

Clearly $E$ is an equivalence relation, $\phi(x, b)$ depends on $E$ and $E$ is over $A$.

For all $i < n - 1$, choose $b_i$ so that $\models \theta(b_i, a)$ and for all $i < j < n - 1$, $\models \neg\forall x(\phi(x, b_i) \leftrightarrow \phi(x, b_j))$. For all $w \subseteq n - 1$, let $E_w = \bigcap_{i \in w} \phi(\mathbf{M}, b_i) \cap \bigcap_{i \in (n-1)-w} \neg\phi(\mathbf{M}, b_i)$. Then for all $w \subseteq n - 1$ and $c, d \in E_w$, $\models E(c, d)$ (exercise). So the number of equivalence classes of $E$ is $\leq 2^{n-1}$. □

**4.4 Exercise.**

*(i) If $\phi(x, b)$ is almost over $A$, then there are $E \in FE(A)$, $n < \omega$ and $a_i$, $i < n$, such that $\models \forall x(\phi(x, b) \leftrightarrow \vee_{i<n} E(x, a_i))$.*

*(ii) If $\phi(x, b)$ is almost over $\mathcal{A}$ and $\mathcal{A}$ is a model, then $\phi(x, b)$ is over $\mathcal{A}$. (Hint: Every equivalence class of a finite equivalence relation over $\mathcal{A}$ is represented in $\mathcal{A}$.)*

**4.5 Lemma.**

*(i) Assume $p \in S(B)$ does not fork over $A \subseteq B$. If $p'$ is almost over $A$ and $p \cup p'$ is consistent, then $p \cup p'$ does not fork over $A$.*

*(ii) Assume $q \in S(A)$, $p$ is almost over $A$ and $q \cup p$ is consistent. Then for all finite $\Delta$, $R_\Delta(q \cup p, \omega) = R_\Delta(q, \omega)$.*

**Proof**. (i): It is easy to see that if $\phi_i$, $i<n$, are almost over $A$ then so does $\wedge_{i<n}\phi_i$. So we may assume that $p'=\{\phi(x,b)\}$. Let $\phi(x,b)$ depend on $E\in FE(A)$ and choose $a$ so that it realizes $p$ and $\models\phi(a,b)$. Clearly $p\cup\{E(x,a)\}\vdash p\cup\{\phi(x,b)\}$, and so by Lemma 2.5, it is enough to show that $p\cup\{E(x,a)\}$ does not fork over $A$.

Let $a_i$, $i<n$, be a maximal sequence such that for all $i<n$, $t(a_i,B)=t(a,B)$, and for $i\neq j$, $\neg E(a_i,a_j)$. Then $p\vdash\vee_{i<n}E(x,a_i)$. By compactness, there is a finite $r\subseteq p$ such that

$$(*)\qquad r\vdash\vee_{i<n}E(x,a_i).$$

Let $q$ be such that it is finite and $r\subseteq q\subseteq p$. Clearly it is enough to show that $(\wedge q)\wedge E(x,a)$ does not fork over $A$. Since $\wedge q$ does not fork over $A$, there is $q'\in S(A)$ such that $q'\cup q$ is consistent and for every finite $\Delta$ there is $\Delta'\supseteq\Delta$ such that $R_{\Delta'}(q'\cup q,\omega)=R_{\Delta'}(q',\omega)$.

So it is enough to show that for all finite $\Delta$, $R_\Delta(q'\cup q\cup\{E(x,a)\},\omega)=R_\Delta(q'\cup q,\omega)$ (this implies that $q'\cup q\cup\{E(x,a)\}$ is consistent). By (*), Exercise 1.9 (ii) and Theorem 1.15, there is $i<n$ such that $R_\Delta(q'\cup q\cup\{E(x,a_i)\},\omega)=R_\Delta(q'\cup q,\omega)$. Since $t(a,B)=t(a_i,B)$, $R_\Delta(q'\cup q\cup\{E(x,a)\},\omega)=R_\Delta(q'\cup q\cup\{E(x,a_i)\},\omega)$.

(ii): As above, we may assume that $p=\{\phi(x,b)\}$ and choose $E\in FE(A)$ and $a$ so that $\phi(x,b)$ depends on $E$ and $a$ realizes $q\cup p$. Then $q\cup\{E(x,a)\}\vdash q\cup p$ and so by Exercise 1.9 (ii), it is enough to show that

(*) $R_\Delta(q\cup\{E(x,a)\},\omega)=R_\Delta(q,\omega)$.

As above we can find $a_i$, $i<n$, such that for all $i<n$, $t(a_i,A)=t(a,A)$ and $q\vdash\vee_{i<n}E(x,a_i)$. By Exercise 1.9 (ii) and Theorem 1.15, there is $i<n$ such that $R_\Delta(q\cup\{E(x,a_i)\},\omega)=R_\Delta(q,\omega)$. Since $t(a,A)=t(a_i,A)$, (*) follows. □

**4.6 Exercise.** *If $p$ is consistent and almost over $A$ then $p$ does not fork over $A$ (Hint: Choose $a$ so that it realizes $p$ and apply Lemma 4.5 (i) to $t(a,A)\cup p$.)*

**4.7 Lemma.** *For all $\phi(x,y)$ there is $m<\omega$ such that for all infinite indiscernible sets $I=\{b_i|\ i<\omega\}$ based on $A$ and $n\geq m$,*

$$\phi_n(x,I)=\bigvee_{w\subseteq 2n-1,\ |w|=n}(\wedge_{i\in w}\phi(x,b_i))$$

*is almost over $A$.*

**Proof**. Let $m$ be the number given by Exercise 3.5 for $\phi$ and $n\geq m$. Let $I=\{b_i|\ i<\omega\}$ be an infinite indiscernible set based on $A$. For a contradiction, assume $\phi_n(x,I)$ is not almost over $A$. Let $\xi=((|A|+2)^{|T|})^{++}$. By compactness, we can find $I_i$, $i<\xi$, copies of $I$ over $A$ such that $\phi_n(x,I_i)$ are pairwise non-equivalent. So for all $i<j$, we can choose $a_{ij}$ such that $\models\phi_n(a_{ij},I_i)\wedge\neg\phi_n(a_{ij},I_j)$. Let $B=A\cup\bigcup_{i<j<\xi}a_{ij}$. Then for all $i<j<\xi$, $Av(I_i,A)=Av(I_j,A)$ and by the choice of $m$, $Av(I_i,B)\neq Av(I_j,B)$. Since $I$ is based on $A$, for all $i<\xi$, $Av(I_i,B)$ does not fork over $A$. This contradicts Exercise 3.13. □

**4.8 Lemma.** *Assume $A \subseteq \mathcal{B}$ and $\mathcal{B}$ is $(|A|+\omega)^+$-saturated. If $p, q \in S^m(\mathcal{B})$, $p \neq q$ and both $p$ and $q$ do not fork over $A$, then there is $E \in FE(A)$ such that $p(x) \cup q(y) \vdash \neg E(x,y)$.*

**Proof**. Choose $\phi(x,b)$, $b \in \mathcal{B}$, such that $\phi(x,b) \in p$ and $\neg\phi(x,b) \in q$.

**Claim.** There is $\psi(x,d)$, $d \in \mathcal{B}$, such that it is almost over $A$ and $\psi(x,d) \in p$ and $\neg\psi(x,d) \in q$.

**Proof**. If $t(b,A)$ is algebraic, then we can let $\psi(x,d) = \phi(x,b)$. So we may assume that $t(b,A)$ is not algebraic. By Theorem 3.9, let $I \subseteq \mathcal{B}$ be an infinite indiscernible set over $A$ such that it is based on $A$ and for all $c \in I$, $t(c,A) = t(b,A)$. Clearly we may assume that $b \in I$. By Lemma 4.7, for some $n$, $\phi_n(x,I)$ is almost over $A$. By Lemma 3.11, $\phi_n(x,I) \in p$ and $\neg\phi_n(x,I) \in q$. □ Claim.

By Lemma 4.3, choose $E \in FE(A)$ so that $\psi(x,d)$ depends on $E$. Clearly $E$ is as wanted. □

**4.9 The finite equivalence relation theorem.** *If $p, q \in S^m(B)$, $p \neq q$ and both $p$ and $q$ do not fork over $A \subseteq B$, then there is $E \in FE(A)$ such that $p(x) \cup q(y) \vdash \neg E(x,y)$.*

**Proof**. Assume not. Then there are $a$ and $b$ such that $a$ realizes $p$, $b$ realizes $q$ and for all $E \in FE(A)$, $\models E(a,b)$. Let $\mathcal{C} \supseteq B$ be $(|A|+\omega)^+$-saturated model. By Exercise 4.2 (ii), Lemma 4.5 (i) and Lemma 2.6, there are $a'$ and $b'$ such that $a'$ realizes $p$, $b'$ realizes $q$, $a' \downarrow_A \mathcal{C}$, $b' \downarrow_A \mathcal{C}$ and for all $E \in FE(A)$, $\models E(a',a) \wedge E(b',b)$. Clearly this contradicts Lemma 4.8. □

**4.10 Definition.**

*(i) We define $stp(a,A)$, the strong type of $a$ over $A$, to be the set*

$$\{E(x,a) |\ E \in FE(A)\}.$$

*By $stp(a,A) = stp(b,A)$ we mean, that for all $E \in FE(A)$, $\models E(a,b)$.*

*(ii) We say that $p \in S(A)$ is stationary, if for all $a,b$ and $B \supseteq A$ the following holds: if $t(a,A) = t(b,A) = p$, $a \downarrow_A B$ and $b \downarrow_A B$, then $t(a,B) = t(b,B)$.*

Notice that $stp(a,A)$ is not over $A$ (but it is almost over $A$).

**4.11 Exercise.**

*(i) If $A \subseteq B$, $stp(a,A) = stp(b,A)$, $a \downarrow_A B$ and $b \downarrow_A B$, then $t(a,B) = t(b,B)$.*

*(ii) $stp(a,A) \vdash t(a,A)$.*

*(iii) If $\mathcal{A}$ is a model, then $t(a,\mathcal{A}) \vdash stp(a,\mathcal{A})$. (Hint: Exercise 4.4 (ii).)*

*(iv) If $\mathcal{A}$ is a model, then every $p \in S(\mathcal{A})$ is stationary.*

*(v) For all $A \subseteq B$ and $a$, there is $b$ such that $stp(b,A) = stp(a,A)$ and $b \downarrow_A B$. (Hint: Exercise 4.6 and Lemma 2.6.)*

*(vi) Suppose $(a_i)_{i<\omega}$ is indiscernible over $A$ and $E \in FE(A)$. Show that $E(a_i,a_j)$ holds for all $i,j < \omega$ and conclude that if $i_0 < ... < i_n < \omega$ and $j_0 < ... < j_n < \omega$, then $stp(\cup_{k\leq n} a_{i_k}/A) = stp(\cup_{k\leq n} a_{j_k}/A)$.*

## 5. Further properties of forking

In this section we collect the rewards of the hard work done in the two previous sections.

**5.1 Theorem.** *For all $A$, $a$ and $b$, $a \downarrow_A b$ implies $b \downarrow_A a$.*

**Proof**. Suppose not. By Lemma 2.6 and Theorem 3.3, we can find sequences $a_i$ and $b_i$, $i<\omega$, so that $a_0=a$, $b_0=b$, $(a_i \cup b_i)_{i<\omega}$ is indicernible over $A$ and for all $i<\omega$, $b_i \downarrow_A \bigcup_{j<i}(a_j \cup b_j)$ and $a_i \downarrow_A b_i \cup \bigcup_{j<i}(a_j \cup b_j)$. By Exercise 4.11 (vi), $stp(a_i/A) = stp(a_0/A)$ for all $i<\omega$ and thus by Exercise 4.11 (i), $t(a_1/A \cup b_0) = t(a_0/A \cup b_0)$ and so $b_0 \not\downarrow_A a_1$. Clearly $b_1 \downarrow_A a_0$. This contradicts the fact that $(a_i \cup b_i)_{i<\omega}$ is indiscernible over $A$. □

**5.2 Exercise.**

*(i) If $a \cup b \downarrow_A B$, then $a \downarrow_A B$.*

*(ii) For all $A \subseteq B$ and $C$, there is an automorphism $f \in Aut(A)$ such that for all $a \in C$, $stp(f(a),A) = stp(a,A)$ and $f(a) \downarrow_A B$. (Hint: For every singleton $a \in C$, choose a new constant $c_a$. For $a=(a_0,...,a_n)$, write $c_a = (c_{a_0},...,c_{a_{n-1}})$. By (v), for $a \in C$, choose $b_a$ so that $stp(b_a,A) = stp(a,A)$ and $b_a \downarrow_A B$. Then, by (i) above, show that the following set is consistent:*

$$\{E(c_a,a) |\ a \in C,\ E \in FE(A)\} \cup \{\phi(c_a,d) |\ \phi(x,d) \in t(b_a,B)\}.)$$

Exercise 5.2 (i) allows us to write $A \downarrow_B C$ if for all finite sequences $a \in A$, $a \downarrow_B C$.

**5.3 Lemma.** *Assume $A \subseteq B$ and $a \downarrow_A B$. Then for all finite $\Delta$ and $1<\xi\leq\omega$,*

$$R_\Delta(t(a,B),\xi) \geq R_\Delta(stp(a,A),\xi).$$

**Proof**. We prove only the case $\xi=2$, the other cases are similar. In order to simplify the notation we assume that $\Delta=\{\phi\}$. By Exercise 1.13, $R_\phi(stp(a,A),2) = n<\omega$. So by compactness, for all $\eta \in 2^n$, there is $a_\eta$ such that

(i) $stp(a_\eta,A) = stp(a,A)$,

(ii) for all $m<n$ and $\xi \in 2^m$, there is $b_\xi$ such that if $\eta,\eta' \in 2^n$, $\eta \restriction m = \eta' \restriction m = \xi$ and $\eta(m) \neq \eta'(m)$, then $\models \neg(\phi(a_\eta,b_\xi) \leftrightarrow \phi(a_{\eta'},b_\xi))$.

By Exercise 5,y (ii), we may assume that for all $\eta \in 2^n$, $a_\eta \downarrow_A B$. But then by Exercise 4.11 (i), for all $\eta \in 2^n$, $t(a_\eta,B) = t(a,B)$ and so (ii) above, implies that $R_\Delta(t(a,B),2) \geq n$. □

**5.4 Theorem.** *Assume $A \subseteq B$. Then $a \downarrow_A B$ iff for all finite $\Delta$,*

$$R_\Delta(t(a,B),\omega) = R_\Delta(t(a,A),\omega).$$

**Proof**. From right to left the claim follows immediately from the definition of forking and Exercise 1.9 (ii). We prove the other direction: By Exercise 1.9 (ii), it is enough to show that for all finite $\Delta$, $R_\Delta(t(a,B),\omega) \geq R_\Delta(t(a,A),\omega)$. By Lemma 5.3, it is enough to show that for all finite $\Delta$, $R_\Delta(stp(a,A),\omega) \geq R_\Delta(t(a,A),\omega)$. This follows from Lemma 4.5 (ii) (and Exercise 1.9 (ii)). □

**5.5 Exercise.**
*(i) Assume* $A \subseteq B \subseteq C$. *Then* $a \downarrow_A C$ *iff* $a \downarrow_A B$ *and* $a \downarrow_B C$.
*(ii)Show that* $a \cup b \downarrow_B C$ *iff* $a \downarrow_B C$ *and* $b \downarrow_{B \cup a} C$.

**5.6 Exercise.**
*(i) For every* $B$ *and* $a$, *there is* $A \subseteq B$ *of power* $< |T|^+$ *such that* $a \downarrow_A B$. *(Hint: By Exercise 1.9 (v), for every finite* $\Delta$ *there is finite* $A_\Delta \subseteq B$ *such that* $R_\Delta(t(a, A_\Delta), \omega) = R_\Delta(t(a, B), \omega)$.*)*
*(ii) There are no increasing continues sequence* $A_i$, $i < |T|^+$, *and* $a$ *such that* $a \not\downarrow_{A_i} A_{i+1}$ *for all* $i$.

