2014.01.13

INTEGRAL EQUATIONS

* What is an integral equation?

It is an equation in which an unknown function appears under an integral sign and the problem of solving the equation is to determine that function.

Examples

1. Laplace transform (1782)

The integral equation for g(t) given by

$$f(x) = \int_{-\infty}^{+\infty} e^{-xt} g(t)\, dt$$

→ it expresses a function as a superposition of moments

we want to find g(t)

Isolated problems involving integral equations occurred long before the subject acquired a distinct status and methodology.

2. Another of the ~~~~ noteworthy results that really belong to the history of integral equations is from Fourier's famous 1811 paper on the theory of heat

$$f(x) = \int_0^{\infty} \cos(xt)\, u(t)\, dt$$

we want to find u(t)

3. Some trivial examples of equations that we are going to study are the following:

↳ 3.1 Determine a continuous function y that satisfies the equation

$$e^x - 1 = \int_0^x y(s)\, ds, \qquad \forall x \in \mathbb{R}.$$

By differentiating both sides, one gets that

$$\boxed{e^x = y(x)}$$

and a direct substitution then shows that this is a solution.

3.2 Determine a bounded and continuous function ϕ satisfying the integral equation

$$e^{-x^2} = \int_{-\infty}^{+\infty} e^{-(x-y)^2} \phi(y)\, dy \quad , \quad \forall x \in \mathbb{R}.$$

Hint: use the convolution theorem and Fourier transformation.

→ The first example is a Volterra equation (because integration bounds depend on x)

→ The second example is a Fredholm equation (integration bounds are fixed)

→ However, these examples are misleading, since usually we cannot solve a given equation explicitly.

In general, one can write integral equations in the following way:

Given a kernel function $k(s,t)$, we define the integral transform

$$(Kx)(s) = \int k(s,t)\, x(t)\, dt \quad , \quad K : X \longrightarrow Y$$

(X, Y Banach or Hilbert spaces)

Then, for a given function f, we want to find x such that

$$\boxed{(\lambda I + K)\, x = f, \quad \lambda \in \mathbb{R}} \qquad (*)$$

* Classification :

- $\lambda = 0$ → integral equation of the first kind
- $\lambda \neq 0$ → (*) is an int. eq. of the second kind
- If the integration bounds of k depend on s

  ⟹ Volterra integral equations
- If the integration bounds are fixed on k

  ⟹ Fredholm integral equations

Examples

(a) $$\int_0^1 k(s,t)x(t)\,dt = g(s), \qquad s\in[0,1]$$

Fredholm int. eq. of the 1^{st} kind

(b) $$\lambda x(s) - \int_0^s k(s,t)\,x(t)\,dt = g(s), \qquad s\in[0,1]$$
$k : [0,1]^2 \to \mathbb{R}$

Volterra int. eq. of the 2^{nd} kind

which is the relation between them ?

*REMARK 1: Any Volterra equation is also Fredholm

set
$$\tilde{k}(s,t) := \begin{cases} k(s,t) & t \leq s \\ 0 & t > s \end{cases}$$

Then

$$(b) \iff \lambda x(s) - \int_0^1 \tilde{k}(s,t)\,x(t)\,dt = g(s)$$

Fredholm int. eq.

* REMARK 2 : Volterra int. eq. $\Leftrightarrow$ initial value problems

Fredholm int. eq. $\Leftrightarrow$ boundary value problems

To see this :

- Ex. Consider an initial value problem

$$\begin{cases} x''(t) = f(x(t)) , & t \in [0,1] \\ x(0) = 1 \\ x'(0) = 0 \end{cases}$$

we integrate the diff. equation between 0 and t :

$$\underbrace{x'(t)}_{\int_0^t x''(s)\,ds = x'(t) - \underbrace{x'(0)}_{0}} = \int_0^t f(x(s))\,ds + c_1$$

we integrate again the above eq :

$$x(t) = x(0) + \int_0^s \int_0^t f(x(\tau))\,d\tau\,dt + c_2 + c_1 t$$

Volt. int. eq.

- Ex. Consider the boundary value problem

$$\begin{cases} x''(t) = f(x(t)) , & t \in [0,1] \\ x(0) = 0 \\ x(1) = 0 \end{cases}$$

with the same idea of the previous example, we get a Fredholm int. eq.

* REMARK 3. As we already said, it is hard to find explicit solution of (*). This will depend a lot depending on the integral transform K.

↳ Note that if $K = A \in M_{n \times n}(\mathbb{C})$, then we have a finite dimensional linear system of equations. And we study this system by analyzing the matrix A, as we did in linear algebra.

In this case, this is like solving infinite dimensional system of equations.

We will see that if K is a compact operator, then we will be able to say something about the solvability of (*).

* What kind of questions can we ask about (*)?

1. Existence ~~and~~ of solution

If the given integral equation has a solution will be the first question. Not all equations have.

↳ For example, if we replace the equation in the first example above by

$$e^x = \int_0^x y(s)\,ds,$$

we get into trouble.

If we ask that y is for example continuous, then evaluating at $x=0$ we get

$$1 = \int_0^0 y(s)\,ds = 0 \quad \#$$

Hence, there is no solution.

2. Uniqueness of solution

If the solution exists, is it unique?

3. Stability (Numerical interest)

Does the solution depend continuously from the data?

↳ $(\lambda I + K)x = g$

$(\lambda I + K)x^\delta = g^\delta$

Assume $\|g^\delta - g\| \le \delta, \quad \delta \to 0$

Then $\quad x^\delta \to x$?

$$\|x^\delta - x\| \le C\|g^\delta - g\| \; ?$$

YES, if $\lambda \neq 0$

NO, if $\lambda = 0$

⇒ • Integral equations of the first kind are "ill-posed"

• Integral equations of the second kind are "well-posed"

* Why are integral equations important?

- One of the main motivation in studying integral equations come from the need to understand differential equations
- As most of you probably know, in many physical theories the crucial equations are (partial) differential equations.
  - ↳ for example, Maxwell's equations which describe the propagation of electromagnetic waves.
- It turns out that understanding the behavior of their solutions is often lot easier if one succeeds in transforming them to integral equations.
  - ↳ Especially, the numerical solution of integral equations is often easier.

So... why are then integral equations easier to analyze than differential equations?

↓

One explanation might be that often integral operators are continuous, whereas differential operators are not:

Consider the linear map

$$Tf(x) = \int_0^x f(s)\,ds.$$

$T: C[0,1] \to C[0,1]$ is a continuous map:

Assume that $|f(x) - g(x)| < \varepsilon$, $\forall x \in [0,1]$.

Then

$$|Tf(x) - Tg(x)| \leq \int_0^x |f(s) - g(s)|\,ds \leq \varepsilon x \leq \varepsilon$$

for every $x \in [0,1]$.

So, if f and g are close, then so are Tf and Tg.