We finish this section by giving two characterizations for non-forking.
We prove the following lemma for Exercise 5.8 (ii) below.

**5.7 Lemma.** *Assume* $A \subseteq B$ *and* $a \cup b \downarrow_A B$. *Then* $a \downarrow_A b$ *iff* $a \downarrow_B b$.

**Proof**. From right to left this follows immediately from Exercise 5.5 (i). So we prove the other direction. By $a \cup b \downarrow_A B$ and Theorem 5.1, $B \downarrow_A a \cup b$. By Exercise 5.5 (i), $B \downarrow_{A \cup b} a$. By Theorem 5.1, $a \downarrow_{A \cup b} B$. By Exercise 5.5 (i) and $a \downarrow_A b$, $a \downarrow_A B \cup b$. By Exercise 5.5 (i) again, $a \downarrow_B b$. □

For all sets $A$, we write $A \downarrow_B C$ if for all $a \in A$, $a \downarrow_B C$. Notice that by Exercise 5.2 (i), if $A = rng(a)$, then $A \downarrow_B C$ iff $a \downarrow_B C$.

**5.8 Exercise.**
*(i) Assume that* $\mathcal{A}$ *is a model, for all* $i < j < \omega$, $t(a_i, \mathcal{A}) = t(a_j, \mathcal{A})$ *and for all* $i < \omega$, $a_i \downarrow_{\mathcal{A}} \cup_{j<i} a_j$. *Show that* $\{a_i |\ i < \omega\}$ *is indiscernible over* $\mathcal{A}$. *(Hint: Show first that for all* $i < \omega$, $a_i \downarrow_{\mathcal{A}} \cup\{a_j |\ j < \omega,\ j \neq i\}$ *and then, by using Exercise 4.11 (iv), prove by induction on* $n$, *that every sequence of length* $n$ *of different elements has the same type over* $\mathcal{A}$ *than* $(a_i)_{i<n}$.*)*
*(ii) If for all* $i < j < \omega$, $stp(a_i, A) = stp(a_j, A)$ *and for all* $i < \omega$, $a_i \downarrow_A \cup_{j<i} a_j$, *then* $\{a_i |\ i < \omega\}$ *is indiscernible over* $A$. *(Hint: By Exercise 5.2 (ii), choose a model* $\mathcal{A} \supseteq A$ *so that* $\mathcal{A} \downarrow_A \cup_{i<\omega} a_i$ *and apply Lemma 5.7 and (i) above.)*
*(iii)* Why cannot we prove (ii) as (i) was proved?*

**5.9 Definition.** *We say that* $\mathcal{A}$ *is strongly* $\xi$*-saturated, if for all* $a$ *and* $A \subseteq \mathcal{A}$ *of power* $< \xi$, *there is* $b \in \mathcal{A}$ *such that* $stp(b, A) = stp(a, A)$.

**5.10 Lemma.** *Assume* $\xi > |T|$. *If* $\mathcal{A}$ *is* $\xi$*-saturated, then* $\mathcal{A}$ *is strongly* $\xi$*-saturated.*

**Proof**. Let $A \subseteq \mathcal{A}$ be of power $< \xi$ and $a$ arbitrary. Choose a model $\mathcal{B} \subseteq \mathcal{A}$ of power $< \xi$ such that $A \subseteq \mathcal{B}$. Choose $b \in \mathcal{A}$ so that $t(b, \mathcal{B}) = t(a, \mathcal{B})$. By Exercise 4.11 (iii), $b$ is as wanted. □

**5.11 Theorem.** *Assume* $A \subseteq B$. *Then* $a \downarrow_A B$ *iff for all* $C \supseteq B$ *there is* $b$ *such that* $t(b, B) = t(a, B)$ *and* $t(b, C)$ *does not split strongly over* $A$.

**Proof**. From left to right this follows from Lemmas 2.6 and 3.11. We prove the other direction: For a contradiction assume $a \not\downarrow_A B$. Let $\xi = |T| + |A|$ and $\mathcal{C} \supseteq B$ be a $\xi^+$-saturated model. Choose $b$ so that $t(b, B) = t(a, B)$ and $t(b, \mathcal{C})$ does not split strongly over $A$. Since $a \not\downarrow_A B$, we can choose $c \in B \subseteq \mathcal{C}$ so that $b \not\downarrow_A c$.

For all $i < \xi^+$, choose $c_i$ so that $stp(c_i, A) = stp(c, A)$ and $c_i \downarrow_A c \cup \bigcup_{j<i} c_j$. Then by Exercise 5.8 (ii), $\{c\} \cup \{c_i |\ i < \xi^+\}$ is indiscernible over $A$. Since $t(b, \mathcal{C})$ does not split strongly over $A$, $b \not\downarrow_A c_i$ for all $i < \xi^+$. Because $c_i \downarrow_A \bigcup_{j<i} c_j$, $b \not\downarrow_{A \cup \bigcup_{j<i} c_j} c_i$ (exercise). This contradicts Exercise 5.6 (ii). □

**5.12 Exercise$^*$.**

*(i) Assume $\mathcal{A}$ is $\lambda(T)$-saturated model and $B \supseteq \mathcal{A}$. Then $a \downarrow_{\mathcal{A}} B$ iff there is $A \subseteq \mathcal{A}$ of power $< \lambda(T)$ such that $t(a, B)$ does not split over $A$. (Hint: Notice that by Theorem 5.11 and Exercises 1.6 and 4.11 (iv), it is enough to show the following: If $A \subseteq \mathcal{A}$ is such that $|A| < \lambda(T)$ and $t(a, \mathcal{A})$ does not split over $A$, then there is $b$ such that $t(b, \mathcal{A}) = t(a, \mathcal{A})$ and $t(b, B)$ does not split over $A$. Furthermore, if $c$ is another such sequence, then $t(c, B) = t(b, B)$. This not easy.)*

*(ii) Suppose $I$ is an infinite indiscernible sequence and $J$ is such that $I \cup J$ is indiscernible. Show that for all $a \in J$, $a \downarrow_I J - \{a\}$.*

*(iii) Assume $I$ is an infinite indiscernible set. Show that $Av(I, I \cup A)$ does not fork over $I$ and that $Av(I, I)$ is stationary. (Hint: Show that it is enough to prove that if $t(a, I) = Av(I, I)$ and $t(a, I \cup b) \neq Av(I, I \cup b)$ then $a \not\downarrow_I b$. For this, for a contradiction, assume that this does not hold and choose $a_i$, $i < \omega$, so that $t(a_i, I \cup a \cup \bigcup_{j<i} a_j) = Av(I, I \cup a \cup \bigcup_{j<i} a_j)$ and $a_i \downarrow_{I \cup a \cup \bigcup_{j<i} a_j} b$. Then prove a contradiction using (ii) above and Exercise 3.5.)*

*(iv) Prove that the claims in Example 2.2 are true. (Hint for (i): Clearly we may assume that $a \notin A$. Let $q'$ be the set of formulas $E_n(x, b)$ such that $b \in B$ and there is $c \in A$, such that $\models E_n(b, c) \wedge E_n(a, c)$. Let $q = t(a, A) \cup q' \cup \{\neg E_n(x, b) |\ E_n(x, b) \notin q'\} \cup \{x \neq b |\ b \in B\}$. Show first that if $p \in S(B)$ and $q \not\subseteq p$, then $p$ forks over $A$. Then show that there is exactly one $p \in S(B)$, such that $q \subseteq p$. Finally apply Lemma 2.6. Notice that above we proved that every $p \in S^1(A)$ is stationary.*

*Hint for (ii): As (i), except now the type $t(a, A)$ need not be stationary. So instead of one, define a set $Q$ of types $q \in S(B)$ such that if $p \in S(B) - Q$ then $p$ forks over $A$ and if some $q \in Q$ forks over $A$, then every $q \in Q$ forks over $A$. Notice that if $t(a, B)$ forks over $A \subseteq B$ and $f \in Aut(A)$, then $t(f(a), f(B))$ forks over $A$.)*

**5.13 Definition.** *Assume $p \in S(B)$. We say that $\psi(y)$ defines $p \restriction \phi(x, y)$, if for all $b \in B$, $\phi(x, b) \in p$ iff $\models \psi(b)$. If in addition, $\psi$ is almost over $A \subseteq B$, we say that $p \restriction \phi$ is definable almost over $A$. If for all $\phi$, $p \restriction \phi$ is definable almost over $A$, then we say that $p$ is definable almost over $A$.*

**5.14 Theorem.**

*(i) If $p \in S(B)$ does not fork over $A \subseteq B$, then $p$ is definable almost over $A$.*

*(ii) $p \in S(B)$ does not fork over $A \subseteq B$ iff for all $C \supseteq B$, there is $q \in S(C)$ such that $p \subseteq q$ and $q$ is definable almost over $A$.*

**Proof**. (i): If $p \restriction A$ is algebraic, then the claim is easy (if $a$ realizes $p$, then $\phi(a,y)$ is almost over $A$). So we assume that $p \restriction A$ is not algebraic. By Lemma 2.4, $p$ is not algebraic. By Theorem 3.9, choose an infinite indiscernible $I$ based on $A$ so that for all $a \in I$, $t(a,B) = p$. Let $\phi = \phi(x,y)$ be arbitrary. By Lemma 4.7, there is $n$ such that $\phi_n(I,y)$ is almost over $A$. Trivially $\phi_n(I,y)$ defines $p \restriction \phi$.

(ii): From left to right this follows from Lemma 2.6 and (i). For the other direction, let $\xi = |T| + |A|$. Choose $\xi^+$-saturated $\mathcal{C} \supseteq B$ and $q \supseteq p$ definable almost over $A$. For a contradiction assume that $q$ forks over $A$. Choose $\phi(x,b) \in q$ so that it forks over $A$. For $i < \xi^+$, choose $b_i$ so that $stp(b_i, A) = stp(b, A)$ and $b_i \downarrow_A \cup_{j<i} b_j$. Since $q \restriction \phi$ is definable almost over $A$ and $stp(b_i, A) = stp(b, A)$, $\phi(x, b_i) \in q$ for all $i < \xi^+$. Let $a$ realize $q$. Then for all $i < \xi^+$, $a \not\downarrow_A b_i$. Because $b_i \downarrow_A \cup_{j<i} b_j$, $a \not\downarrow_{A \cup \bigcup_{j<i} b_j} b_i$. This contradicts Exercise 5.6 (ii). □

Theorem 5.14 (ii) is often used as a definition of forking.

**5.15 Exercise$^*$.**

*(i) If $\mathcal{B}$ is a model and $p \in S(\mathcal{B})$ is definable almost over $A \subseteq \mathcal{B}$, then for all $C \supseteq \mathcal{B}$, there is $q \in S(C)$ such that $p \subseteq q$ and $q$ is definable almost over $A$. (Hint: For all $\phi$, choose $\psi_\phi$ so that it is almost over $A$ and defines $p \restriction \phi$ and choose $E_\phi \in FE(A)$ so that $\psi_\phi$ depends on $E_\phi$. Show that*

$$\{\phi(x,c) |\ \exists b \in \mathcal{B}(\models E_\phi(b,c) \wedge\ \phi(x,b) \in p)\}$$

*is the wanted $q$.)*

*(ii) If $\mathcal{B}$ is a model, then $p \in S(\mathcal{B})$ does not fork over $A \subseteq \mathcal{B}$ iff $p$ is definable almost over $A$.*

**5.16 Definition.** *Suppose $A \subseteq B$. We say that $t(a,B)$ Lascar splits over $A$ if there are $b, c \in B$ such that $stp(b,A) = stp(c,A)$ but $t(b, A \cup a) \neq t(c, A \cup a)$.*

**5.17 Exercise$^*$.** *Suppose $A \subseteq B$. Show that $a \downarrow_A B$ iff for all $C \supseteq B$, there is $b$ such that $t(b,B) = t(a,B)$ and $t(b,C)$ does not Lascar split over $A$.*

## 6. An example of the use of forking

To give an example of the use of forking we prove a structure theorem for a class of theories. Since our knowledge of classification theory is still somewhat limited, the class must be very simple. Our class will be the class of theories which are trivial, superstable and unidimensional. An example of such theory is the theory of an equivalence relation which says that the number of equivalence classes is infinite and each equivalence class has size $n$, $n < \omega$. Although our class of theories is as simple as one can think of, in the proof of the structure theorem, many ideas from the proofs of 'the proper structure theorems' are present.

**6.1 Definition.**

*(i) A theory is superstable if it is stable and there are no* $A_i$ , $i < \omega$ , *and* $a$ *such that for all* $i < \omega$ , $A_i \subseteq A_{i+1}$ *and* $a \not\downarrow_{A_i} A_{i+1}$ .

*(ii) A stable theory is trivial if for all* $a, b, c$ *and* $A$ , $a \not\downarrow_A b \cup c$ *and* $b \downarrow_A c$ *imply that* $a \not\downarrow_A b$ *or* $a \not\downarrow_A c$ .

*(iii) Assume* $p, q \in S(A)$ . *We say that* $p$ *is almost orthogonal to* $q$ *if for all* $a$ *and* $b$ *the following holds: If* $a$ *realizes* $p$ *and* $b$ *realizes* $q$ *then* $a \downarrow_A b$ . *We say that* $p$ *is orthogonal to* $q$ *if for all* $a, b$ *and* $B \supseteq A$ *the following holds: If* $a$ *realizes* $p$ , $b$ *realizes* $q$ , $a \downarrow_A B$ *and* $b \downarrow_A B$ *then* $a \downarrow_B b$ .

*(iv) A stable theory is unidimensional if for all* $A$ *and* $p, q \in S(A)$ , *the following holds: If* $p$ *and* $q$ *are not algebraic, then* $p$ *is not orthogonal to* $q$ .

**6.2 Exercise$^*$.**

*(i) Show that* $T_2$ *is superstable but* $T_\omega$ *is not.*

*(ii) Assume* $T = T_\omega$ . *Show that non-algebraic types* $p, q \in S(A)$ *are orthogonal iff there are* $n < \omega$ *and* $a \in A$ *such that* $E_n(x, a) \in p$ *but* $E_n(x, a) \notin q$ *or vice versa (i.e.* $p \neq q$ *). Conclude that* $T_\omega$ *is not unidimensional.*

*(iii) Show that* $T_\omega$ *is trivial. (Hint: Modify Example 2.2 so that it holds for all finite sequences* $a$ *.)*

**6.3 Fact.** *([Hr]) Every unidimensional stable theory is superstable.*

**6.4 Lemma.** *Assume* $T$ *is trivial. If* $p, q \in S(A)$ *are almost orthogonal, then they are orthogonal.*

**Proof**. Assume not. Choose $a, b$ and $B \supseteq A$ so that $a$ realizes $p$, $b$ realizes $q$, $a \not\downarrow_B b$ and

(*) $a \downarrow_A B$ and $b \downarrow_A B$ .

Then $a \not\downarrow_A B \cup b$ and so triviality and (*) imply that $a \not\downarrow_A b$, a contradiction. □

**6.5 Lemma.** *Assume* $T$ *is superstable and* $C \subseteq B$ . *If* $C \neq B$ , *then there is a singleton* $b \in B - C$ *and* $\phi(x, c)$ , $c \in C$ , *such that* $\models \phi(b, c)$ *and for all* $b' \in B$ *and* $c' \in C$ , *if* $t(c', \emptyset) = t(c, \emptyset)$ , $\models \phi(b', c')$ *and* $b' \not\downarrow_{c'} C$ , *then* $b' \in C$ .

**Proof**. If not then we can easily find $\phi_i(x, c_i)$ , $i < \omega$ , such that for all $i < \omega$ , $\wedge_{j \leq i} \phi_i(x, c_i)$ is consistent and $\phi_i(x, c_i)$ forks over $\cup_{j<i} c_j$ . Clearly this contradicts the assumption that $T$ is superstable. □

**6.6 Definition.** *Assume* $\kappa$ *is a cardinal, not necessarily infinite. We write* $A \subseteq_\kappa B$ , *if for all* $C \subseteq A$ *of power* $< \kappa$ *and* $b \in B$ , *there is* $a \in A$ *such that* $t(a, C) = t(b, C)$ .