On the other hand, let

$$h_\varepsilon(x) = \varepsilon \sin\left(\frac{x}{\varepsilon}\right), \quad x \in [0,1], \quad \varepsilon > 0.$$

Then

$$|h_\varepsilon(x)| \leq \varepsilon \quad , \quad \forall x \in [0,1],$$

but for the derivative $h'_\varepsilon(x) = \cos\left(\frac{x}{\varepsilon}\right)$ we have

$$\sup_{0 \leq x \leq 1} |h'_\varepsilon(x)| = 1,$$

when $0 < \varepsilon < \frac{2}{\pi}$.

So even if h_ε is close to zero, the derivative is not.

This is also well known in numerical analysis:

↳ numerical differentiation is unstable, whereas integration is not.

→ Another motivation or relation of why integral equations are more "important" than differential equations is related to the uniqueness and existence of solutions for ordinary differential equations:

The result is proved by transforming the problem into a integral equation, and then solving this by using an iterative algorithm.

INTEGRAL VOLTERRA EQUATIONS

* VOLTERRA EQUATIONS OF THE SECOND KIND

As we already saw, Volterra integral equations are connected to initial value problems.

→ Let us consider the initial value problem

$$y' + p(x)y = f(x), \qquad y(x_0) = y_0 \qquad (1)$$

By integrating both sides ~~we get~~ between x_0 and x, we get the int. eq.

$$\boxed{y(x) - y_0 + \int_{x_0}^{x} p(s)\,y(s)\,ds = F(x), \qquad (2)}$$

where $\quad F(x) = \int_{x_0}^{x} f(s)\,ds.$

We could also proceed in a different manner: namely, let $z(x) = y'(x)$. Then, by integrating we get

$$y(x) - y_0 = \int_{x_0}^{x} z(s)\,ds$$

and substituting these into (1) we get the integral equation

$$\boxed{z(x) + p(x)\int_{x_0}^{x} z(s)\,ds = f(x) - p(x)y_0 \qquad (3)}$$

integration bounds depend on x. And also ~~note~~ that the unknown in both cases is under the integral sign and outside

Equations (2) and (3) are Volterra integral equations and are both included in the following definition.

⇓ it's second kind!

• DEFINITION: Let $I = [a,b] \subset \mathbb{R}$ be an interval, $k : I \times I \to \mathbb{C}$ and $f : I \to \mathbb{C}$ continuous function. Equation of the type

$$(*) \quad \boxed{\phi(x) - \lambda\int_{a}^{x} k(x,y)\,\phi(y)\,dy = f(x), \quad x \in I},$$

for a continuous function $\phi : I \to \mathbb{C}$ is called a Volterra equation of the second kind.

* REMARKS

Esau hitzet; et idotti!!

~~(i) the could equ.~~

(i) This equation is linear in the unknown function ϕ.

(ii) It is classical terminology to first consider equations of the second kind. The reason for that is that actually in general equations of the first kind will be reduced to equations of the second kind.

*EXAMPLE 1: Consider the equation

$$\phi(x) - \lambda \int_0^x y\,\phi(y)\,dy = f(x), \qquad (*)$$

where $\lambda \in \mathbb{C}$. Lets assume that f is differentiable. Then the above equation implies that also ϕ is, and by differentiating both sides we get the linear first order equation

$$\phi' - \lambda x \phi = f'$$

$$\left(e^{-\frac{\lambda x^2}{2}}\phi\right)'(x) = e^{-\frac{\lambda x^2}{2}}\left[\phi' - \lambda x \phi\right] = e^{-\frac{\lambda x^2}{2}} f'$$

integrating:

$$e^{-\frac{\lambda x^2}{2}}\phi(x) - \phi(0) = \int_0^x e^{-\frac{\lambda y^2}{2}} f'(y)\,dy$$

$$\Rightarrow \boxed{\phi(x) = e^{\frac{\lambda x^2}{2}} f(0) + \int_0^x e^{\frac{\lambda}{2}(x^2-y^2)} f'(y)\,dy}$$

with initial condition

$$\phi(0) = f(0).$$

This can be easily solved and the solution is

$$\phi(x) = e^{\lambda\frac{x^2}{2}} f(0) + \int_0^x e^{\frac{\lambda}{2}(x^2-y^2)} f'(y)\,dy.$$

If $f(x) = \frac{x^2}{2}$, then $f'(x) = x$ and we can evaluate the above integral explicitly:

$$\phi(x) = \int_0^x e^{\frac{\lambda}{2}(x^2-y^2)}\, y\,dy = -\frac{1}{\lambda}\left(1 - e^{\frac{\lambda x^2}{2}}\right)$$

$$z = -\frac{\lambda}{2}y^2 \qquad \phi(x) = e^{\frac{\lambda}{2}x^2}\int_0^x e^{-\frac{\lambda}{2}y^2} y\,dy = -\frac{1}{\lambda} e^{\frac{\lambda}{2}x^2}\int_0^{-\frac{\lambda}{2}x^2} e^{z}\,dz$$

$$dz = -\lambda y\,dy$$

It is easy to check that this is a solution of $(*)$ in the special case $f(x) = \frac{x^2}{2}$. However, we have not shown that it is unique.

$$-\frac{1}{\lambda}\left(1 - e^{\frac{\lambda x^2}{2}}\right) + \int_0^x y\left(1 - e^{\frac{\lambda y^2}{2}}\right) dy = -\frac{1}{\lambda}\left(1 - e^{\frac{\lambda x^2}{2}}\right) + \frac{x^2}{2} - \frac{1}{\lambda}\left(e^{\frac{\lambda x^2}{2}} - 1\right) = \frac{x^2}{2} \checkmark$$

*EXISTENCE AND UNIQUENESS OF SOLUTIONS

The main result of this section is the following:

THEOREM: The Volterra equation of the second kind (*) has a unique solution $\phi \in C([a,b])$ for any continuous f.

Proof: The proof is an iterative argument, actually it is the Picard iteration, but with slightly different notation.

We write equation (*) as

$$\phi(x) = f(x) + \lambda \int_a^x K(x,y)\phi(y)\,dy \qquad (++)$$

Let

$$\phi_0(x) = f(x)$$

$$\phi_k(x) = \int_a^x K(x,y)\phi_{k-1}(y)\,dy \,, \qquad k = 1, 2, \dots$$

Then, if the series

$$\phi(x) = \sum_{k=0}^{\infty} \lambda^k \phi_k(x)$$

converges uniformly, it is easy to see that it is a solution of (++); formally we can compute

$$f(x) + \lambda \int_a^x K(x,y)\phi(y)\,dy = \phi_0(x) + \lambda \int_a^x K(x,y) \sum_{k=0}^{\infty} \lambda^k \phi_k(y)\,dy$$

$$= \phi_0(x) + \lambda \sum_{k=0}^{\infty} \lambda^k \int_a^x K(x,y)\phi_k(y)\,dy$$

$$\begin{pmatrix} k \to 0 \\ k-1 \to 1 \end{pmatrix} \nearrow \quad = \phi_0(x) + \sum_{k=1}^{\infty} \lambda^k \underbrace{\int_a^x K(x,y)\phi_{k-1}(y)\,dy}_{=\,\phi_k(x)}$$

$$= \sum_{k=0}^{\infty} \lambda^k \phi_k(x) = \phi(x)$$

To justify the above computation we have to show that the functions ϕ_k are continuous and that the sum $\sum_{k=0}^{\infty} \phi_k(x)$ is uniformly convergent.

- The continuity of ϕ_k is trivial: since f is continuous, also $\phi_0 = f$ is continuous, and then inductively using the expression for ϕ_k and the continuity of K, we get that also ϕ_k is continuous for any k.

- Next we prove the uniform convergence.

Let
$$M = \max_{a \le x \le b} |f(x)| \quad , \quad N = \max_{a \le x, y \le b} |K(x,y)|$$

Then trivially
$$|\phi_0(x)| \le M \, , \quad a \le x \le b .$$

Plugging this estimate into the first iteration step we get

$$|\phi_1(x)| \le \int_a^x |K(x,y)|\,|\phi_0(y)|\,dy \le MN(x-a) \, , \quad a \le x \le b .$$

Using this we can estimate

$$|\phi_2(x)| \le \int_a^x |K(x,y)|\,|\phi_1(y)|\,dy \le MN^2 \int_a^x (y-a)\,dy = \frac{MN^2}{2}(x-a)^2$$

Inductively we then get for $k \ge 1$ the estimate

$$|\phi_k(x)| \le \int_a^x |K(x,y)|\,|\phi_{k-1}(y)|\,dy \le \frac{MN^k}{(k-1)!} \int_a^x (y-a)^{k-1}\,dy = \frac{MN^k}{k!}(x-a)^k$$

And note that
$$\sum_{k=0}^{\infty} \lambda^k \frac{MN^k}{k!}(x-a)^k = M e^{\lambda N(x-a)}$$

$\Big(|\phi_k(x)| \le \frac{MN^k}{k!}(x-a)^k$

$\sum |\phi_k(x)|$ converges

$\Rightarrow \sum \phi_k$ converges $\Big)$

and the convergence is uniform. Hence by the majorant principle also the series $\sum \lambda^k \phi_k(x)$ is uniformly convergent.

- So, we have shown the existence.

we have seen that $\phi(x) = \sum_{k=0}^{\infty} \lambda^k \phi_k(x)$ is solution of (*).

$\phi_0(x) = f(x)$

$\phi_k(x) = \int_a^x k(x,y)\,\phi_{k-1}(y)\,dy$

· Now it remains to prove that there are no other solutions.

Uniqueness of solution:

Assume that [illegible] ϕ and ψ are two solutions of ⊛,

and we consider the difference $h(x) = \phi(x) - \psi(x)$ we want to see that $h(x)=0$

By linearity, $h(x)$ solves the homogeneous equation

$$h(x) = \lambda \int_a^x K(x,y) h(y)\, dy \qquad \boxplus$$

and we have to show that this implies that $h=0$.

We do this using classical bootstrap principle

Set
$$R = \max_{a \le x \le b} |h(x)|$$

Then $\boxplus$ implies

$$|h(x)| \le |\lambda| \int_a^x |k(x,y)|\,|h(y)|\, dy \le |\lambda| N R (x-a), \quad a \le x \le b.$$

Now substituting this in the right hand side of $\boxplus$, we obtain

$$|h(x)| \le |\lambda|^2 \left\{ NR \int_a^x |k(x,y)|(y-a)dy \le |\lambda|^2 RN^2 \int_a^x (y-a)dy \right.$$

$$\le |\lambda|^2 \frac{RN^2 (x-a)^2}{2}, \quad a \le x \le b$$

Repeating the same procedure, we obtain

$$|h(x)| \le \frac{|\lambda|^k RN^k}{k!} (x-a)^k, \quad a \le x \le b$$

for any $k = 1, 2, \ldots$. Letting $k \to \infty$, since

$$\lim_{k \to \infty} \frac{|\lambda|^k RN^k}{k!} (x-a)^k = 0,$$

we obtain $h(x) = 0$.

I finished here the first part

This finishes the proof. □

→ Now we go back to Example 1 and we check that actually, the solution that we constructed in the proof is solution of the equation:

- CONTINUATION OF EXAMPLE 1:

We consider equation

$g(x) = \frac{x^2}{2}$

$$\phi(x) - \lambda \int_0^x y\,\phi(y)\,dy = \frac{x^2}{2}$$

Recall that we saw that the solution is $\phi(x) = -\frac{1}{\lambda}\left(1 - e^{\frac{\lambda x^2}{2}}\right)$

Let us compute the first three terms of the iteration:

$$\phi_0(x) = \frac{x^2}{2}$$

$$\phi_1(x) = \lambda \int_0^x y\,\frac{y^2}{2}\,dy = \frac{\lambda}{8}x^4$$

$$\phi_2(x) = \lambda \int_0^x y\,\frac{\lambda}{8}y^4\,dy = \frac{\lambda^2 x^6}{48}.$$

On the other hand, we know that the correct solution is

$$\phi(x) = -\frac{1}{\lambda}\left(1 - e^{\frac{\lambda x^2}{2}}\right).$$

If we use the Taylor expansion of $e^{\frac{\lambda x^2}{2}}$,

$$e^{\frac{\lambda x^2}{2}} = 1 + \frac{\lambda}{2}x^2 + \frac{\lambda^2}{2!}\frac{x^4}{2^2} + \frac{\lambda^3}{3!}\frac{x^6}{2^3} + \dots,$$

we get

$$\phi(x) = +\frac{1}{\lambda}\left(\frac{\lambda}{2}x^2 + \frac{\lambda^2}{8}x^4 + \frac{\lambda^3 x^6}{48} + \dots\right)$$

$$= \frac{x^2}{2} + \frac{\lambda}{8}x^4 + \frac{\lambda^2 x^6}{48} + \dots$$

⇒ The iteration gives just the beginning of the Taylor expansion of the solution.

* RESOLVENT KERNELS

We will give another expression of the solution $\phi(x)$ as follows.