**6.7 Exercise$^*$.**

*(i) For all* $B$ *and regular (infinite)* $\kappa$ , *there is* $A$ *such that* $A \subseteq_\kappa B$ *and* $|A| \leq \kappa^{|T|}$ .

*(ii) If* $\mathcal{B}$ *is a model and* $\mathcal{A} \subseteq_\omega \mathcal{B}$ , *then* $\mathcal{A}$ *is a model.*

**6.8 Theorem.** *Assume $T$ is trivial, superstable and unidimensional and $\mathcal{B}$ is a model. Choose any $A \subseteq_1 \mathcal{B}$ and $a_i \in \mathcal{B} - A$, $i < \alpha$, so that $(a_i)_{i<\alpha}$ is a maximal sequence satisfying the following: for all $i < \alpha$, $a_i \downarrow_A \cup_{j<i} a_j$. Then $\mathcal{B} = acl(A \cup \bigcup_{i<\alpha} a_i)$.*

**Proof.** Let $C = acl(A \cup \bigcup_{i<\alpha} a_i)$. For a contradiction, assume $C \neq \mathcal{B}$. Choose $b$ and $\phi(x, c)$ as in Lemma 6.5. Choose $c' \in A$ so that $t(c', \emptyset) = t(c, \emptyset)$. Since $C$ is algebraicly closed, $t(b, C)$ is not algebraic. So we can find $a \notin C$ such that $\models \phi(a, c')$ and $a \downarrow_A C$. Then $t(a, C)$ is not algebraic and since $T$ is unidimensional, $t(a, C)$ is not orthogonal to $t(b, C)$. By Lemma 6.4, we may assume that $a \not\downarrow_C b$. Choose $\psi(x, d, b)$, $d \in C$, so that it forks over $C$ and $\models \psi(a, d, b)$. Since $\mathcal{B}$ is a model, we can choose $a' \in \mathcal{B}$ so that $\models \phi(a', c') \wedge \psi(a', d, b)$. Then $a' \not\downarrow_C b$ and so $a' \notin C$. So by the choice of $\phi(x, c)$, $a' \downarrow_{c'} C$. Since $c' \in A$, $a' \downarrow_A C$. This contradicts the maximality of $(a_i)_{i<\alpha}$. $\square$

**6.9 Exercise$^*$.** *We write $I(\kappa, T)$ for the number of non-isomorphic models in $\{\mathcal{A} \models T | \ |\mathcal{A}| = \kappa\}$. Assume $T$ is trivial, superstable and unidimensional theory. Then for all $\beta$, $I(\aleph_\beta, T) \leq |\omega + \beta|^{(2^{|T|})}$. (Hint: Use Theorem 6.8 and show first that the isomorphism type of $\mathcal{B}$ is determined by the isomorphism type of $A \cup \bigcup_{i<\alpha} a_i$. Show then that if $A$ is a model then the isomorphism type of $A \cup \bigcup_{i<\alpha} a_i$ is determined by the isomorphism type of $A$ and the cardinals $\kappa_p$, $p \in S(A)$, where $\kappa_p = |\{i < \alpha | \ a_i$ realizes $p\}|$. Finally count the number of possible choices of $A$ and $(\kappa_p)_{p \in S(A)}$, in the case $A$ is chosen to be as small as possible.)*

Notice that usually $|\omega + \beta|^{(2^{|T|})}$ is very small compared to $\aleph_\beta$, and so it is also very small compared to $2^{\aleph_\beta}$, which is the maximal number of models any theory can have in power $\aleph_\beta$.

**6.10 Fact.** *Our structure theorem and the estimate of the number of models are very weak (in every cardinality the number of models is $\leq 2^{(2^{|T|})}$). The idea in this section was to demonstrate the use of forking.*

## PART II: PRIME MODELS

In many cases, as in section 6, by using the independence notion studied in the previous part, we can find a 'base' for every model of $T$. To get a structure theorem, we need to show that this 'base' determines the structure of the model. Prime (primary) models provide a method to do this. In section 6, we assumed triviality in order to be able to use algebraic closure instead of prime models.

### 7. General isolation notion

We will construct the required prime models by using isolation as a tool. It depends on the properties of $T$, which isolation notion $F$ is the right one. So in order to avoid repeating same arguments several times, our approach is axiomatic. When reading the axioms, one may keep in his mind the following two examples: $(p, A) \in F^s_\lambda$ if $(p, A) \in P_\lambda$ (see below) and $p \restriction A \vdash p$ and $(p, A) \in F^f_\lambda$ if $(p, A) \in P_\lambda$ and $p$ does not fork over $A$. In the next section we give more examples.

Let $\lambda$ be an infinite cardinal and $P_\lambda$ be the class of those pairs $(p, A)$ such that $|A| < \lambda$ and for some $B \supseteq A$, $p \in S(B)$. Let $F_\lambda \subseteq P_\lambda$ be such that Axioms I-IX below are satisfied. We write $(t(C, B), A) \in F_\lambda$ if for all $c \in C$, $(t(c, B), A) \in F_\lambda$.

Ax I: If $rng(a) = rng(b)$, then $(t(a, B), A) \in F_\lambda$ iff $(t(b, B), A) \in F_\lambda$ and for all automorphisms $f$, $(p, A) \in F_\lambda$ iff $(f(p), f[A]) \in F_\lambda$.

Ax II: If $a \in A \subseteq B$ and $|A| < \lambda$, then $(t(a, B), A) \in F_\lambda$.

Ax III: If $A \subseteq B \subseteq C \subseteq dom(p)$, $|B| < \lambda$ and $(p, A) \in F_\lambda$, then $(p \restriction C, B) \in F_\lambda$.

Ax IV: If $(t(a \cup b, B), A) \in F_\lambda$, then $(t(a, B), A) \in F_\lambda$.

Ax V: If $|C| < \lambda$ and $(t(a \cup C, B), A) \in F_\lambda$, then $(t(a, B \cup C), A \cup C) \in F_\lambda$.

Ax VI: If $A, B \subseteq C$, $(t(b, C \cup a), B) \in F_\lambda$ and $(t(a, C), A) \in F_\lambda$, then $(t(a, C \cup b), A) \in F_\lambda$.

Ax VII: If $A \subseteq B$, $(t(a, B \cup C), A \cup C) \in F_\lambda$ and $(t(C, B), A) \in F_\lambda$, then $(t(a \cup C, B), A) \in F_\lambda$.

Ax VIII: If $B_i$, $i < \delta$, is increasing sequence of sets, $p \in S(\cup_{i<\delta} B_i)$ and for all $i < \delta$, $(p \restriction B_i, A) \in F_\lambda$, then $(p, A) \in F_\lambda$.

Ax IX: If $(p, A) \in F_\lambda$ and $dom(p) \subseteq B$, then there are $A' \subseteq B$ and $q \in S(B)$ such that $p \subseteq q$ and $(q, A') \in F_\lambda$.

Notice that $\emptyset$ satisfies all the axioms except Ax II and $\{(t(a, B), A) \in P_\lambda | a \in A\}$ satisfies them all. So the axioms alone do not guarantee a good behaviour of an isolation notion.

**7.1 Definition.**

*(i) We say that* $(A, (a_i, B_i)_{i<\alpha})$ *is an* $F_\lambda$*-construction over* $A$ *if for all* $i < \alpha$, $(t(a_i, A_i), B_i) \in F_\lambda$, *where* $A_i = A \cup \bigcup_{j<i} a_j$. *We say that* $C$ *is* $F_\lambda$*-constructible over* $A$ *if there is an* $F_\lambda$*-construction* $(A, (a_i, B_i)_{i<\alpha})$ *over* $A$ *such that* $C = A \cup \bigcup_{i<\alpha} a_i$.

*(ii) We say that $C$ is $(F_\lambda,\kappa)$-saturated if for all $B\subseteq C$ of power $<\kappa$ and $p\in S(B)$ the following holds: if for some $A$, $(p,A)\in F_\lambda$, then $p$ is realized in $C$. We say that $C$ is $F_\lambda$-saturated if it is $(F_\lambda,|C|^+)$-saturated.*

*(iii) We write $\mu(F_\lambda)$ for the least cardinal $\mu$ such that for all $\kappa\geq\mu$ and $C$, if $C$ is $(F_\lambda,\mu)$-saturated then it is $(F_\lambda,\kappa)$-saturated. If such $\mu$ does not exist, then we write $\mu(F_\lambda)=\infty$.*

*(iv) We say that $C$ is $F_\lambda$-primary ($(F_\lambda,\kappa)$-primary) over $A$ if it is $F_\lambda$-constructible over $A$ and $F_\lambda$-saturated ($(F_\lambda,\kappa)$-saturated).*

*(v) We say that $C$ is $F_\lambda$-primitive over $A$ if for all $F_\lambda$-saturated $B\supseteq A$ there is an elementary embedding $f: C\to B$ such that $f\restriction A=id_A$. We say that $C$ is $F_\lambda$-prime over $A$ if it is $F_\lambda$-primitive and $F_\lambda$-saturated.*

**7.2 Exercise.**

*(i) Show that for all $A$ and $\kappa$, there is an $(F_\lambda,\kappa)$-primary set over $A$ and if $\mu(F_\lambda)<\infty$ then there is also an $F_\lambda$-primary set over $A$. (Hint: Use Ax IX.)*

*(ii) Show that if $C$ is $F_\lambda$-constructible over $A$, then it is $F_\lambda$-primitive over $A$ and so $F_\lambda$-primary sets over $A$ are $F_\lambda$-prime over $A$.*

**7.3 Fact.** *([Sh]) In many cases, $F_\lambda$-prime models are $F_\lambda$-primary. E.g. If $T$ is superstable, then for all $\lambda$ and $A$, $F^a_\lambda$-prime models over $A$ are $F^a_\lambda$-primary over $A$. (For $F^a_\lambda$, see section 10.)*

Notice that from Exercise 7.2 (ii) it follows that if $(A,(a_i,B_i)_{i<\alpha})$ is an $F_\lambda$-construction over $A$, for all $i<j<\alpha$, $a_i\neq a_j$ and $C\supseteq A$ is an infinite $F_\lambda$-saturated set, then $\alpha<|C|^+$.

Notice also that in Exercise 7.2, only axioms AX I and Ax IX and the assumption $\mu(F_\lambda)<\infty$ were used. (In (ii) only Ax I is needed.) In most cases this exercise together with Lemma 10.7 and Exercise 10.9 are all we need to know about primary models to prove a structure theorem. However, if all the axioms are satisfied and $\lambda$ is regular, then a lot more is known about $F_\lambda$-primary models. In the case of our structure theorem in section 11, all the axioms are satisfied and $\lambda=\omega$, which is a regular cardinal and this is used in order to make the proof short. For an alternative way of proving a structure theorem, see [HS2]. See also Exercise 11.9 (i).

**Assumption.** From now on in this section, we assume that $\lambda$ is regular.

Let $(A,(a_i,B_i)_{i<\alpha})$ be an $F_\lambda$-construction. We say that $X\subseteq\alpha$ is closed if for all $i\in X$, $B_i\subseteq A\cup\bigcup_{j\in X,j<i}a_j$. We say that $X\subseteq\alpha$ is strongly closed if for all $i\in X$ and elements $a\in B_i$ the following holds:

(*) if there is $j<i$ such that $a\in a_j$, then there is $k<i$ such that $a\in a_k$ and $k\in X$.

**7.4 Lemma.** *If $(A,(a_i,B_i)_{i<\alpha})$ is an $F_\lambda$-construction and $X'\subseteq\alpha$ is of power $<\lambda$, then there is strongly closed $X\subseteq\alpha$ such that $X'\subseteq X$ and $|X|<\lambda$.*

**Proof**. We construct a tree (forest) $R$ such that it's first level consists of elements of $X'$ and if $i\in R$ then the set of the immediate successors of $i$ is (a copy of) of a minimal set that satisfies (*) for $i$ in the definition of strongly closed. Clearly $R$ does not have infinite branches and since $\lambda$ is regular, each level of $R$ is of power $< \lambda$. But then $|R| < \lambda$ (if $\lambda = \omega$ use König's lemma). Clearly $R$ is closed. $\square$

**7.5 Definition.** *We say that $C$ is $F_\lambda$-atomic over $A$ if for all $c \in C$, there is $B \subseteq A$ such that $(t(c, A), B) \in F_\lambda$.*

**7.6 Theorem.** *($\lambda$ regular) If $C$ is $F_\lambda$-constructible over $A$ then it is $F_\lambda$-atomic over $A$.*

**Proof**. Let $(A, (a_i, B_i)_{i<\alpha})$ be an $F_\lambda$-construction of $C$. By Ax IV, we may assume that for all $i < \alpha$, $a_i \cap A_i = \emptyset$ ($A_i = A \cup \bigcup_{j<i} a_j$).

**Claim.** If $X \subseteq \alpha$ is closed and $|X| < \lambda$, then there is $B \subseteq A$ such that $(t(\cup_{i \in X} a_i, A), B) \in F_\lambda$.

**Proof**. We prove this by induction on $i = \cup\{j + 1 |\ j \in X\} \leq \alpha$. The case $i$ is limit is left as an exercise. (Hint: Use Ax III, the assumption that $\lambda$ is regular and the fact that for all $j < i$, $X \cap j$ is closed.) So assume that the claim holds for $i$. We prove it for $i+1$. For this let $X \subseteq \alpha$ be closed, $|X| < \lambda$ and $\cup\{j+1 |\ j \in X\} = i+1$. Let $D = \cup\{a_j | j \in X \cap i\}$. By the induction assumption, there is $B'$ such that $(t(D, A), B') \in F_\lambda$. Let $B = B' \cup (B_i \cap A)$. By Ax VII and Ax III, $B$ is as wanted. $\square$ Claim.

Now let $c \in C$. Choose $a \in A$ and $b \in C - A$ such that $c = a \cup b$. By Ax IV, we may assume that there is finite $X' \subseteq \alpha$ such that $b = \cup_{i \in X'} a_i$. By Lemma 7.4, there is closed $X \subseteq \alpha$ such that $X' \subseteq X$ and $|X| < \lambda$. By Claim, we can choose $B'$ so that $(t(b, A), B') \in F_\lambda$. Let $B = B' \cup a$. By Ax VII, Ax II and Ax III, $(t(c, A), B) \in F_\lambda$. $\square$

**7.7 Lemma.** *Let $(A, (a_i, B_i)_{i<\alpha})$ be an $F_\lambda$-construction.*

*(i) For all $\beta < \alpha$, $(A_\beta, (a_i, B_i)_{\beta \leq i < \alpha})$ is an $F_\lambda$-construction ($A_\beta = A \cup \bigcup_{i<\beta} a_i$).*

*(ii) If $D \subseteq A \cup \bigcup_{i<\alpha} a_i$ is of power $< \lambda$, then there are $C_i$, $i < \alpha$, such that $(A \cup D, (a_i, C_i)_{i<\alpha})$ is an $F_\lambda$-construction.*

**Proof**. (i) is immediate so we prove (ii): By (i), Theorem 7.6, Ax III and the assumption that $\lambda$ is regular, for all $i < \alpha$ we can find $C_i'$ such that $(t(a_i \cup D, A_i), C_i') \in F_\lambda$. Let $C_i = C_i' \cup D$. Then by Ax V, $(t(a_i, A_i \cup D), C_i) \in F_\lambda$. $\square$

**7.8 Exercise.** *For $l \in \{1, 2\}$, let $(A^l, (a_i^l, B_i^l)_{i<\alpha^l})$ be an $F_\lambda$-construction of an $F_\lambda$-primary set $C^l$ over $A^l$. Assume that $f$ is an elementary function such that $A^1 \subseteq dom(f) \subseteq C^1$, $A^2 \subseteq rng(f) \subseteq C^2$, $|dom(f) - A^1| < \lambda$ and $|rng(f) - A^2| < \lambda$.*

*(i) For all $i < \alpha^1$, there is an elementary function $g \supseteq f$ such that $dom(g) = dom(f) \cup a_i^1$ and $rng(g) \subseteq C^2$. (Hint: Use Lemma 7.7 (ii) and Theorem 7.6.)*

*(ii) For all $i < \alpha^1$, there is an elementary function $g \supseteq f$ and strongly closed $X \subseteq \alpha^1$ and $Y \subseteq \alpha^2$ such that $i \in X$, $dom(g) = A^1 \cup \bigcup_{i \in X} a_i^1$ and $rng(g) = A^2 \cup \bigcup_{i \in Y} a_i^2$. (Hint: Use (i) and Lemma 7.4.)*

**7.9 Lemma.** *Let $(A, (a_i, B_i)_{i<\alpha})$ be an $F_\lambda$-construction and $s : \beta \to \alpha$ be one-one and onto. If for all $i < \beta$, $B_{s(i)} \subseteq A \cup \bigcup_{j<i} a_{s(j)}$, then $(A, (a_{s(i)}, B_{s(i)})_{i<\beta})$ is an $F_\lambda$-construction.*

**Proof**. Let $i < \beta$. For all $j \le \alpha$, we write $D_j = A \cup B_{s(i)} \cup \bigcup\{a_{s(k)} |\ k < i,\ s(k) < j\}$. By induction on $j \le \alpha$, we show that $(t(a_{s(i)}, D_j), B_{s(i)}) \in F_\lambda$. This is enough, since $D_\alpha = A \cup \bigcup_{k<i} a_{s(k)}$.