In the proof,

$$\phi_0(x) = f(x)$$

$$\phi_k(x) = \int_a^x k(x,y)\,\phi_{k-1}(y)\,dy$$

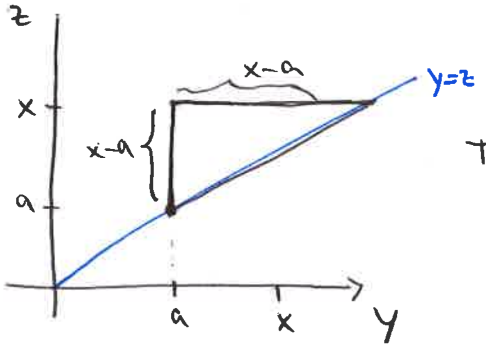
Indeed,

$$\phi_1(x) = \int_a^x k(x,y)\,f(y)\,dy$$

$$\phi_2(x) = \int_a^x k(x,y)\,\phi_1(y)\,dy$$

and hence,

$$\phi_2(x) = \int_a^x dz \int_a^z k(x,z)\,k(z,y)\,f(y)\,dy$$



The integration is taken over this triangular set

Set $F(x,z,y) = k(x,z)\,k(z,y)$. Then it holds

$$\int_a^x dz \int_a^z F(x,z,y)\,dy = \int_a^x dy \int_y^x F(x,z,y)\,dz$$

~~Let us consider the iterated kernels~~

Accordingly, if we define the iterated kernel

$$k^{(2)}(x,y) = \int_y^x k(x,z)\,k(z,y)\,dz ,$$

then we can write

$$\phi_2(x) = \int_a^x k^{(2)}(x,y)\,f(y)\,dy .$$

Repeating the same argument, we obtain the iterated kernels

$$k^{(1)}(x,y) = k(x,y)$$

$$k^{(k)}(x,y) = \int_y^x k(x,z)\, k^{(k-1)}(z,y)\, dz\,, \qquad k = 2, 3, \dots$$

and we can write $\phi_k(x)$ as follows

$$\boxed{\phi_k(x) = \int_a^x k^{(k)}(x,y) f(y)\, dy}\ .$$

~~* EXERCISE: Show that~~ $\cancel{\phi(x) = \sum_{k=0}^{\infty} \lambda^k \phi_k(x)}$ ~~is solution of ⊛~~

$$\cancel{\phi(x) = f(x) + \underbrace{k(x,y) + \lambda k^{(2)}(x,y) + \dots + \lambda^{k-1} k^{(k)}(x,y) + \dots}_{= \Gamma(x,y;\lambda)}}$$

Set

⊠ $\Gamma(x,y;\lambda) = k(x,y) + \lambda k^{(2)}(x,y) + \dots + \lambda^{k-1} k^{(k)}(x,y) + \dots$

↳ is called resolvent kernel.

Then

$$\phi(x) = f(x) + \lambda \int_a^x \Gamma(x,y;\lambda) f(y)\, dy$$

is solution of ⊛ .

EXERCISE: • Prove that ⊠ converges uniformly with respect to (x,y) for $|\lambda| < \infty$.

• Show that $k^{(n)}(x,y) = \int_y^x k^{(n-1)}(x,z) k(z,y)\, dz$

* VOLTERRA EQUATIONS OF THE FIRST KIND

We start with the exact definition.

- **DEFINITION:** Let $I = [a,b]$ be an interval of the real line, and $k : I \times I \to \mathbb{C}$ and $f : I \to \mathbb{C}$ continuous function. An integral equation for $\phi : I \to \mathbb{C}$ of the form

$$\boxed{f(x) = \int_a^x k(x,y)\,\phi(y)\,dy, \qquad a \leq x \leq b,} \quad \oplus$$

is a Volterra equation of the first kind.

→ Again, we would ideally like to have a unique continuous solution ϕ like for Volterra equations of the second kind. However, this is not always possible.

↳ Assume that above f is just continuous, but not differentiable and assume that the kernel k has continuous partial derivative with respect to the first variable.

Aha!

Then for any continuous ϕ (it would be enough to assume just that ϕ is integrable and bounded for example), $\oplus$ would then imply differentiability of f!

Hence the above equation has a smoothing property.

→ Motivated by this comment, let us assume that f is differentiable, and that K has a continuous partial derivative with respect to the first variable.

↳ Differentiating $\oplus$, we get

$$\boxed{+} \qquad f'(x) = k(x,x)\,\phi(x) + \int_a^x \frac{\partial k(x,y)}{\partial x}\,\phi(y)\,dy.$$

Assuming that $k(x,x) \neq 0$ for all $x \in [a,b]$ we can then transform this into a Volterra equation of the second kind:

$$\boxed{g(x) = \phi(x) + \int_a^x k_1(x,y)\,\phi(y)\,dy,} \qquad \boxed{++}$$

with $\displaystyle g(x) = \frac{f'(x)}{k(x,x)}, \quad k_1(x,y) = \frac{1}{k(x,x)}\,\frac{\partial k(x,y)}{\partial x}.$

→ Now we can prove the following.

• **THEOREM**: Assume that f is differentiable, k has a continuous partial derivative with respect to the first variable and that k is not vanishing on the diagonal: $k(x,x) \neq 0$ for all $x \in [a,b]$ and that $f(a)=0$. Then the Volterra equation of the first kind ⊕ has a unique continuous solution.

Proof

• Let's first show existence.

(we know that this exists)

Let ϕ be the unique solution of the Volterra equation ++.

Multiplying both sides of ++ by $k(x,x)$ we get +, and finally integrating both sides from a to x we get

$$f'(x) = \frac{d}{dx}\left[\int_a^x k(x,y)\phi(y)dy\right]$$

$$f(x) - f(a) = \int_a^x k(x,y)\phi(y)dy$$

+

$$f'(x) = \frac{d}{dx}\left[\int_a^x k(x,y)\phi(y)dy\right]$$

$$\int_a^x f'(x)dx = \int_a^x \frac{d}{ds}\left[\int_a^s k(s,y)\phi(y)dy\right]ds = \int_a^x k(x,y)\phi(y)dy$$

Since $f(a)=0$, we have that ϕ solves ⊕.

• Uniqueness is also easy.

As before, it is enough to show that if h is a continuous solution of ⊕ with $f=0$, then it is zero.

↳ But this follows by the unique solvability of ++.

□.

***EXAMPLE**: Consider equation

$$\frac{e^{2x}}{2} - 1 = \int_0^x e^{x+y}\phi(y)dy$$

This is a Volterra equation of first kind with $k(x,y) = e^{x+y}$.

The corresponding Volterra equation of the second kind is

$$\phi(x) + \int_0^x e^{y-x}\phi(y)\,dy = 1$$

$k(x,x) = e^{2x} \neq 0$

$f(x) = \frac{1}{2}e^{2x} - 1$

$f'(x) = e^{2x}$

$\frac{f'(x)}{k(x,x)} = 1$

$\frac{\partial k}{\partial x}(x,y) = e^{x+y}$

↪ $\frac{1}{k(x,x)}\frac{\partial k}{\partial x}(x,y) = e^{y-x}$

Multiplying both sides by e^x, we get

$$\phi(x)e^x + \int_0^x e^y\phi(y)dy = e^x$$

and differentiating this, we obtain

$$\phi' + 2\phi = 1.$$

This equation has a general solution

$$\phi(x) = \frac{(1 - Ce^{-2x})}{2}$$

Note that $\phi(0) = 1$ (we get this from ⊗).

Using this, we get $C = 1$, and hence, the solution is

$$\phi(x) = \frac{1}{2}(1 + e^{-2x})$$

I arrived here in the second hour

→ The assumption that the kernel is non vanishing on the diagonal is often too strong: for example, any kernel containing the factor $x-y$ obviously fails this condition.

↳ Any convolution operator, does.

In this cases, the following trick sometimes helps:

Assume that $k(x,x) = 0$ for all $x \in [a,b]$.

Then equation $\boxed{+}$ → $f'(x) = k(x,x)\phi(x) + \int_a^x \frac{\partial k(x,y)}{\partial x}\phi(y)dy$

becomes

$$f'(x) = \int_a^x \frac{\partial k(x,y)}{\partial x}\phi(y)dy.$$

This is again a Volterra equation of the first kind.

Assuming that f is twice differentiable and that also the second partial derivative of k with respect to the first variable exists and is continuous, we can differentiate the above equation to get

$$\boxed{++} \quad f''(x) = \frac{\partial k(x,x)}{\partial x}\phi(x) + \int_0^x \frac{\partial^2 k(x,y)}{\partial x^2}\phi(y)dy.$$

Hence, if $f'(a) = 0$, equation $\boxed{+}$ is equivalent to $\boxed{++}$.

At the same time, equation $\boxed{+}$ was equivalent to ⊕.

Therefore, equation ⊕ can be reduced to an equation of the second kind if $\frac{\partial K(x,x)}{\partial x} \neq 0$:

$$\frac{f''(x)}{\frac{\partial k}{\partial x}(x,x)} = \phi(x) + \int_a^x k_2(x,y)\,\phi(y)\,dy$$

with
$$k_2(x,y) = \frac{1}{\frac{\partial k}{\partial x}(x,x)} \frac{\partial^2 k(x,y)}{\partial x^2} .$$

Now, if $f(a) = f'(a) = 0$, then, as in the previous case, we can deduce that our original equation ⊕ admits a unique solution $\phi(x)$.

BOUNDED AND COMPACT OPERATORS

Our next goal is to provide tools for establishing the existence ~~and~~ of solutions to a wider class of integral equations; in general, integral equations of the second kind. $K\phi(x) = \int k(x,y)\phi(y)dy$, $\boxed{\phi - K\phi = f}$

In order to do that, we want to introduce and investigate compact operators; compactness of the integral transform K will be very important for the solvability of integral equations. K compact

However, before doing that, we give some review of bounded linear operators relating them with integral ~~equations~~ operators and showing some particular cases when a integral equation has solution.

If every Cauchy sequence converges to an element in the space

An inner product space that is complete with respect to the norm defined by the inner product

* BOUNDED LINEAR OPERATORS (in Hilbert spaces)

We start with the definition of a bounded operator

• DEFINITION: Let H_1 and H_2 be Hilbert spaces. A linear map $A : H_1 \to H_2$ is called bounded if there exists a positive number C such that

$$\|Ax\|_2 \leq C\|x\|_1 \quad , \quad x \in H_1 .$$

Here $\|\cdot\|_1$ and $\|\cdot\|_2$ are the Hilbert norms of H_1 and H_2, respectively.

* REMARK. The infimum of the constants C (that one can have above) is called the norm of the operator and is denoted by $\|A\|$.

$\Rightarrow$ $\|Ax\|_2 \leq \|A\| \, \|x\|_1 \quad , \quad \forall x \in H_1 .$

There are several equivalent definitions for a norm of the operator A:

$$\|A\| = \inf \{ C \geq 0 : \|Ax\|_2 \leq C\|x\|_1 \text{ for all } x \in H_1 \}$$

$$= \sup \{ \|Ax\|_2 : x \in H_1 \text{ with } \|x\|_1 \leq 1 \}$$

$$= \sup \{ \|Ax\|_2 : x \in H_1 \text{ with } \|x\|_1 = 1 \}$$

$$= \sup \left\{ \frac{\|Ax\|_2}{\|x\|_1} : x \in H_1 \text{ with } x \neq 0 \right\}$$

We next want to study continuous linear maps between two Hilbert spaces.

- PROPOSITION : The linear operator A is continuous if and only if it is bounded.

Proof:

$\Leftarrow$ Assume A is bounded. Let $\{x_n\}$ be a sequence in H_1 such that $\lim_{n\to\infty} x_n = x$. Then

$$\|Ax - Ax_n\|_2 \overset{A \text{ linear}}{=} \|A(x-x_n)\|_2 \overset{A \text{ bounded}}{\leq} C\,\|x-x_n\|_1 \to 0 \text{ as } n\to\infty,$$

i.e. A is continuous.

$\Rightarrow$ Assume that A is continuous but not bounded. Then for any $n\in\{1,2,\dots\}$ we can find $x_n\in H_1$ such that $\|x_n\|_1 = 1$ but

$$\|Ax_n\|_2 \geq n\|x_n\|_1 = n.$$

Let $y_n = \frac{x_n}{n}$. Then $\|y_n\|_1 = \frac{1}{n}$ and hence $y_n \xrightarrow{n\to\infty} 0$.

And since A is continuous, $Ay_n \to A(0) = 0$

But

$$\|Ay_n\| \geq 1 \quad \text{for all } n$$

and hence A is not continuous at 0 #

This means that A must be bounded. $\square$.

We will also see that the product of two bounded operators is bounded.

- PROPOSITION: Let H_1, H_2 and H_3 be Hilbert spaces and let $A: H_1\to H_2$ and $B: H_2\to H_3$ be bounded linear operators. Then the product $BA: H_1 \to H_3$, defined by $(BA)x := B(Ax)$ for all $x\in H_1$, is a bounded linear operator with $\|BA\| \leq \|A\|\,\|B\|$.

Proof:

This follows from

$$\|(BA)x\|_3 = \|B(Ax)\|_3 \overset{B \text{ bounded}}{\leq} \|B\|\,\|Ax\|_2 \overset{A \text{ bounded}}{\leq} \|B\|\,\|A\|\,\|x\|_1$$

$\square$.

we next go to study the case when our linear map is an integral operator

* INTEGRAL OPERATORS

Let us study the particular case when our linear map is an integral operator.

we want to see when an integral operator is bounded

• **THEOREM:** Let $I = [a,b] \subset \mathbb{R}$ and $k : I \times I \to \mathbb{C}$ be a continuous function. Then the linear operator $K : C([a,b]) \to C([a,b])$ defined by

$$(1) \qquad (Kf)(x) = \int_I k(x,y) f(y)\,dy , \qquad x \in I ,$$

called integral operator with continuous kernel k, is a bounded operator with

$$\|K\| = \sup_{x\in I} \int_I |k(x,y)|\,dy .$$

Proof:

Clearly (1) defines a linear operator $K : C([a,b]) \to C([a,b])$.

Now,

$$|Kf(x)| \leq \int_I |k(x,y)|\,|f(y)|\,dy$$

$$\leq \sup_{y\in I} |f(y)| \int_I |k(x,y)|\,dy$$

$$\leq \|f\|_\infty \int_I |k(x,y)|\,dy$$

Hence,

The norm in the continuous space $C([a,b])$

$$(2) \quad \|Kf\|_\infty = \sup_{x\in I} |Kf(x)| \leq \|f\|_\infty \sup_{x\in I} \int_I |k(x,y)|\,dy$$

$\Rightarrow K$ is a bounded operator.