If $j \le s(i)+1$, then $D_j \subseteq A \cup \bigcup_{k<s(i)} a_k$ and so by Ax III, the claim follows. If $j$ is limit, then the claim follows from the induction assumption and Ax VIII. So assume $j = k+1$ and $k > s(i)$. We may also assume that $D_j = D_k \cup \{a_k\}$, since otherwise there is nothing to prove. Then there is $m < i$ such that $s(m) = k$. By the assumption on $s$, $B_k \subseteq D_k$. Then by Ax III, $(t(a_k, D_k \cup a_{s(i)}), B_k) \in F_\lambda$. By the induction assumption and Ax VI, the claim follows. □

**7.10 Lemma.** *For $l \in \{1,2\}$, let $(A, (a^l_i, B^l_i)_{i<\alpha^l})$ be an $F_\lambda$-construction of an $F_\lambda$-primary set $C^l$ over $A$. Assume that $f$ is an elementary function and $X^l \subseteq \alpha^l$, $i \in \{1,2\}$, are closed sets such that $dom(f) = A \cup \bigcup_{i\in X^1} a^1_i$, $rng(f) = A \cup \bigcup_{i \in X^2} a^2_i$ and $f \restriction A = id_A$. Then for all $i^* < \alpha^1$, there are an elementary function $g \supseteq f$ and closed $Y^l \subseteq \alpha^l$ such that $X^1 \cup \{i^*\} \subseteq Y^1$, $X^2 \subseteq Y^2$, $dom(g) = A \cup \bigcup_{i\in Y^1} a^1_i$ and $rng(g) = A \cup \bigcup_{i\in Y^2} a^2_i$.*

**Proof**. Clearly we may assume that $i^* \notin X^1$ For $l \in \{1,2\}$, let $\beta^l$ be the order type of $X^l$ and $\gamma^l$ be the order type of $\alpha^l - X^l$. Let $\delta^l = \beta^l + \gamma^l$ and $s^l : \delta^l \to \alpha^l$ be such that for all $i < \beta^l$, $s^l(i)$ is the $i$:th member of $X^l$ and for all $i < \gamma^l$, $s^l(\beta^l + i)$ is the $i$:th member of $\alpha^l - X^l$. Then $s^1$ and $s^2$ satisfy the assumptions of Lemma 7.9 and so by Lemma 7.9, 7.7 (i) and Exercise 7.8 (ii), we can find an elementary function $g \supseteq f$ and strongly closed $Z^l \subseteq \delta^l - \beta^l$ in the sense of the $F_\lambda$-construction

$$(A \cup \bigcup_{j<\beta^l} a_{s^l(j)}, (a^l_{s^l(i)}, B^l_{s^l(i)})_{\beta^l \le i < \delta^l})$$

such that $(s^1)^{-1}(i^*) \in Z^1$, $dom(g) = dom(f) \cup \bigcup_{i\in Z^1} a^1_{s^1(i)}$ and $rng(g) = rng(f) \cup \bigcup_{i\in Z^2} a^2_{s^2(i)}$. Let $Y^l = X^l \cup s^l[Z^l]$. Clearly, if the sets $Y^l$ are closed, $g$ and $Y^l$, $l \in \{1,2\}$, are as wanted.

So let $i \in Y^l$ and $a \in B^l_i$ be an element. We needs to show that $a \in A \cup_{j \in Y^l, j<i}$ $a^l_j$. If

(*) $i \in X^l$ or $a \in A \cup \bigcup_{j\in s^l[Z^l], j<i} a^l_j$,

there is nothing to prove. So we assume that (*) is not true. By the definition of $F_\lambda$-construction there is $j < i$ such that $a \in a^l_j$. Since (*) fails, by the definition of strong closeness, $j \in X^l$. □

**7.11 Theorem.** *($\lambda$ regular) $F_\lambda$-primary sets over $A$ are unique up to isomorphism over $A$.*

**Proof**. Let $(A, (b_i, B_i)_{i<\beta})$ be an $F_\lambda$-construction of an $F_\lambda$-primary set $B$ over $A$ and let $(A, (c_i, C_i)_{i<\gamma})$ be an $F_\lambda$-construction of an $F_\lambda$-primary set $C$ over $A$. By induction on $i \le \alpha = max\{\beta, \gamma\}$, we choose elementary functions $f_i$ and closed sets $X_i \subseteq \beta$ and $Y_i \subseteq \gamma$ such that

(i) $f_0 = id_A$, $X_0 = Y_0 = \emptyset$,

(ii) for all $i < j$, $f_i \subseteq f_j$, $X_i \subseteq X_j$ and $Y_i \subseteq Y_j$,

(iii) $dom(f_i) = A \cup \bigcup_{k \in X_i} b_k$ and $rng(f_i) = A \cup \bigcup_{k \in Y_i} c_k$,

(iv) if $i < \beta$, then $i \in X_{i+1}$ and if $i < \gamma$, then $i \in Y_{i+1}$.

If $i$ is limit, we let $f_i = \cup_{j<i} f_j$, $X_i = \cup_{j<i} X_j$ and $Y_i = \cup_{j<i} Y_j$. Clearly these are as wanted. If $i = j+1$, then the existence of $f_i$, $X_i$ and $Y_i$ follows from Lemma 7.10. Clearly $f_\alpha$ is an elementary function from $B$ onto $C$ and $f_\alpha \restriction A = id_A$. □

## 8. Examples of isolatation notions

We recall that we have assumed that $T$ is stable.

**8.1 Definition.**

*(i) As already mentioned, we define $F^s_\lambda$ to be the set of all pairs $(p, A) \in P_\lambda$ such that $p \restriction A \vdash p$.*

*(ii) We define $F^t_\lambda$ to be the set of all pairs $(p, A) \in P_\lambda$ which satisfy the following: there is $q \subseteq p \restriction A$ such that $|q| < \lambda$ and $q \vdash p$.*

Notice that $F^t_\omega$-isolation is the usual isolation notion.

**8.2 Lemma.** *If $\lambda > |T|$, then $F^s_\lambda$ satisfies Ax IX.*

**Proof**. Assume not. Let $p$, $A$ and $B$ exemplify this. Then for all $\eta \in 2^{\leq\lambda}$, we can find $p_\eta$ and $A_\eta \subseteq B$ such that

(i) $p_{()} = p \restriction A$ and $A_{()} = A$,

(ii) for all $\eta$, $p_\eta \in S(A_\eta)$, $A_{\eta\frown(0)} = A_{\eta\frown(1)}$ and $|A_{\eta\frown(0)} - A_\eta| < \omega$,

(iii) if $\eta$ is an initial segment of $\xi$, then $p_\eta \subseteq p_\xi$,

(iv) if $\alpha = length(\eta)$ is limit, then $p_\eta = \cup_{\beta<\alpha} p_{\eta\restriction\beta}$,

(v) for all $\eta$, $p_{\eta\frown(0)}$ is contradictory with $p_{\eta\frown(1)}$.

By Exercise 1.11 (ii), we can find $\eta \in 2^\lambda$ such that for all $\alpha < \lambda$ there is a singleton $\Delta$ for which $R_\Delta(p_{\eta\restriction(\alpha+1)} \restriction \Delta, 2) < R_\Delta(p_{\eta\restriction\alpha} \restriction \Delta, 2)$. Since $\lambda > |T|$, there are infinite $X \subseteq \lambda$ and a singleton $\Delta$ such that for all $\alpha \in X$, $R_\Delta(p_{\eta\restriction(\alpha+1)} \restriction \Delta, 2) < R_\Delta(p_{\eta\restriction\alpha} \restriction \Delta, 2)$, a contradiction. □

**8.3 Exercise.**

*(i) Show that $F^s_\lambda$ satisfies the axioms Ax I-VIII.*

*(ii) Show that $F^t_\lambda$ satisfies the axioms Ax I-VIII and if $T$ is $\lambda$-stable, then it satisfies also Ax IX. (Hint for Ax IV: If $q(x, y) \vdash t(a \cup b, B)$, then $\{\exists y \wedge r |\ r \subseteq q$ finite$\} \vdash t(a, B)$. This idea works also in the case of Ax VII. Hint for Ax VIII: Notice that $p \restriction B_0 \vdash p$. Hint for Ax IX: Assume not. Essentially as in the proof of Lemma 8.2, construct a tree of height $\kappa$, where $\kappa$ is the least cardinal such that $2^\kappa > \lambda$. Use the tree to show that $T$ is not $\lambda$-stable.)*

**8.4 Lemma.** $\mu(F^s_\lambda) \leq \lambda$.

**Proof**. Assume $C$ is $(F^s_\lambda, \lambda)$-saturated. We show that $C$ is $(F^s_\lambda, |C|^+)$-saturated. For this let $(p, A) \in F^s_\lambda$ be such that $dom(p) \subseteq C$. Then $(p \restriction A, A) \in F^s_\lambda$ and so there is $c \in C$ which realizes $p \restriction A$. But then $c$ realizes $p$. □

**8.5 Exercise.**

*(i)* $\mu(F^t_\lambda) \le \lambda$.

*(ii) Show that every* $F_\lambda$*-saturated set is a model, if the following holds: For all* $B$ *and a formula* $\phi(x)$ *over* $B$*, if* $\models \exists x\phi$*, then there are* $A \subseteq B$ *and* $p \in S(B)$ *such that* $\phi \in p$ *and* $(p, A) \in F_\lambda$*.*

*(iii)* $C$ *is an* $F^s_\lambda$*-saturated set iff it is a* $\lambda$*-saturated model.*

*(iv) Assume* $T$ *is* $\omega$*-stable. Then* $C$ *is an* $F^t_\omega$*-saturated set iff it is a model.*

## 9. Spectrum of stability

To continue our studies of prime models, we need more knowledge on stability.

**9.1 Definition.** *Let* $\kappa(T)$ *be the least cardinal* $\kappa$ *such that there are no* $A_i$*,* $i < \kappa$*, and* $a$ *such that for all* $i < j$*,* $A_i \subseteq A_j$ *and* $a \not\downarrow_{A_i} A_j$*.*

In Exercise 5.6 we showed:

**9.2 Recall.**

*(i)* $\kappa(T) \le |T|^+$.

*(ii) For all* $A$ *and* $p \in S(A)$*, there is* $B \subseteq A$ *of power* $< \kappa(T)$ *such that* $p$ *does not fork over* $B$*.*

**9.3 Lemma.** *If* $\xi^{<\kappa(T)} > \xi$*, then* $T$ *is not* $\xi$*-stable.*

**Proof**. Choose $\kappa < \kappa(T)$ so that $\xi^{<\kappa} = \xi < \xi^\kappa$. Then there are $a_i$, $i < \kappa$, and $a$ such that for all $i < \kappa$, $a \not\downarrow_{\cup_{j<i} a_j} a_i$. Let $<$ be a well-ordering of $\xi^{\le\kappa}$ such that if $\eta$ is an initial segment of $\eta'$, then $\eta < \eta'$. For all $\eta \in \xi^{\le\kappa}$, choose $a_\eta$ so that

(i) for all $\eta \in \xi^\kappa$, the function that takes $a_i$ to $a_{\eta\restriction i}$ and $a$ to $a_\eta$ is elementary,

(ii) for all $\eta \in \xi^{\le\kappa}$, if $\alpha = length(\eta)$, then $a_\eta \downarrow_{\cup_{\beta<\alpha} a_{\eta\restriction\beta}} \cup\{a_{\eta'} |\ \eta' < \eta\}$.

Then the following holds: If $\eta \in \xi^\kappa$ and $\alpha < \kappa$ and $A$ is the set of those $a_{\eta'}$ such that $\eta' \in \xi^{\le\kappa}$ and $\eta \restriction \alpha$ is not an initial segment of $\eta'$, then $a_\eta \downarrow_{\cup_{\beta<\alpha} a_{\eta\restriction\beta}} A$. (Exercise, prove by induction on $<$.) So if $\eta, \eta' \in \xi^\kappa$ and $\eta \ne \eta'$, then $t(a_\eta, B) \ne t(a_{\eta'}, B)$, where $B = \cup_{\tau\in\xi^{<\kappa}} a_\tau$. By the choice of $\kappa$, $T$ is not $\xi$-stable. □

**9.4 Exercise.**

*(i) If* $T$ *is* $\xi$*-stable, then* $cf(\xi) \ge \kappa(T)$*, especially* $\kappa(T) \le cf(\lambda(T))$*.*

*(ii) If* $T$ *is* $\xi$*-stable, then for all* $A$ *of power* $\le \xi$ *there is a model* $\mathcal{B} \supseteq A$ *of power* $\le \xi$*. (Hint: For all* $i < \omega$*, choose* $A_i$ *of power* $\le \xi$ *so that* $A_0 = A$*, every* $p \in A_i$ *is realized in* $A_{i+1}$ *and if* $i < j$*, then* $A_i \subseteq A_j$*. Then* $\cup_{i<\omega} A_i$ *is as wanted.)*

*(iii)* Show that for all* $A$ *there is a model* $\mathcal{A}$ *such that* $A \subseteq \mathcal{A}$ *and* $|\mathcal{A}| \le |A| + \lambda(T)$*.*

*(iv) Assume* $p \in S(A)$ *and* $B \supseteq A$*. Then*

$$|\{q \in S(B) |\ p \subseteq q,\ q \text{ does not fork over } A\}| \le \lambda(T).$$

*(Hint: By (ii) and Exercise 4.11 (iv), prove the claim first under the additional assumption* $|A| \le \lambda(T)$*.)*

**9.5 Theorem.** *$T$ is $\xi$-stable iff $\xi = \lambda(T) + \xi^{<\kappa(T)}$.*

**Proof**. From left to right this follows from Lemma 9.3 and the definition of $\lambda(T)$. By Recall 9.2 (ii) and Exercises 9.4 (i) and 9.4 (iii), for all $A$, $|S(A)| \leq \lambda(T) \times \lambda(T) \times |A|^{<\kappa(T)}$, from which the other direction follows. □

**9.6 Lemma.** *If $T$ is $\xi$-stable, then there is a saturated model of power $\xi$.*

**Proof**. Choose an increasing continuous sequence $A_i$, $i \leq \xi$, of models of power $\leq \xi$ so that for all $i < \xi$ and $a$, there is $b \in A_{i+1}$ such that $t(b, A_i) = t(a, A_i)$. We show that $\mathcal{A} = A_\xi$ is as wanted. For this let $B \subseteq \mathcal{A}$ be of power $< \xi$ and $b$ be arbitrary. We show that $t(b, B)$ is realized in $\mathcal{A}$.