Let us next prove that

$$\|K\| = \sup_{x\in I} \int_I |k(x,y)|\,dy .$$

From (2), we have

$$\|K\| = \sup_{\|g\|_\infty \le 1} \|Kg\|_\infty \le \sup_{x\in I} \int_I |k(x,y)|\,dy \;, \quad \forall g \in C([a,b]).$$

So lets go to see the other part

Let $x_0 \in I$ such that

$$\int_I |k(x_0,y)|\,dy = \sup_{x\in I} \int_I |k(x,y)|\,dy$$

For $\varepsilon > 0$ choose $g \in C([a,b])$ with

$$g(y) = \frac{\overline{k(x_0,y)}}{|k(x_0,y)| + \varepsilon} \;, \quad y \in I$$

Then $\|g\|_\infty \le 1$ and

$$\underset{\text{its the supremum}}{\|Kg\|_\infty} \ge |(Kg)(x_0)| \overset{= \left|\int_I k(x_0,y)\, g(y)\,dy\right|}{=} \int_I \frac{|k(x_0,y)|^2}{|k(x_0,y)| + \varepsilon}\,dy$$

$$\ge \int_I \frac{|k(x_0,y)|^2 - \varepsilon^2}{|k(x_0,y)| + \varepsilon}\,dy = \int_I \frac{\cancel{(|k(x_0,y)| + \varepsilon)}(|k(x_0,y)| - \varepsilon)}{\cancel{|k(x_0,y)| + \varepsilon}}\,dy$$

$$= \int_I |k(x_0,y)|\,dy - \varepsilon(b-a)$$

Hence,

$$\|K\| = \sup_{\|g\|_\infty \le 1} \|Kg\|_\infty \ge \int_I |k(x_0,y)|\,dy - \varepsilon(b-a) \quad \forall \varepsilon > 0.$$

Letting $\varepsilon \to 0$, we get

$$\|K\| \ge \int_I |k(x_0,y)|\,dy = \sup_{x\in I} \int_I |k(x,y)|\,dy$$

and the proof is complete.

Hemen bukatu dot!! $\square$.

• **REMARK**: An integral operator can be seen as a generalization of ordinary matrix multiplication.

Let $A \in M_{m \times n}(\mathbb{C})$ with entries a_{ij} and let $u \in \mathbb{C}^n$ be given. Then $Au \in \mathbb{C}^m$ and its components are

$$\begin{pmatrix} a_{11} \cdots a_{1m} \\ \vdots \\ a_{n1} \cdots a_{nm} \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

$$(Au)_i = \sum_{j=1}^{n} a_{ij} u_j , \quad i = 1, \dots, m.$$

Thus, the function values $\underset{\text{the kernel is like } A}{k(x,y)}$ are analogous to the entries a_{ij} of the matrix A and the values $\underset{\text{the product}}{(kf)(x)}$ are analogous to the entries $(Au)_i$.

* **EXAMPLE** (Tensor Product kernels)

The tensor product of two functions g, h on $\mathbb{R}$ is the function $g \otimes h$ on $\mathbb{R}^2$ defined by

$$(g \otimes h)(x,y) = g(x)\overline{h(y)} , \quad x, y \in \mathbb{R}.$$

An important special case of an integral operator is where the kernel k is a tensor product:

$$k = g \otimes h.$$

If we assume that $g, h \in L^2(\mathbb{R})$, then for $f \in L^2(\mathbb{R})$ we have for all x for which $g(x)$ is defined that

$$\underline{\underline{(kf)(x)}} = \int g(x)\overline{h(y)} f(y)\,dy = \underline{\underline{\langle f, h \rangle_{L^2} g(x)}}$$

One can also extend this notion of the operator $g \otimes h$ to arbitrary Hilbert spaces by simply replacing $L^2(\mathbb{R})$ with H above.

Another example of an integral operator in the Hilbert space $L^2([0,1])$

- **EXAMPLE:** The Volterra Operator on $L^2([0,1])$

Let us define

$$(Kf)(x) = \int_0^x f(y)\,dy\,, \qquad f \in L^2([0,1]).$$

This is a bounded integral operator in $L^2([0,1])$ with kernel

$$k(x,y) = \begin{cases} 1\,, & y \leq x \\ 0\,, & y > x\,. \end{cases}$$

Note that $k \in L^2([0,1]\times[0,1])$.
Then we can easily shown that

$$\|Kf\|_{L^2}^2 = \int_0^1 |Kf(x)|^2 dx$$

$$= \int_0^1 \left| \int_0^x f(y)\,dy \right|^2 dx$$

$\int ab \leq \left(\int |a|^2\right)^{1/2} \left(\int |b|^2\right)^{1/2} \leftarrow$ Cauchy-Schwarz

$$\leq \int_0^1 \left(\int_0^x |f(y)|^2 dy \right)\left(\int_0^x 1^2\,dy \right) dx$$

$$\leq \int_0^1 \|f\|_{L^2}^2\, x\, dx$$

$$= \frac{1}{2}\|f\|_{L^2}^2\,.$$

Hence, K is bounded and $\|K\| \leq \dfrac{1}{\sqrt{2}}$.

In general, integral operators with kernels $k \in L^2(\mathbb{R}^2)$ are called Hilbert-Schmidt operators; they are very useful in the theory of PDE's.

this theorem provides a sufficient condition under which K is a bounded operator in $L^2(\mathbb{R})$

- THEOREM 1 (Hilbert-Schmidt integral operators)

Let $k \in L^2(\mathbb{R}^2)$ be fixed. Then the integral operator K given by (1) defines a bounded linear mapping of $L^2(\mathbb{R})$ into itself, with operator norm $\|K\| \leq \|k\|_{L^2}$.

$\boxed{\text{Proof:}}$

Assume that $k \in L^2(\mathbb{R}^2)$ and we define $k_x(y) = k(x,y)$.
Then $k_x \in L^2(\mathbb{R})$ for a.e. x. Hence, if $f \in L^2(\mathbb{R})$, then

$(Kf)(x) = \langle k_x, \bar{f} \rangle_{L^2}$ exists for almost every x.

We want to see that Kf is measurable and square integrable function of x.

if the preimage of each measurable set is measurable

We first consider the case where f and k are nonnegative.
Then $k(x,y)f(y)$ is a measurable function on $\mathbb{R}^2$ and by Tonelli's Theorem, we have that

Fubini's theorem for non-negative functions

$$(Kf)(x) = \int k(x,y) f(y)\, dy \quad \text{is a measurable function of } x.$$

Now, by Cauchy-Schwarz inequality, we get the estimate

$$\|Kf\|_{L^2}^2 = \int_{\mathbb{R}} |Kf(x)|^2 dx$$

$$= \int \left| \int k(x,y) f(y) dy \right|^2 dx$$

$$\leq \int \left(\int |k(x,y)|^2 dy \right) \left(\int |f(y)|^2 dy \right) dx$$

$$= \|f\|_{L^2}^2 \iint |k(x,y)|^2 dy\, dx$$

$$= \|f\|_{L^2}^2 \|k\|_{L^2}^2 < +\infty \quad \Rightarrow Kf \in L^2(\mathbb{R}).$$

Now suppose that $f \in L^2(\mathbb{R})$ and $k \in L^2(\mathbb{R}^2)$ are arbitrary.
We write

$$f = (f_1^+ - f_1^-) + i(f_2^+ - f_2^-)$$

$$k = (k_1^+ - k_1^-) + i(k_2^+ - k_2^-)$$

positive part of a function f

$$f^+(x) = \max\{f(x), 0\} = \begin{cases} f(x) & \text{if } f(x) > 0 \\ 0 & \text{otherwise} \end{cases}$$

negative part of f

$$f^-(x) = -\min\{f(x), 0\} = \begin{cases} -f(x) & \text{if } f(x) < 0 \\ 0 & \text{otherwise} \end{cases}$$

Note that both f^+ and f^- are nonnegative!

$$\left(\begin{array}{l} f = f^+ - f^- \\ |f| = f^+ + f^- \end{array} \right)$$

Each function $Kf_j^{\pm}(x) = \int k_j^{\pm}(x,y) f_j^{\pm}(y)\,dy$

is measurable and belongs to $L^2(\mathbb{R})$.

Because they are all square integrable, we can add them together appropriately to obtain $Kf \in L^2(\mathbb{R})$. Now that we know that Kf is measurable, we can apply the exact same estimates as above to conclude that

$$\|Kf\|_2 \le \|k\|_{L^2} \|f\|_{L^2}.$$

Hence K is a bounded mapping of $L^2(\mathbb{R})$ into itself, with operator norm $\|K\| \le \|k\|_{L^2}$.

But what happens if the kernel $k \notin L^2(\mathbb{R}^2)$? The above theorem does not say that K is not bounded. ? $\square$.

→ The following result gives a criteria for continuity of an integral operator on L^2; this will be improved later to allow also singular kernels.

• PROPOSITION (Schur's lemma or Schur's Test)

Assume that a continuous kernel function k satisfies

$$\sup_x \int |k(x,y)|\,dy \le C_1, \quad \sup_y \int |k(x,y)|\,dx \le C_2$$

Then the integral operator

$$(Kf)(x) = \int_{\mathbb{R}} k(x,y) f(y)\,dy$$

extends to a bounded operator $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ with norm bound $\|K\| \le (C_1 C_2)^{1/2}$

$\boxed{\text{Proof}}$

This follows from the Cauchy-Schwarz inequality.
As in the proof of Theorem 1, measurability of Kf is most easily shown by showing that Kf is measurable and square-integrable when f, k are nonnegative, and then extending to the general case.
For simplicity, we will omit the details and simply assume that Kf is measurable for arbitrary $f \in L^2(\mathbb{R})$. Then by applying the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
\|Kf\|_{L^2}^2 &= \int |Kf(x)|^2 dx \\
&= \int \left| \int k(x,y) \cdot f(y)\, dy \right|^2 dx \\
&\leq \int \left(\int |k(x,y)|^{1/2} |k(x,y)|^{1/2} |f(y)|\, dy \right)^2 dx \\
&\leq \int \left(\int |k(x,y)|\, dy \right) \left(\int |k(x,y)|\, |f(y)|^2 dy \right) dx \\
&\leq \int \underbrace{\sup_x \int |k(x,y)|\, dy}_{\leq C_1} \int |k(x,y)|\, |f(y)|^2 dy\, dx
\end{aligned}
$$

Tonelli (to interchange the order of integration)

$$
\begin{aligned}
&= \cancel{\int} C_1 \int |f(y)|^2 \int |k(x,y)|\, dx\, dy \\
&\leq C_1 \int |f(y)|^2 \underbrace{\sup_y \int |k(x,y)|\, dx}_{\leq C_2} dy \\
&\leq C_1 C_2 \|f\|_{L^2}^2
\end{aligned}
$$

Then K is bounded and $\|K\| \leq (C_1 C_2)^{1/2}$. $\square$.

• REMARK: We can generalize all the above results to $\mathbb{R}^n$.

* REMARK: If the kernel $k \notin L^2(\mathbb{R}^2)$, then the integral operator K may still be bounded in $L^2(\mathbb{R})$. Schur test is a very useful tool to check the boundedness of K in $L^2(\mathbb{R})$.

EXAMPLE: Let us consider the kernel $k : [0,1] \times [0,1] \to \mathbb{C}$ defined by

$$k(x,y) = \frac{1}{|x-y|^{1/2}}.$$

Note that $k \notin L^2([0,1]^2)$. However,

$$\sup_{y \in [0,1]} \int_0^1 |k(x,y)| \, dx < +\infty$$

and

$$\sup_{x \in [0,1]} \int_0^1 |k(x,y)| \, dy < +\infty .$$

Therefore, we can conclude that the integral operator

$$(Kf)(x) = \int_0^1 k(x,y) f(y) \, dy$$

is bounded in $L^2([0,1])$.

BOUNDED AND COMPACT OPERATORS

(PART II)

Today we continue working with bounded linear operators, introducing their kernel, image and cokernel, showing some properties related to them and also the relation with projections, an important class of bounded operators in Hilbert spaces. We will also see an special case where an integral equation of the second kind is solvable.

- **DEFINITION:** $A: H_1 \to H_2$ linear, H_1, H_2 Hilbert spaces.

  (i) $\operatorname{Ker}(A) = \{x \in H_1 : Ax = 0\}$ "kernel of A" or "null-space of A"

  (ii) $\operatorname{Im}(A) = \{y \in H_2 : \exists x \in H_1 \text{ s.t. } Ax = y\}$ "Image of A"

  (iii) $\operatorname{Coker}(A) = H_2 / \operatorname{Im}(A)$ "cokernel of A"

We leave the following easy result: If A is bounded $\Rightarrow$ $\operatorname{Ker}(A)$ closed, but $\operatorname{Im}(A)$ may not be closed.

- **PROPOSITION:** $A: H_1 \to H_2$ linear ; A injective $\Leftrightarrow$ $\operatorname{Ker}(A) = \{0\}$

  A surjective (i.e. onto) $\Leftrightarrow$ $\operatorname{Coker}(A) = \{0\}$

  Proof:

  Always $\{0\} \subset \operatorname{Ker}(A)$: $A0 = 0$. If A inj. $\Rightarrow Ax = 0 \Rightarrow x = 0$

  Conversely, assume $\operatorname{Ker}(A) = \{0\}$; let $Ax_1 = Ax_2 \Leftrightarrow A(x_1 - x_2) = 0 \Leftrightarrow$ $\Leftrightarrow x_1 - x_2 \in \operatorname{Ker} A \Leftrightarrow x_1 = x_2$.

  Finally, A surj. $\Leftrightarrow H_2 = \operatorname{Im}(A) \Leftrightarrow \operatorname{Coker}(A) = \{0\}$.