By Exercise 9.4 (i), there is $\alpha < \xi$ such that $b \downarrow_{A_\alpha} \mathcal{A}$.

**Claim.** There is $\beta < \xi$ such that $\beta \geq \alpha$ and $B \downarrow_{A_\beta} A_{\beta+1}$.

**Proof**. Assume not. Then by the pigeon hole principle, we can find $d \in B$ such that

$$|\{\gamma < \xi |\ d \ \not\downarrow_{A_\gamma} A_{\gamma+1}\}| \geq cf(\xi).$$

This is impossible by Exercise 9.4 (i). □ Claim.

Choose $c \in A_{\beta+1}$ so that $t(c, A_\beta) = t(b, A_\beta)$. By Claim, $c \downarrow_{A_\beta} B$ and so by stationarity, $c$ realizes $t(b, B)$. □

**9.7 Exercise.** *Let $\kappa$ be the least regular cardinal $\geq \kappa(T)$. If $B$ is $F^s_\lambda$-constructible over $A$, then $|B| \leq \lambda(T) + (|A| + \lambda)^{<\kappa}$.*

## 10. $a$-prime models

**10.1 Definition.**

*(i) We define $F^a_\lambda$ to be the set of the pairs $(p, A) \in P_\lambda$ such that for some $a \models p$, $stp(a, A) \vdash p$.*

*(ii) We say that $f$ is a strong automorphism over $A$, $f \in Saut(A)$, if $f \in Aut(A)$ and for all $a$ and $E \in FE(A)$, $a\ E\ f(a)$.*

**10.2 Lemma.**

*(i) Assume $f \in Aut(A)$ and for all $c \in C$ and $E \in FE(A)$, $c\ E\ f(c)$, then there is $g \in Saut(A)$ such that $f \restriction C \subseteq g$.*

*(ii) Assume $(p, A) \in P_\lambda$. Then $(p, A) \in F^a_\lambda$ iff for all $a \models p$, $stp(a, A) \vdash p$.*

*(iii) Assume $A \subseteq B$. If $stp(a, A) = stp(b, A)$, $a \downarrow_A B$ and $b \downarrow_A B$, then $stp(a, B) = stp(b, B)$.*

*(iv) If $(t(a, B), A) \in F^a_\lambda$, then $stp(a, A) \vdash stp(a, B)$.*

**Proof**. (i): By Exercise 5.2 (ii), choose a model $\mathcal{B} \supseteq A$ so that $\mathcal{B} \downarrow_A C \cup f(C)$. Then $t(C, \mathcal{B}) = t(f(C), \mathcal{B})$ and so there is $g \in Aut(\mathcal{B})$ such that $f \restriction C \subseteq g$. Clearly $g$ is as wanted.

(ii): Assume not. Then there are $a, b \models p$ and $c$ such that $stp(a, A) \vdash p$, stp(b,A)=stp(c,A) and $c \not\models p$. Choose $f \in Aut(dom(p))$ such that $f(b) = a$. Let $a' = f(c)$. Then $stp(a', A) = stp(a, A)$ but $a' \not\models p$, a contradiction.

(iii) Assume not. Choose a model $\mathcal{C} \supseteq B$ such that $\mathcal{C} \downarrow_B a \cup b$. Then by Exercise 4.11 (iii), $t(a,\mathcal{C}) \neq t(b,\mathcal{C})$. Since $a \downarrow_A \mathcal{C}$ and $b \downarrow_A \mathcal{C}$, we have a contradiction.

(iv) Immediate by (ii), (iii) and Exercise 4.11 (iv). $\square$

**10.3 Exercise.** *Show that $stp(a \cup b, A) = stp(a' \cup b, A)$ does NOT imply $stp(a, A \cup b) = stp(a', A \cup b)$. (Hint: $P^{\mathbf{M}} \subseteq \mathbf{M}$ infinite, $R^{\mathbf{M}} \subseteq P^{\mathbf{M}} \times (\mathbf{M} - P^{\mathbf{M}})$, for all $a \in P^{\mathbf{M}}$, $|R(a,\mathbf{M})| = 2$ and $\{R(a,\mathbf{M}) | \, a \in P^{\mathbf{M}}\}$ is a partition of $\mathbf{M} - P^{\mathbf{M}}$.)*

**10.4 Theorem.** *$F^a_\lambda$ satisfies Ax I-VIII and if $\lambda \geq \kappa(T)$, then it satisfies also Ax IX.*

**Proof**. We show Ax VII, the rest is left as an exercise. Assume Ax VII does not hold. By Lemmas 10.2 (i) and (ii), choose $a'$ and $C'$ so that there is $f \in Saut(A)$ such that $f(a' \cup C') = a \cup C$ but $t(a' \cup C', B) \neq t(a \cup C, B)$. Since $(t(C,B), A) \in F^a_\lambda$, $B \downarrow_A C$ and $B \downarrow_A C'$. Let $B' = f(B)$. By Lemma 10.2 (iii) and (i), there is $g \in Saut(A \cup C)$ such that $g(B') = B$. Let $a'' = g(a)$. Then $t(a'', B \cup C) \neq t(a, B \cup C)$ but $stp(a'', A \cup C) = stp(a, A \cup C)$, a contradiction. $\square$

By $\kappa_r(T)$ we mean the least regular cardinal $\geq \kappa(T)$.

**10.5 Lemma.**

*(i) If $A$ is $(F^a_\lambda, \kappa)$-saturated for any (infinite) $\kappa$, then it is a model.*

*(ii) If $\lambda \geq \kappa(T)$, then $\mu(F^a_\lambda) \leq \lambda + |T|^+$.*

*(iii) If for all $B \subseteq A$ of power $< \lambda$ and $a$ there is $b \in A$ such that $stp(b,B) = stp(a,B)$, then $A$ is $F^a_\lambda$-saturated. And if $\lambda \geq \kappa(T)$, then the other direction is true also.*

*(iv) If $T$ is $\lambda$-stable and $A$ is a $\lambda$-saturated model, then $A$ is $F^a_\lambda$-saturated.*

*(v) If $A$ is $F^a_\lambda$-primary over $B$, then $|A| \leq \lambda(T) + (\lambda + |B|)^{<\kappa_r(T)}$.*

**Proof**. (i): Trivial.

(ii): Let $\mu = \lambda + |T|^+$ and $A$ be $(F^a_\lambda, \mu)$-saturated. Assume $B \subseteq A$ and $(t(a,B), C) \in F^a_\lambda$. We show that $t(a,B)$ is realized in $A$. By Ax IX, we may assume that $B = A$. Since the number of formulas over $C$ (modulo equivalence) is $< \mu$ and $A$ is a model, we can find $D$ such that $C \subseteq D \subseteq A$, $|D| < \mu$ and $t(a,D) \vdash stp(a,C)$. Since $(t(a,D), C) \in F^a_\lambda$, there is $b \in A$ such that $t(b,D) = t(a,D)$. Clearly $b$ realizes $t(a,B)$.

(iii): The first claim is trivial, so we prove the second: Let $a$ be arbitrary and $B \subseteq A$ be of power $< \lambda$. We show that $stp(a,B)$ is realized in $A$. Since $\lambda \geq \kappa(T)$, we can choose $C$ and $b$ such that $B \subseteq C \subseteq A$, $|C| < \lambda$, $stp(b,B) = stp(a,B)$ and $stp(b,C) \vdash t(b,A)$. Then $t(b,A)$ is realised in $A$. Clearly this implies the claim.

(iv): We prove the following claim. It is easy to see (exercise) that this suffices.

**Claim.** If $T$ is $\lambda$-stable, $p \in S(A)$, $|A| \leq \lambda$ and $(a_i)_{i<\alpha}$ is a sequence of realizations of $p$ such that for all $i < j < \alpha$, $stp(a_i, A) \neq stp(a_j, A)$, then $|\alpha| \leq \lambda$.

**Proof**. By Exercises 9.4 (ii) and 5.2 (ii), choose a model $\mathcal{B} \supseteq A$ such that $|\mathcal{B}| = \lambda$ and $\mathcal{B} \downarrow_A \cup_{i<\alpha} a_i$. By Exercise 4.11 (iii), for all $i < j < \alpha$, $t(a_i, \mathcal{B}) \neq t(a_j, \mathcal{B})$. Since $T$ is $\lambda$-stable, $|\alpha| \leq \lambda$. $\square$ Claim.

(v): Immediate by (iv) and Lemma 9.6. $\square$

**10.6 Exercise.** *Assume $T$ is $\lambda$-stable, $\mathcal{A}$ is $\lambda$-saturated and $A \subseteq \mathcal{A}$ and $B$ are of power $< \lambda$. Then there is $f \in Saut(A)$ such that $f[B] \subseteq \mathcal{A}$. (Hint: Use Lemma 10.5 and the fact that if $stp(a,A) = stp(b,A)$, then $t(a, A \cup b) \vdash stp(a,A)$.)*

**10.7 Lemma.** *Assume $x = a$ and $\lambda \geq \kappa_r(T)$ or $x = s$ and $\lambda > |T|$. If $\mathcal{A}$ is $F^x_\lambda$-saturated, $\mathcal{A} \subseteq B \cap D$, $D \downarrow_{\mathcal{A}} B$ and $(B, (c_i, C_i)_{i<\alpha})$ is an $F^x_\lambda$-construction over $B$, then $(B \cup D, (c_i, C_i)_{i<\alpha})$ is an $F^x_\lambda$-construction over $B \cup D$.*

**Proof**. We prove the first case, the other is similar. Assume not. Then we can find $F^a_\lambda$-saturated $\mathcal{A}$, $B$, $B'$, $D$, $a$ and $b$ such that $\mathcal{A} \subseteq B \cap D$, $D \downarrow_{\mathcal{A}} B$, $(t(a,B), B') \in F^a_\lambda$, $stp(b,B') = stp(a,B')$ and $t(b, B \cup D) \neq t(a, B \cup D)$. Clearly we may assume that $d = D - \mathcal{A}$ is finite, $B' \downarrow_{\mathcal{A} \cap B'} \mathcal{A}$ and $t(b, B' \cup d) \neq t(a, B' \cup d)$. By Lemma 10.5 (iii), choose $d' \in \mathcal{A}$ such that $stp(d', \mathcal{A} \cap B') = stp(d, \mathcal{A} \cap B')$. By Lemma 10.2 (iii) and (i), there is $f \in Saut(B')$ such that $f(d) = d'$. Then $t(f(b), B) \neq t(a, B)$ or $t(f(a), B) \neq t(a, B)$. Clearly this contradicts the assumption that $(t(a,B), B') \in F^a_\lambda$. □

**10.8 Definition.** *We write $A \triangleright_B C$ ($A$ dominates $C$ over $B$) if for all $d$, $d \downarrow_B A$ implies $d \downarrow_B C$.*

**10.9 Exercise.**

*(i) Assume $x = a$ and $\lambda \geq \kappa_r(T)$ or $x = s$ and $\lambda > |T|$. If $\mathcal{A}$ is $F^x_\lambda$ saturated and $C$ is $F^x_\lambda$-constructible over $\mathcal{A} \cup B$, then $B \triangleright_{\mathcal{A}} C$. (Hint: Use Lemma 10.7.)*

*(ii) Assume $B \subseteq A$ and $a \cup b \downarrow_B A$. Then $a \triangleright_A b$ iff $a \triangleright_B b$.*

## 11. Structure of $a$-saturated models

In this chapter we prove a structure theorem for $a$-saturated models assuming that $T$ is superstable and does not have dop.

Through out this section we assume that $T$ is superstable (i.e. $\kappa(T) = \omega$). We write $a$-primary, $a$-saturated etc. for $F^a_{\kappa(T)}$-primary, $F^a_{\kappa(T)}$-saturated etc. If $(P, <)$ is a tree without branches of length $> \omega$ and $t \in P$ is not the root, then by $t^-$ we mean the unique immediate predecessor of $t$.

**11.1 Definition.**

*(i) We say that $p \in S(A)$ is (almost) orthogonal to $B \subseteq A$ if $p$ is (almost) orthogonal to every $q \in S(A)$ which does not fork over $B$.*

*(ii) We say that $\{a_i |\ i < \alpha\}$ is $A$-independent if for all $i < \alpha$ $a_i \downarrow_A \cup\{a_j |\ j < \alpha,\ j \neq i\}$.*

*(iii) We say that $(P, f, g)$ is a decomposition of $\mathcal{A}$ if the following holds:*

*(a) $P = (P, <)$ is a tree without branches of length $> \omega$, $f : P - \{r\} \to \mathcal{A}$, where $r$ is the root of $P$, and $g : P \to \{A |\ A \subseteq \mathcal{A}\}$,*

*(b) $g(r)$ is an $a$-primary model over $\emptyset$,*

*(c) for all $t \in P$, $\{f(u) |\ u^- = t\}$ is a maximal $g(t)$ independent set of sequences from $\mathcal{A}$ such that $t(f(u), g(t))$ is not algebraic and if $t \neq r$, then also (e) below holds,*

*(d) for all $t, u \in P$, if $u^- = t$, then $g(u)$ is $a$-primary over $g(t) \cup f(u)$,*

*(e) for all $t$, $u$ and $v$ from $P$, if $u^- = t$ and $t^- = v$, then $t(f(u), g(t))$ is orthogonal to $g(v)$.*

**11.2 Exercise.**

*(i) If $\mathcal{A}$ is $a$-saturated, $B \subseteq \mathcal{A}$ and $p \in S(\mathcal{A})$, then $p$ is orthogonal to $B$ iff $p$ is almost orthogonal to $B$. (Hint: See the proof of Lemma 10.7.)*

*(ii) Show that $\{a_i \mid i < \alpha\}$ is $A$-independent iff for all $i < \alpha$ $a_i \downarrow_A \cup\{a_j \mid j < i\}$.*

*(iii) Show that for all $a$-saturated $\mathcal{A}$, there exists a decomposition of $\mathcal{A}$.*

*(iv) Assume $(P, f, g)$ is a decomposition of $\mathcal{A}$. If $t \in P$ is not the root, then $g(t) \downarrow_{g(t^-)} \cup\{g(u) \mid u \in P,\ t \not\leq u\}$. (Hint: Clearly it is enough to show that for all finite downwards closed $P' \subseteq P$, the claim holds for $(P', f \restriction P', g \restriction P')$. Prove this by induction on $|P'|$.)*

**11.3 Definition.**

*(i) Assume $\mathcal{A}$ is $a$-saturated. We say that a non-algebraic type $t(a, \mathcal{A})$ is a c-type (c for compulsion) if the following holds: If $\mathcal{B} \subseteq \mathcal{A}$ is $a$-saturated and $t(a, \mathcal{A})$ is not orthogonal to $\mathcal{B}$, then there is $b \notin \mathcal{A}$ such that $b \downarrow_{\mathcal{B}} \mathcal{A}$ and $a \triangleright_{\mathcal{A}} b$.*

*(ii) We say that $p \in S(A)$ is regular if the following holds: if $q \in S(B)$ is a non-forking extension of $p$ and $r \in S(B)$ is a forking extension of $p$, then $q$ is orthogonal to $r$.*

Given $A \subseteq \mathcal{A}$ and $p \in S(A)$, it would be nice if we could define a dimension of $p(\mathcal{A})$ by using forking as a dependence relation. However, this is not possible, since not all the axioms of the general dependence relation are satisfied, transitivity is lacking. Regularity is a property designed to give the transitivity.

We want to mention also, that if in the definition of c-type we replace domination by compulsion (whatever it is) we can give a marginally simpler proof for the structure theorem. We do not do this because domination is a widely used concept and compulsion is not. The notion of c-type is used only by the author.

**11.4 Lemma.** *If $\mathcal{A} \subseteq \mathcal{B}$ are $a$-saturated, $\mathcal{A} \neq \mathcal{B}$, then there is $a \in \mathcal{B}$ such that $t(a, \mathcal{A})$ is a c-type.*

**Proof**. Since $T$ is superstable, we can find finite $A \subseteq \mathcal{A}$ and $a \in \mathcal{B}$ such that $a \notin \mathcal{A}$ and for all $A' \subseteq \mathcal{A}$ and $a' \in \mathcal{B}$, if $t(a' \cup A', \emptyset) = t(a \cup A, \emptyset)$ and $a' \notin \mathcal{A}$, then $a' \downarrow_{A'} \mathcal{A}$. We show that $t(a, \mathcal{A})$ is a c-type. For this let $\mathcal{C} \subseteq \mathcal{A}$ be $a$-saturated and assume that $t(a, \mathcal{A})$ is not orthogonal to $\mathcal{C}$. By Exercise 11.2 (i), choose $c$ so that $c \downarrow_{\mathcal{C}} \mathcal{A}$ and $c \not\downarrow_{\mathcal{A}} a$. Without loss of generality we may assume that $A \downarrow_{A \cap \mathcal{C}} \mathcal{C} \cup c$ and $c \not\downarrow_A a$. Notice that then $A \cup a \downarrow_{A \cap \mathcal{C}} \mathcal{C}$.

By Exercise 11.2 (ii), choose $(A \cap \mathcal{C}) \cup c$-independent set $I = \{a_i \cup A_i \mid i < \omega\}$ of realizations of $t(a \cup A, (A \cap \mathcal{C}) \cup c)$ such that $a_0 = a$ and $A_0 = A$. Then $I$ is not $A \cap \mathcal{C}$-independent, since otherwise for all $i < \omega$, $c \not\downarrow_{\cup_{j<i} a_j \cup A_j} a_i \cup A_i$. Let $n < \omega$ be the least number such that every $J \subseteq I$ of power $n$ is $A \cap \mathcal{C}$-independent. Without loss of generality we may assume that $a_0 \cup A_0 \not\downarrow_{\cup_{0<i<n} a_i \cup A_i} a_n \cup A_n$. Then

$$(*) \quad a_0 \not\downarrow_{A_0} \cup_{0<i\leq n} a_i \cup A_i.$$

By the choice of $n$, $a_0 \cup A_0 \downarrow_{A\cap\mathcal{C}} A_n \cup \bigcup_{0<i<n} a_i \cup A_i$.

For all $0 < i < n$, choose $b_i \in \mathcal{C}$ and $B_i \subseteq \mathcal{C}$ and $B_n \subseteq \mathcal{C}$ such that

$$stp(B_n \cup \bigcup_{0<i<n} b_i \cup B_i, A \cap \mathcal{C}) = stp(A_n \cup \bigcup_{0<i<n} a_i \cup A_i, A \cap \mathcal{C}).$$

Then

$$t(B_n \cup \bigcup_{0<i<n} b_i \cup B_i, A \cup a) = t(A_n \cup \bigcup_{0<i<n} a_i \cup A_i, A \cup a).$$

Let $\mathcal{D} \subseteq \mathcal{B}$ be $a$-primary over $\mathcal{A} \cup a$. Then we can find $b \in \mathcal{D}$ such that

$$t(b \cup B_n \cup \bigcup_{0<i<n} b_i \cup B_i, A \cup a) = t(a_n \cup A_n \cup \bigcup_{0<i<n} a_i \cup A_i, A \cup a).$$

By (*), $b \notin \mathcal{A}$. By the choice of $A$ and $a$, $b \downarrow_{B_n} \mathcal{A}$, especially $b \downarrow_{\mathcal{C}} \mathcal{A}$. Since $b \in \mathcal{D}$, by Exercise 10.8 (i) $a \triangleright_{\mathcal{A}} b$. □

**11.5 Exercise$^*$.** *Let $a$ and $\mathcal{A}$ be as in the proof of Lemma 11.4. Show that $t(a, \mathcal{A})$ is regular. (Hint: Show first that $t(a, A)$ is regular.)*

**11.6 Fact.** *([Sh]) Regular types over $a$-saturated models are c-types.*

**11.7 Definition.** *We say that $T$ has dop (dimensional order property) if there are $a$-saturated $\mathcal{A}_i$, $i < 4$, and $p \in S(\mathcal{A}_3)$ such that*
*(i) $\mathcal{A}_0 \subseteq \mathcal{A}_1 \cap \mathcal{A}_2$ and $\mathcal{A}_1 \downarrow_{\mathcal{A}_0} \mathcal{A}_2$,*
*(ii) $\mathcal{A}_3$ is $a$-primary over $\mathcal{A}_1 \cup \mathcal{A}_2$,*
*(iii) $p$ is orthogonal to $\mathcal{A}_1$ and to $\mathcal{A}_2$.*
*We say that $T$ has ndop if it does not have dop.*

**11.8 Fact.** *([Sh]) Assume $T$ is $\lambda$-stable, has dop and $\lambda > \mu \geq \kappa_r(T)$. Then $T$ has $2^\lambda$ non-isomorphic $F^a_\mu$-saturated models of cardinality $\lambda$.*

**11.9 Theorem.** *Assume $T$ is superstable with ndop, $\mathcal{A}$ is $a$-saturated and $(P, f, g)$ is a decomposition of $\mathcal{A}$. If $\mathcal{B} \subseteq \mathcal{A}$ is $a$-primary over $\cup_{t\in P} g(t)$, then $\mathcal{B} = \mathcal{A}$.*

**Proof**. Assume not. Choose $a \in \mathcal{A}$ such that $a \notin \mathcal{B}$. By Theorem 7.6, we can find finite downwards closed $P^* \subseteq P$ and $\mathcal{C} \subseteq \mathcal{B}$ such that $\mathcal{C}$ is $a$-primary over $\cup_{t\in P^*} g(t)$ and $a \downarrow_{\mathcal{C}} \mathcal{B}$. So choose $a$ so that in addition $|P^*|$ is minimal. Let $\mathcal{D} \subseteq \mathcal{A}$ be $a$-primary over $\mathcal{C} \cup a$. By Lemma 11.4, pick $b \in \mathcal{D}$ such that $t(b, \mathcal{C})$ is a c-type. Then $b \downarrow_{\mathcal{C}} \mathcal{B}$ and $b \notin \mathcal{B}$. There are three cases:

1. There is no $t \in P^*$ such that $P^* = \{u \in P^* | \ u \leq t\}$. Let $t$ be a leaf of $P^*$ and $P' = P^* - \{t\}$. By Theorem 7.11 and Lemma 10.7, we can find $\mathcal{C}' \subseteq \mathcal{C}$ such that it is $a$-primary over $\cup_{u\in P'} g(u)$ and $\mathcal{C}$ is $a$-primary over $g(t) \cup \mathcal{C}'$. By ndop, $t(b, \mathcal{C})$ is not orthogonal to $\mathcal{C}'$ or to $g(t)$. We assume that $t(b, \mathcal{C})$ is not orthogonal to $\mathcal{C}'$, the other case is similar. Since $t(b, \mathcal{C})$ is a c-type, we can find $c' \notin \mathcal{C}$ such

that $c' \downarrow_{\mathcal{C}'} \mathcal{C}$ and $b \triangleright_{\mathcal{C}} c'$. By Exercise 10.9 (ii), we can find $c$ from $\mathcal{A}$ so that $c \downarrow_{\mathcal{C}'} \mathcal{B}$ and $c \notin \mathcal{B}$. This contradicts the choice of $a$ and $P^*$.

2. There is $t \in P^*$ such that $P^* = \{u \in P^* | \ u \le t\}$, $t$ is not the root of $P$ and $t(b, \mathcal{C})$ is not orthogonal to $g(t^-)$. As in case 1 above, we get a contradiction with the choice of $a$ and $P^*$.

3. There is $t \in P^*$ such that $P^* = \{u \in P^* | \ u \le t\}$ and $t$ is the root of $P$ or $t(b, \mathcal{C})$ is orthogonal to $g(t^-)$. Clearly this contradicts (c) in the definition of decomposition. □

**11.10 Exercise$^*$.**

*(i) Show, without using Theorem 7.11, that if Theorem 11.9 holds, $\mathcal{A}$ and $\mathcal{B}$ are $a$-saturated and $(P, f, g)$ is a decomposion of both $\mathcal{A}$ and $\mathcal{B}$, then $\mathcal{A} \cong \mathcal{B}$.*

*(ii) Assume $(P, f, g)$ is a decomposition of $\mathcal{A}$, $(P', f', g')$ is a decomposition of $\mathcal{A}'$, $h : (P, <) \to (P', <')$ is an isomorphism and $H : \cup_{t \in P} g(t) \to \cup_{t \in P'} g(t)$ is such that for all $t \in P$, $H \restriction g(t)$ is an isomorphism onto $g(h(t))$. Then $H$ is elementary.*

*(iii) Show that we can add (f) below to the definition of decomposition and still prove Theorem 11.9:*

*(f) if $t \in P$ is not the root, then $t(f(t), g(t^-))$ is regular.*

## 12. A non-structure theorem for strictly stable theories

In this chapter we prove the following theorem:

**12.1 Theorem.** *Assume $T$ is a stable unsuperstable theory and $\kappa = cf(\kappa) > (2^{|T|})^+$. Then there are models $\mathcal{A}_i$, $i < 2^\kappa$, such that for all $i < 2^\kappa$, $|\mathcal{A}_i| = \kappa$ and for all $i < j < 2^\kappa$, $\mathcal{A}_i \not\cong \mathcal{A}_j$.*

Theorem 12.1 holds for all unsuperstable theories (even $\kappa = cf(\kappa) > (2^{|T|})^+$ replaced by $\kappa > |T|$). We assume stability since this makes it possible for us to prove the theorem by using forking and primary models which are the topic of this paper. The proof is from [HS1]. Notice that below we construct the models $\mathcal{A}_i$ so that they are $F^a_\omega$-saturated (and more).

Through out this section, we assume that $T$ is a stable unsuperstable theory. Let $\lambda = (2^{|T|})^+$ We write $s$-primary, $s$-saturated etc. for $F^s_\lambda$-primary, $F^s_\lambda$-saturated etc. We say that $t(a, A)$ $s$-isolates $t(a, B)$ if $(t(a, B), A) \in F^s_\lambda$.

Let $J \subseteq \kappa^{\le\omega}$ be such that it is closed under initial segments. If $\eta, \xi \in J$ then by $r'(\eta, \xi)$ we mean the longest element of $J$ which is an initial segment of both $\eta$ and $\xi$. If $u, v \in I = P_\omega(J)$ (=the set of all finite subsets of $J$) then by $r(u, v)$ we mean the largest set $R$ which satisfies

(i) $R \subseteq \{r'(\eta, \xi) | \ \eta \in u, \ \xi \in v\}$

(ii) if $\eta \in R$, $\xi \in u$, $\tau \in v$ and $\eta$ is an initial segment of $r'(\xi, \tau)$, then $\eta = r'(\xi, \tau)$.

We order $I$ by $u \le v$ if for every $\eta \in u$ there is $\xi \in v$ such that $\eta$ is an initial segment of $\xi$ i.e. $r(u, v) = r(u, u)$ $(= \{\eta \in u | \ \neg\exists \xi \in u(\eta \text{ is a proper initial segment of } \xi)\})$.

**12.2 Definition.** *Assume $J \subseteq \kappa^{\leq\omega}$ is closed under initial segments and $I = P_\omega(J)$. Let $\Sigma = \{A_u | \ u \in I\}$ be an indexed family of sets. We say that $\Sigma$ is strongly independent if*

*(i) for all $u, v \in I$, $u \leq v$ implies $A_u \subseteq A_v$,*

*(ii) if $u, u_i \in I$, $i < n$, and $B \subseteq \cup_{i<n} A_{u_i}$ has power $< \lambda$, then there is an automorphism $f = f^{\Sigma,B}_{(u,u_0,\dots,u_{n-1})}$ (of $\mathbf{M}$) such that $f \restriction (B \cap A_u) = id_{B\cap A_u}$ and $f(B \cap A_{u_i}) \subseteq A_{r(u,u_i)}$.*

The model construction in Lemma 12.3 below is a generalized version of the construction used in [Sh1] XII.4.

**12.3 Lemma.** *Assume that $\Sigma = \{A_u | \ u \in I\}$, $I = P_\omega(J)$, is strongly independent. Then there are sets $\mathcal{A}_u$, $u \in I$, such that*

*(i) for all $u, v \in I$, $u \leq v$ implies $\mathcal{A}_u \subseteq \mathcal{A}_v$,*

*(ii) for all $u \in I$, $\mathcal{A}_u$ is $s$-primary over $A_u$, (and so by (i), $\cup_{u\in I}\mathcal{A}_u$ is a model),*

*(iii) if $v \leq u$, then $\mathcal{A}_u$ is $s$-atomic over $\cup_{u\in I} A_u$ and $s$-primary over $\mathcal{A}_v \cup A_u$,*

*(iv) if $J' \subseteq J$ is closed under initial segments and $u \in P_\omega(J')$, then $\cup_{v\in P_\omega(J')}\mathcal{A}_v$ is $s$-constructible over $\mathcal{A}_u \cup \bigcup_{v\in P_\omega(J')} A_v$.*

**Proof**. Let $\{u_i | \ i < \alpha^*\}$ be an enumeration of $I$ such that $u \leq v$ and $v \not\leq u$ implies $i < j$. It is easy to see that we can choose $\alpha$, $\gamma_i < \alpha$ for $i < \alpha^*$, $a_\gamma$ and $B_\gamma$ for $\gamma < \alpha$, and $s : \alpha \to I$ so that

(a) $\gamma_0 = 0$ and $(\gamma_i)_{i<\alpha^*}$ is increasing and continuous,

(b) if $\gamma_i \leq \gamma < \gamma_{i+1}$, then $s(\gamma) = u_i$,

(c) for all $\gamma < \alpha$, $|B_\gamma| < \lambda$ and if we write for $\gamma \leq \alpha$, $A^\gamma_u = A_u \cup \{a_\delta | \ \delta < \gamma,\ s(\delta) \leq u\}$, then $B_\gamma \subseteq A^\gamma_{s(\gamma)}$,

(d) for all $\gamma < \alpha$, if we write $A^\gamma = \cup_{u\in I} A^\gamma_u$, then $t(a_\gamma, B_\gamma)$ $s$-isolates $t(a_\gamma, A^\gamma)$,

(e) for all $i < \alpha^*$, there are no $a$ and $B \subseteq A^{\gamma_{i+1}}_{u_i}$ of power $< \lambda$ such that $t(a, B)$ $s$-isolates $t(a, A^{\gamma_{i+1}})$,

(f) if $a_\delta \in B_\gamma$, then $B_\delta \subseteq B_\gamma$.

For all $u \in I$, we define $\mathcal{A}_u = A^\alpha_u$. We show that these are as wanted.

(i) follows immediately from the definitions and for (ii) it is enough to prove the following claim (Claim (III) implies (ii) easily).

**Claim.** For all $i < \alpha^*$,

(I) $\Sigma_i = \{A^{\gamma_i}_u | \ u \in I\}$ is strongly independent, we write $f^{i,B}_{(u,u_0,\dots,u_{n-1})}$ instead of $f^{\Sigma_i,B}_{(u,u_0,\dots,u_{n-1})}$,

(II) the functions $f^{i,B}_{(u,u_0,\dots,u_{n-1})}$ can be chosen so that if $j < i$, $u, u_k \in I$, $k < n$, $B \subseteq \cup_{i<n} A^{\gamma_i}_{u_k}$ has power $< \lambda$ and $a_\gamma \in B$ implies $B_\gamma \subseteq B$ and $B' = B \cap A^{\gamma_j}$, then $f^{i,B}_{(u,u_0,\dots,u_{n-1})} \restriction B' = f^{j,B'}_{(u,u_0,\dots,u_{n-1})} \restriction B'$,

(III) if $j < i$, then $A^{\gamma_{j+1}}_{u_j}$ is $s$-saturated,

**Proof**. Notice that if $a_\gamma \in A^\delta_u \cap A^\delta_v$, then $a_\gamma \in A^\delta_{r(u,v)}$. Similarly we see that the first half of (I) in the claim is always true (i.e. if $u \leq v$ then for all $\delta < \alpha$,

$A^{\delta}_u \subseteq A^{\delta}_v$.) We prove the rest by induction on $i < \alpha^*$. We notice first that it is enough to prove the existence of $f^{i,B}_{(u,u_0,...,u_{n-1})}$ only in the case when $B$ satisfies

(*) if $a_\gamma \in B$, then $B_\gamma \subseteq B$.

For $i = 0$, there is nothing to prove. If $i$ is limit, then the claim follows easily from the induction assumption (use (II) in the claim). So we assume that the claim holds for $i$ and prove it for $i+1$. We prove first (I) and (II). For this let $u, u_k \in I$, $k < n$, and $B \subseteq \cup_{k<n} A^{\gamma_{i+1}}_{u_k}$ be of power $< \lambda$ such that (*) above is satisfied. If for all $k < n$, $s(\gamma_i) \not\leq u_k$, then (I) and (II) in the claim follow immediately from the induction assumption. So we may assume that $s(\gamma_i) \leq u_0$. Let $B' = B \cap (\cup_{k<n} A^{\gamma_i}_{u_k})$. By the induction assumption there is an automorphism $f = f^{i,B'}_{(u,u_0,...,u_{n-1})}$ such that $f \restriction (B' \cap A^{\gamma_i}_u) = id_{B' \cap A^{\gamma_i}_u}$ and $f(B' \cap A^{\gamma_i}_{u_k}) \subseteq A^{\gamma_i}_{r(u,u_k)}$. If $s(\gamma_i) \leq u$, then, by (*) and (d) in the construction, we can find an automorphism $g = f^{i+1,B}_{(u,u_0,...,u_{n-1})}$ such that $g \restriction B' = f \restriction B'$ and $g \restriction (B - B') = id_{B-B'}$. Clearly this is as wanted.

So we may assume that $s(\gamma_i) \not\leq u$. Since $s(\gamma_i) \leq u_0$, $u_0 \not\leq r(u, u_0)$. By the choice of the enumeration of $I$ there is $j < i$ such that $u_j = r(u, u_0)$. Then by the induction assumption (part (III)), $A^{\gamma_{i+1}}_{u_j} = A^{\gamma_i}_{u_j} = A^{\gamma_{j+1}}_{u_j}$ is $s$-saturated and by the choice of $f$, $f(B' \cap A^{\gamma_i}_{u_0}) \subseteq A^{\gamma_i}_{u_j}$. So by (d) in the construction and (*) above, there are no difficulties in finding the required automorphism $f^{i+1,B}_{(u,u_0,...,u_{n-1})}$.

So we need to prove (III): For this it is enough to show that $A^{\gamma_{i+1}}_{u_i}$ is $s$-saturated. Assume not. Then there are $a$ and $B$ such that $B \subseteq A^{\gamma_{i+1}}_{u_i}$, $|B| < \lambda$ and $t(a, B)$ is not realized in $A^{\gamma_{i+1}}_{u_i}$. Since $\lambda \geq \lambda(T)$, there are $b$ and $C$ such that $B \subseteq C \subseteq A^{\gamma_{i+1}}_{u_i}$, $|C| < \lambda$, $t(b, B) = t(a, B)$ and $t(b, C)$ $s$-isolates $t(b, A^{\gamma_{i+1}}_{u_i})$. But since (I) in the claim holds for $i+1$, $t(b, C)$ $s$-isolates $t(b, A^{\gamma_{i+1}})$. This contradicts (e) in the construction. □ Claim

(iii) and (iv) follow immediately from the construction, Claim (III) and Lemma 7.9. □

Since $T$ is unsuperstable, there are $a$ and sets $\mathcal{A}_i$, $i < \omega$, such that

(i) if $j < i < \omega$, then $\mathcal{A}_j \subseteq \mathcal{A}_i$,

(ii) for all $i < \omega$, $a \not\downarrow_{\mathcal{A}_i} \mathcal{A}_{i+1}$.

It is easy to see that we may choose the sets $\mathcal{A}_i$ so that they are $s$-saturated models and of power $\lambda$. Let $\mathcal{A}_\omega$ be $s$-primary over $a \cup \bigcup_{i<\omega} \mathcal{A}_i$. As in the proof of Lemma 9.3, for all $\eta \in \kappa^{\leq\omega}$, we can find $\mathcal{A}_\eta$ such that

(a) for all $\eta \in \kappa^{\leq\omega}$, there is an automorphism $f_\eta$ such that $f_\eta(\mathcal{A}_{length(\eta)}) = \mathcal{A}_\eta$,

(b) if $\eta$ is an initial segment of $\xi$, then $f_\xi \restriction \mathcal{A}_{length(\eta)} = f_\eta \restriction \mathcal{A}_{length(\eta)}$,

(c) if $\eta \in \kappa^{<\omega}$, $\alpha \in \kappa$ and $X$ is the set of those $\xi \in \kappa^{\leq\omega}$ such that $\eta \frown (\alpha)$ is an initial segment of $\xi$, then

$$\cup_{\xi\in X} \mathcal{A}_\xi \downarrow_{\mathcal{A}_\eta} \cup_{\xi\in(\kappa^{\leq\omega}-X)} \mathcal{A}_\xi.$$

For all $\eta \in \kappa^\omega$, we let $a_\eta = f_\eta(a)$.

**12.4 Exercise.** *Assume* $\eta \in \kappa^{<\omega}$, $\alpha \in \kappa$ *and* $X$ *is the set of those* $\xi \in \kappa^{<\omega}$ *such that* $\eta \frown (\alpha)$ *is an initial segment of* $\xi$. *Let* $B \subseteq \cup_{\xi \in (\kappa^{\leq\omega} - X)} \mathcal{A}_\xi$ *and* $C \subseteq \cup_{\xi \in X} \mathcal{A}_\xi$ *be of power* $< \lambda$. *Then there is* $C' \subseteq \mathcal{A}_\eta$ *such that* $t(C', B) = t(C, B)$. *(Hint: Use Exercise 10.6.)*

**12.5 Lemma.** *Assume* $J \subseteq \kappa^{\leq\omega}$ *and* $I = P_\omega(J)$. *For all* $u \in I$, *define* $A_u = \cup_{\eta \in u} \mathcal{A}_\eta$. *Then* $\{A_u | \ u \in I\}$ *is strongly independent.*

**Proof.** Follows immediately from Exercise 12.4. □

For each $\alpha < \kappa$ of cofinality $\omega$, let $\eta_\alpha \in \kappa^\omega$ be a strictly increasing sequence such that $\cup_{i<\omega} \eta_\alpha(i) = \alpha$. Let $S \subseteq \{\alpha < \kappa | \ cf(\alpha) = \omega\}$. By $J_S$ we mean the set

$$\kappa^{<\omega} \cup \{\eta_\alpha | \ \alpha \in S\}.$$

Let $I_S = P_\omega(J_S)$ and $\mathcal{A}_S$ be the model given by Lemmas 12.3 and 12.5 for $\{A_u | \ u \in I_S\}$.

**12.6 Exercise.**

*(i) Assume* $\eta \in \kappa^{<\omega}$, $u \in I_S$, $\alpha < \kappa$, $\{\eta\} \leq u$ *and* $\{\eta \frown (\alpha)\} \not\leq u$. *Let* $X$ *be the set of those* $\xi \in J_S$ *such that* $\eta \frown (\alpha)$ *is an initial segment of* $\xi$. *Then*

$$\cup_{\xi \in X} \mathcal{A}_\xi \downarrow_{\mathcal{A}_u} \cup_{\xi \in J_S - X} \mathcal{A}_\xi.$$

*(ii) Assume* $\alpha \in \kappa$, $u \in I_S$ *and* $v \in P_\omega(J_S \cap \alpha^{\leq\omega})$ *is maximal such that* $v \leq u$. *Then*

$$\mathcal{A}_u \downarrow_{\mathcal{A}_v} \cup_{w \in P_\omega(J_S \cap \alpha^{\leq\omega})} \mathcal{A}_w.$$

*(Hint: Use Lemma 12.3 and Exercise 10.8.)*

**12.7 Lemma.** *Assume* $S, R \subseteq \{\alpha < \kappa | \ cf(\alpha) = \omega\}$ *are such that* $(S - R) \cup (R - S)$ *is stationary. Then* $\mathcal{A}_S$ *is not isomorphic to* $\mathcal{A}_R$.

**Proof.** Assume not. Let $f : \mathcal{A}_S \to \mathcal{A}_R$ be an isomorphism. We write $I_S^\alpha$ for the set of those $u \in I_S$, which satisfy that for all $\xi \in u$, $\cup_{i<length(\xi)} \xi(i) < \alpha$. $I_R^\alpha$ is defined similarly. Then we can find $\alpha$ and $\alpha_i$, $i < \omega$, such that $(\alpha_i)_{i<\omega}$ is strictly increasing, for all $i < \omega$, $f(\cup_{u \in I_S^{\alpha_i}} \mathcal{A}_u) = \cup_{u \in I_R^{\alpha_i}} \mathcal{A}_u$ and $\alpha = \cup_{i<\omega} \alpha_i \in (S - R) \cup (R - S)$. Without loss of generality we may assume that $\alpha \in S - R$, and so $\eta_\alpha \in J_S$. Let $\mathcal{A}_S^{\alpha_i} = \cup_{u \in I_S^{\alpha_i}} \mathcal{A}_u$ and $\mathcal{A}_R^{\alpha_i} = \cup_{u \in I_R^{\alpha_i}} \mathcal{A}_u$. Then it easy to see that for all $i < \omega$ there is $j < \omega$ such that $a_{\eta_\alpha} \not\downarrow_{\mathcal{A}_S^{\alpha_i}} \mathcal{A}_S^{\alpha_j}$. So there is $u \in I_R$ such that for all $i < \omega$ there is $j < \omega$ such that $\mathcal{A}_u \not\downarrow_{\mathcal{A}_R^{\alpha_i}} \mathcal{A}_R^{\alpha_j}$. Since $\alpha \notin R$, this contradicts Exercise 12.6 (ii). □

We can now prove Theorem 12.1: By [Sh1] Appendix 1 Theorem 1.3 (2) and (3), there are stationary $S_i \subseteq \{\alpha < \kappa | \ cf(\alpha) = \omega\}$, $i < \kappa$, such that for all $i < j < \kappa$, $S_i \cap S_j = \emptyset$. For all $X \subseteq \kappa$, let $\mathcal{A}_X = \mathcal{A}_{\cup_{i \in X} S_i}$. Then by Lemma 12.7, if $X \neq X'$, then $\mathcal{A}_X$ is not isomorphic to $\mathcal{A}_{X'}$. Since clearly $|J_{\cup_{i \in X} S_i}| = \kappa$, $|\mathcal{A}_X| = \kappa$. □ Theorem 12.1.

# APPENDIX

## A. $M^{eq}$ and canonical bases

In this section, in order to simplify the notations, we assume that $L$ is relational and that every formula is equivalent either to some atomic formula or $\exists v_0(v_0 = v_0)$ or $\neg\exists v_0(v_0 = v_0)$. This assumption is w.o.l.g. (change the vocabulary if necessary - this is know as Morleyization).

We start by noticing that if $\phi(x, y)$ is a formula and in $\mathcal{A} \models T$ it defines an equivalence relation on $\mathcal{A}^n$, then it defines an equivalence relation in every model of $T$.

Let $EQ^n$ be the set of all equivalence relations on $M^n$ definable over $\emptyset$ and $EQ = \bigcup_{n<\omega} EQ^n$. For every model $\mathcal{A}$ we define $\mathcal{A}^{eq}$ as follows: We let $L^{eq} = L \cup \{S_E, F_E | \ E \in EQ\}$ where $S_E$ is a new unary relation symbol, $F_E$ is a new function symbol of arity $n$ if $E \in EQ^n$. The universe of $\mathcal{A}^{eq}$ consists of $\mathcal{A}$ together with the equivalence classes $a/E$ where $E \in EQ^n$, $E$ is not an identity, and $a \in \mathcal{A}^n$ and still assuming that $E$ is not the identity, $S_E$ is interpreted as the set $\{a/E | \ a \in \mathcal{A}\}$ and $F_E(a) = a/E$ if $a \in \mathcal{A}$ and otherwise $F(a) = a_1$, where $a = (a_1, ..., a_n)$. The interpretation of $S_=$ is $\mathcal{A}$ and $F_=(a) = a$. Finally, the interpretations of relation symbols $R \in L$ are the same as in $\mathcal{A}$. We let $T^{eq} = Th(M^{eq})$.

**A.1 Exercise.**

*(i) Show that for all $f \in Aut(\mathcal{A})$ there is unique $g \in Aut(\mathcal{A}^{eq})$ such that $f \subseteq g$.*

*(ii) If $\mathcal{A} \preceq \mathcal{B}$, then there is a unique elementary embedding $f : \mathcal{A}^{eq} \to \mathcal{B}^{eq}$ such that $f \restriction \mathcal{A} = id$. Also if $\mathcal{A}^{eq} \preceq \mathcal{B}^{eq}$, then $\mathcal{A} \preceq \mathcal{B}$. Conclude that for all $\mathcal{A}$, $\mathcal{A}^{eq} \models T^{eq}$. (Hint: Use Ehrenfeuch-Fraïssé games, see e.g. [Hy2].)*

*(iii) Show that $M^{eq}$ is not saturated.*

*(iv) Show that there is saturated $M'$ such that $M^{eq} \preceq M'$ and for all $E \in EQ$, the interpretation of $S_E$ in $M'$ is the same as in $M^{eq}$.*

*(v) Show that if $T$ is $\xi$-stable, then so is $T^{eq}$.*

*(vi) Let $p$ be an $L$-type over $B \subseteq M$ such that it is realized in $M$ and $A \subseteq M$. Show that $p$ does not fork over $A$ in the sense of $M$ iff $p$ does not fork over $A$ in the sense of $M^{eq}$.*

We use $M^{eq}$ as the monster model for $T^{eq}$ and not $M'$ from Exercise A.1 (iv).

**A.2 Exercise.**

*(i) Show that every $L^{eq}$-formula is equivalent to a boolean combination of formulas of the form:*

*(a) $\exists v_0(v_0 = v_0)$,*

*(b) $x = y$,*

*(c) $S_E(x)$,*

*(d) $\wedge_{i\leq n} S_{E_i}(x_i) \to \forall y_0 ... \forall y_n(\wedge_{i\leq n} F_{E_i}(y_i) = x_i \to R(y_0, ..., y_n))$, where $R \in L$. (Hint: Standard proof of the elimination of quantifiers, see e.g. [Hy2].)*

*(ii) Show that for all $L^{eq}$-formulas $\phi(x)$, $x = (x_0, ..., x_n)$, and $E_i$, $i \leq n$, there is an $L$-formula $\phi^*(y_0, ..., y_n)$ such that for all $a_i \in M$, $M \models \phi^*(a_0, ..., a_n)$ iff $M^{eq} \models \phi(a_0/E_0, ..., a_n/E_n)$.*

**A.3 Definition.**

*(i) We say that $A \subseteq M$ is a canonical base of $p \in S(M)$ if for all $f \in Aut(M)$, $f \restriction A = id$ iff $f(p) = p$.*

*(ii) Suppose $p \in S(B)$ is stationary. We say that $A \subseteq M$ is a canonical base of $p$ if $A$ is a canonical base of the unique non-forking extension $q \in S(M)$ of $p$.*

*(iii) For $A \subseteq M$, by definable closure $dcl(A)$ of $A$ we mean the set of all elements $a \in M$, which are definable using parameters from $A$.*

Some authors require that canonical bases $A$ are definably closed i.e. $dcl(A) = A$.

**A.4 Exercise.**

*(i) Show that if $A$ is a canonical base of $p \in S(M)$, then $p$ does not fork over $A$ and $p \restriction A$ is stationary. (Hint: First show that $p$ does not split over $A$ and then e.g. see the hint for Exercise 5.12 (ii).)*

*(ii) Show that $dcl(dcl(A)) = dcl(A) \supseteq A$ and that if a sequence $a = (a_0, ..., a_n)$ is definable with parameters from $A$, then for all $i \leq n$, $a_i \in dcl(A)$.*

*(iii) Show that if $A$ is a canonical base of $p \in S(M)$, then so is $dcl(A)$ and if $B$ is another canonical base of $p$, then $dcl(A) = dcl(B)$.*

*(iv) Show that if $p \in S(M)$, $p$ does not fork over $B \subseteq M$ and $A$ is a canonical base of $p$, then $A \subseteq acl(B)$.*

**A.5 Theorem.** *Every $p \in S(M^{eq})$ has a canonical base.*

**Proof**. In order to simplify the notations, we assume that $p \in S^1(M^{eq})$. Let $M' \supseteq M^{eq}$ be a saturated model of power $> |M^{eq}|$ and $a \in M'$ such that it realizes $p$. Let $E$ be such that $M' \models S_E(a)$. Again in order to simplify the notations, we assume that $E \in EQ^1$ and so we can find $b \in M'$ such that $M' \models S_=(b) \wedge F_E(b) = a$.

Clearly, it is enough to find for each $\phi(x, y)$ and element $a_\phi \in M^{eq}$ such that for all $f \in Aut(M^{eq})$, $f(p \restriction \phi(x, y)) = p \restriction \phi(x, y)$ iff $f(a_\phi) = a_\phi$. We fix $\phi(x, y)$.

Choose a model $\mathcal{A} \subseteq M^{eq}$ so that $t(ab, M^{eq})$ does not fork over $\mathcal{A}$ and for all $i < \omega$, choose $a_i$ and $b_i$ from $M^{eq}$ such that $t(a_i b_i, \mathcal{A} \cup \{a_j, b_j |\ j < i\}) = t(ab, \mathcal{A} \cup \{a_j, b_j |\ j < i\})$. Then $(a_i b_i)_{i<\omega}$ and $(a_i)_{i<\omega}$ are indiscernible sequences based on $\mathcal{A}$ (see Exercise 5.8 (i) and the proof of Theorem 3.9) and thus $p = Av((a_i)_{i<\omega}, M^{eq})$.

Then, as in the proof of Theorem 5.11, letting $n$ be as in Exercise 3.5,

$$\psi(y, a_0, ..., a_{2(n-1)}) = \bigvee_{w \subseteq 2n-1,\ |w|=n} (\wedge_{i \in w} \phi(a_i, y))$$

defines $p \restriction \phi(x, y)$ and thus so does $\psi(y, F_E(b_0), ..., F_E(b_{2(n-1)}))$.

Let $E'$ be the equivalence relation on $M^{eq}$ such that for all $a'_i, a''_i \in M^{eq}$, $i < 2n-1$, $(a'_0, ..., a'_{2(n-1)}) E' (a''_0, ..., a''_{2(n-1)})$ if for all $i < 2n-1$, $a'_i = a''_i$ or for all $i <$

$2n-1$, $a'_i, a''_i \in S_E$ and for all $c \in M^{eq}$, $(\psi(c, a'_1, ..., a'_{2(n-1)}) \leftrightarrow \psi(c, a''_1, ..., a''_{2(n-1)}))$. By Exercise A.2 (ii), there is $E^* \in EQ^{2n-1}$ such that for all $b'_i, b''_i \in M$, $i < 2n-1$, $(b'_0, ..., b'_{2(n-1)})E^*(b''_0, ..., b''_{2(n-1)})$ iff

$$(F_E(b'_0), ..., F_E(b'_{2(n-1)}))E'(F_E(b''_0), ..., F_E(b''_{2(n-1)})).$$

We let $a_\phi = (b_0, ..., b_{2(n-1)})/E^*$.

Let $f \in Aut(M^{eq})$. If $f(p) = p$, then for all $c \in M^{eq}$,
$M^{eq} \models \psi(c, F_E(b_0), ..., F_E(b_{2(n-1)}))$
iff $\phi(x, c) \in p$ iff $\phi(x, c) \in f(p)$
iff $M^{eq} \models \psi(c, F_E(f(b_0)), ..., F_E(f(b_{2(n-1)})))$
and so $f(a_\phi) = a_\phi$.

On the other hand, if $f(a_\phi) = a_\phi$, then $\phi(x, c) \in p$
iff $M^{eq} \models \psi(c, F_E(b_0), ..., F_E(b_{2(n-1)}))$
iff $M^{eq} \models \psi(c, F_E(f(b_0)), ..., F_E(f(b_{2(n-1)})))$
iff $\phi(x, c) \in f(p)$. □

**A.6 Exercise.** *Let $T = T_\omega$, $A \subseteq M$ and $a$ an element in $M - A$.*
*(i) Show that $t(a/A)$ is stationary.*
*(ii) Find a canonical base for $t(a, A)$ in $M^{eq}$.*
*(iii) Show that if $C$ is a canonical base for $t(a, A)$ (in $M^{eq}$), then $C \cap M = \emptyset$.*

## B. Morley's theorem

Though out this section we assume that $T$ is a countable complete theory and $\lambda$-categorical for some uncountable $\lambda$ (i.e. upto isomorphism $T$ has exactly one model of power $\lambda$).

The following fact can be proved using Ehrenfeuch-Mostowski models, see e.g. [Hy2] Exercise 12.11.

**B.1 Fact.** *$T$ is $\omega$-stable.*

**B.2. Lemma.** *Every uncountable model of $T$ is $\omega_1$-saturated and thus $T$ is $\omega_1$-categorical.*

**Proof**. Let $A \subseteq \mathcal{A}$ and $p \in S(A)$ be such that $A$ is a countable set and $\mathcal{A}$ is an uncountable model. We need to show that $p$ is realized in $\mathcal{A}$. Let $a_i \in \mathcal{A}$, $i < \omega_1$, be distinct elements. By Theorem 3.3 we may assume that $(a_i)_{i<\omega_1}$ is indiscernible over $A$. Let $a_i$, $\omega_1 \le i < \lambda$, be such that $(a_i)_{i<\lambda}$ is indiscernible over $A$. Let $\mathcal{B}$ be $F^t_\omega$-primary model over $A \cup \bigcup_{i<\lambda} a_i$. Since $T$ is $\lambda$-stable, $T$ has a saturated model of power $\lambda$ and since $T$ is $\lambda$-categorical, $\mathcal{B}$ is saturated. And thus $p$ is realized in $\mathcal{B}$. Let $b \in \mathcal{B}$ be the realization. By Lemma 7.4, one finds a finite $X = \{i_0, ..., i_n\} \subseteq \lambda$ and $B \subseteq \mathcal{B}$ such that $b \in B$ and $B$ is $F^t_\omega$-constructible over $A \cup \bigcup_{k \le n} a_{i_k}$. Since $(a_i)_{i<\lambda}$ is indiscernible over $A$, we may assume that for all $k \le n$, $i_k = k$. But then, since $F^t_\omega$-constructible sets are $F^t_\omega$-primitive and models are $F^t_\omega$-saturated, there is elementary $f : B \to \mathcal{A}$ such that $f \restriction A = id$. Then $a = f(b) \in \mathcal{A}$ realizes $p$. □

**B.3 Lemma.** *Suppose $\mathcal{A}$ is a countable saturated model and $p, q \in S(\mathcal{A})$ are not algebraic. Then $p$ and $q$ are not orthogonal (and so $T$ is unidimensional, see Section 6).*

**Proof**. Suppose they are. By induction on $i \leq \omega_1$ we find realizations $a_i$ of $p$ and models $\mathcal{A}_i$ as follows:

(i) $\mathcal{A}_0 = \mathcal{A}$ (and $a_0$ is any realization of $p$),

(ii) $\mathcal{A}_{i+1}$ is $F^s_\omega$-primary over $\mathcal{A}_i \cup a_i$ and $a_{i+1} \downarrow_{\mathcal{A}} \mathcal{A}_{i+1}$,

(iii) if $i$ is a limit, then $\mathcal{A}_i = \cup_{j<i} \mathcal{A}_j$ and $a_i \downarrow_{\mathcal{A}} \mathcal{A}_i$.

Let $b$ be any realization of $q$. Using Exercise 10.9, an easy induction on $i \leq \omega_1$ shows that $b \downarrow_{\mathcal{A}} \mathcal{A}_{\omega_1}$ and so $b \notin \mathcal{A}_{\omega_1}$. Thus $\mathcal{A}_{\omega_1}$ does not realize $q$, which contradicts Lemma B.2. □

**B.4 Definition.** *We say that $t(a, A)$ is minimal if it is not algebraic but for all $B \supseteq A$, if $a \not\downarrow_A B$, then $t(a, B)$ is algebraic.*

**B.5 Exercise.** *Let $\mathcal{A}$ be a countable saturated model. Show that there is minimal $p \in S(\mathcal{A})$.*

**B.6 Lemma.** *Let $\mathcal{A}$ be an uncountable model, $\mathcal{B} \subseteq \mathcal{A}$ be a countable saturated model, $p \in S(\mathcal{B})$ be minimal and $\{a_i |\ i < \alpha\}$ be a maximal independent (i.e. $a_i \downarrow_{\mathcal{B}} \bigcup_{j<\alpha, j\neq i} a_j$) set of realizations of $p$ from $\mathcal{A}$. Then $\mathcal{A}$ is $F^s_\omega$-primary over $\mathcal{B} \cup \bigcup_{i<\alpha} a_i$.*

**Proof**. Let $\mathcal{C} \subseteq \mathcal{A}$ be $F^s_\omega$-primary model over $\mathcal{B} \cup \bigcup_{i<\alpha} a_i$. It is enough to show that $\mathcal{C} = \mathcal{A}$. Suppose not. Let $b \in \mathcal{A} - \mathcal{C}$.

Then we can find a finite $X \subseteq \alpha$ and $F^s_\omega$-primary model $\mathcal{B}' \subseteq \mathcal{C}$ over $\mathcal{B} \cup \bigcup_{i\in X} a_i$ such that $b \downarrow_{\mathcal{B}'} \mathcal{C}$. If $\mathcal{C}' \subseteq \mathcal{C}$ is $F^s_\omega$-primary over $\mathcal{B}' \cup \bigcup_{i<\alpha} a_i$ it is $F^s_\omega$-primary over $\mathcal{B} \cup \bigcup_{i<\alpha} a_i$ (exercise, hint: Exercise 10.9 (i)) and thus we may assume that $\mathcal{C}' = \mathcal{C}$ (since $b \in \mathcal{A} - \mathcal{C}'$ and $b \downarrow_{\mathcal{B}'} \mathcal{C}'$). Let $p' \in S(\mathcal{B}')$ be the non-forking extension of $p$. Then

(*) $\{a_i |\ i \in X\}$ is a maximal independent set of realizations of $p'$ from $\mathcal{A}$ (exercise).

Let $\mathcal{D} \subseteq \mathcal{A}$ be $F^s_\omega$-primary over $\mathcal{B}' \cup b$. By Exercise 11.2 (i) and Lemma B.3, there is a realization $a$ of $p'$ such that $a \not\downarrow_{\mathcal{B}'} b$. Since $p$ was minimal, $a \in \mathcal{D}$. Since $b$ dominates $\mathcal{D}$ over $\mathcal{B}'$, $a \downarrow_{\mathcal{B}'} \{a_i |\ i \in X\}$, a contradiction with (*) above. □

**B.7 Morley's theorem.** *If $T$ is countable and $\lambda$-categorical for some uncountable $\lambda$, then $T$ is $\kappa$-categorical for all uncountable $\kappa$.*

**Proof**. So suppose $|\mathcal{A}| = |\mathcal{B}| = \kappa$. Let $\mathcal{C} \subseteq \mathcal{A}$ and $\mathcal{C}' \subseteq \mathcal{B}$ be countable saturated models. By taking an isomorphic copy of $\mathcal{B}$, we may assume that $\mathcal{C} = \mathcal{C}'$. Let $p \in S(\mathcal{C})$ be minimal and $\{a_i |\ i < \alpha\}$ and $\{b_i |\ i < \beta\}$ be maximal independent set of realizations of $p$ in $\mathcal{A}$ and $\mathcal{B}$, respectively. By Exercise 9.7, $|\alpha| = |\mathcal{A}| = |\mathcal{B}| = |\beta|$ and thus by taking an isomorphic copy of $\mathcal{B}$, we may assume that $\alpha = \beta$ and that for all $i < \alpha$, $a_i = b_i$. But then the claim follows from Lemma B.6 and Theorem 7.11. □

## C. Properties of forking

We collect together the most important properties of the independence notion $\downarrow$. Let $A \subseteq B \subseteq C \subseteq D$ and $a$ and $b$ be arbitrary.

Monotonicity: If $a \downarrow_A D$, then $a \downarrow_B C$.

Finite character: If $a \not\downarrow_A B$, then there is $c \in B$ such that $a \not\downarrow_A c$.

Locality 1: There is $A' \subseteq A$ of power $< \kappa(T)$ ($\leq |T|^+$), such that $a \downarrow_{A'} A$.

Locality 2: If $(A_i)_{i<\kappa(T)}$ is a $\subseteq$-increasing sequence of sets, then there is $i < \kappa(T)$ such that $a \downarrow_{A_i} A_{i+1}$.

Symmetry: If $a \downarrow_A b$, then $b \downarrow_A a$.

Transitivity: $a \downarrow_A C$ iff $a \downarrow_A B$ and $a \downarrow_B C$.

Existence: There is $c$ such that $stp(c, A) = stp(a, A)$ and $c \downarrow_A B$.

Reflexivity 1: If $t(a, B)$ is algebraic and $t(a, A)$ is not, then $a \not\downarrow_A B$.

Reflexivity 2: If $t(a, A)$ is algebraic, then $a \downarrow_A B$.

Stationarity 1: If $stp(a, A) = stp(b, A)$, $a \downarrow_A B$ and $b \downarrow_A B$, then $stp(a, B) = stp(b, B)$.

Stationarity 2: If $A$ is a model, $t(a, A) = t(b, A)$, $a \downarrow_A B$ and $b \downarrow_A B$, then $stp(a, B) = stp(b, B)$.

## References

[Ba] J. Baldwin, Fundamentals of Stability Theory, Springer-Verlag, Berlin, 1988.

[Bu] S. Buechler, Essential Stability Theory, Springer-Verlag, Berlin, 1996.

[Hr] E. Hrushovski, Unidimensional theories are superstable, Annals of Pure and Applied Logic, vol. 50, 1990, 117-138.

[HS1] T. Hyttinen and S. Shelah, On the number of elementary submodels of an unsuperstable homogeneous structure, Mathematical Logic Quarterly, vol. 14, 1998, 354-358.

[HS2] T. Hyttinen and S. Shelah, Strong splitting in stable homogeneous models, Annals of Pure and Applied Logic, vol. 103, 2000, 201-228.

[Hy1] T. Hyttinen, Stability and general logics, Mathematical Logic Quarterly, vol. 45, 1999, 219-240.

[Hy2] T. Hyttinen, Model Theory, Lecture notes, University of Helsinki, 2013.

[Hy3] T. Hyttinen, A short introduction to classification theory, Graduate Texts in Mathematics, vol. 2, University of Helsinki, 1997.

[La] D. Lascar, Stability in Model Theory, Longman, Essex, 1987.

[Pi] A. Pillay, Geometric Stability Theory, Oxford University Press, New York, 1996.

[Sh] S. Shelah, Classification Theory, Stud. Logic Found. Math. 92, North-Holland, Amsterdam, 2nd rev. ed., 1990.

Department of Mathematics and Statistics
P.O. Box 68
00014 University of Helsinki
Finland