  $\square$.

* ORTHOGONAL PROJECTIONS

We begin by describing some algebraic properties of projections:

- If M and N are subspaces of a linear space X such that every $x \in X$ can be written uniquely as

$$x = y + z \quad \text{with } y \in M \text{ and } z \in N,$$

then we say that $X = M \oplus N$ is the direct sum of M and N, and we call N a complementary subspace of M in X.

- The decomposition $x=y+z$ with $y\in M$ and $z\in N$ is unique if and only if $M\cap N=\{0\}$

- If $X=M\oplus N$, then we define the projection $P: X\to X$ of X onto M along N by $\quad Px=y$, where $x=y+z$ with $y\in M, z\in N$.

The projection is linear, with $\operatorname{Im} P = M$ and $\ker P = N$,

and satisfies $P^2=P$.

↪ this property characterizes projections

- DEFINITION: A projection on a linear space X is a linear map $P: X\to X$ such that
$$P^2=P.$$

Any projection is associated with a direct sum decomposition.

- THEOREM: Let X be a linear space.

(a) If $P: X\to X$ is a projection, then $X=\operatorname{Im} P\oplus\ker P$.

(b) If $X=M\oplus N$, where M and N are linear subspaces of X, then there is a projection $P: X\to X$ with $\operatorname{Im} P = M$ and $\ker P = N$

$\boxed{\text{Proof:}}$

- To prove (a), we first show that $x\in\operatorname{Im} P$ if and only if $x=Px$.

$\begin{cases} \text{If } x=Px \Rightarrow x\in\operatorname{Im}P. \\ \text{If } x\in\operatorname{Im}P, \text{ then } x=Py \text{ for some } y\in X \Rightarrow Px=P^2y\overset{P\text{ proj.}}{=}Py=x. \end{cases}$

If $x\in\operatorname{Im}P\cap\ker P$, then $x=Px$ and $Px=0$, so $\operatorname{Im}P\cap\ker P=\{0\}$.

If $x\in X$, then we have
$$x=Px+(x-Px),$$
where $Px\in\operatorname{Im}P$ and $(x-Px)\in\ker P$, since
$$P(x-Px)=Px-P^2x=Px-Px=0.$$
Thus $X=\operatorname{Im}P\oplus\ker P$.

- To prove (b), we observe that if $X=M\oplus N$, then $x\in X$ has the unique decomposition $x=y+z$ with $y\in M$, $z\in N$.

Then $Px=y$ defines the required projection. $\square$

When using Hilbert spaces, we are particularly interested in orthogonal subspaces. Let's start with some basic definitions.

• <u>DEFINITION</u>: (i) Let H be a Hilbert space. A subset E of H is <u>convex</u>, if for all x and y in E the line sequent joining them lies also in E, i.e.

$$tx+(1-t)y \in E \quad \text{for all } t\in[0,1].$$

(ii) Elements $x,y\in H$ are orthogonal, if $\langle x,y\rangle=0$; then we often write $x\perp y$. Now, let

$$x^{\perp}=\{y\in H : \langle x,y\rangle=0\}.$$ → the set of all elements of H orthogonal to x

Similarly, if M is a subspace of H, then we let

$$M^{\perp}=\{y\in H : \langle x,y\rangle=0 \text{ for all } x\in M\}.$$

→ It is easy to see that $x^{\perp}$ and $M^{\perp}$ are always <u>closed</u> vectorsubspaces of H.
(A subspace L is closed, if it is closed with respect to the topology determined by the inner product; i.e. if a sequence of elements (x_i) of L converges to $x\in H$, then in fact $x\in L$)

→ The next theorem is very important.

• <u>THEOREM</u>: A nonempty convex and closed subset E of Hilbert H space contains a unique element of smallest norm. → this says that the distance to the origin gets minimized at some point. However, we know more, namely that there is exactly one point which is closest to the origin

$\boxed{\text{Proof:}}$

For $x,y\in H$, the parallelogram law

$$\|x+y\|^2+\|x-y\|^2 \leq 2\|x\|^2+2\|y\|^2 \qquad (*)$$

holds.

Let $\delta=\inf\{\|x\|, x\in E\}$, E convex and closed subset of H.
We first show that there is at most one element that minimizes the distance: For any $x,y\in E$, we can apply $(*)$ to $\frac{x}{2}$ and $\frac{y}{2}$ and get

$$\frac{1}{4}\|x-y\|^2=\frac{1}{2}\|x\|^2+\frac{1}{2}\|y\|^2-\left\|\frac{x+y}{2}\right\|^2.$$

Assume now that x and y both minimize the distance to the origin, i.e. $\|x\| = \|y\| = \delta$.

↳ Applying this to above we get

$$\frac{1}{4}\|x-y\|^2 = \frac{1}{2}\delta^2 + \frac{1}{2}\delta^2 - \left\|\frac{x+y}{2}\right\|^2 = \delta^2 - \left\|\frac{x+y}{2}\right\|^2 .$$

By convexity $\frac{x+y}{2} \in E$, and hence $\delta^2 \leq \left\|\frac{x+y}{2}\right\|^2$. Therefore, the above inequality implies

$$\|x-y\|^2 \leq 0, \quad \Rightarrow \quad \underline{\underline{x=y}}.$$

The geometric idea behind this proof is that if points x and y are at the same distance from the origin, then their midpoint is closer.

Next we prove the existence.

↳ This is a simple limiting argument.

By the definition of δ we know that there exists a sequence (x_n) of points of E such that $\|x_n\| \to \delta$ as $n \to \infty$.

Applying $(*)$ to elements of this sequence we get

$$\|x_n - x_m\| \leq 2\|x_n\|^2 + 2\|x_m\|^2 - 4\delta^2,$$

since by convexity $\frac{x_n+x_m}{2} \in E$, so $\left\|\frac{x_n+x_m}{2}\right\| \leq \delta$.

This shows that (x_n) is a Cauchy sequence, and hence by completeness there is a limit $x = \lim x_n$, and since E is closed, $x \in E$. $\|x\| = \delta$?

↳ $\|x\| = \lim_{n\to\infty} \|x_n\| = \delta$

and hence x is a distance minimizing element. $\square$.

We will use this to prove the following very important result:

- THEOREM (Existence of orthogonal projections)

Let H be a Hilbert space and let M be a closed subspace. Then there exists a unique mappings P and Q such that $P: H \to M$ and $Q: H \to M^\perp$ with

$$(+) \quad x = Px + Qx, \quad x \in H.$$

Furthermore, these maps have the following properties:

(i) If $x \in M$, then $Px = x$ and $Qx = 0$

(ii) If $x \in M^\perp$, then $Px = 0$ and $Qx = x$

(iii) $\|x - Px\| = \inf\{\|x-y\| : y \in M\}$, for all $x \in H$

(iv) $\|x\|^2 = \|Px\|^2 + \|Qx\|^2$, $x \in H$

(v) Both P and Q are linear.

$\boxed{\text{Proof:}}$

we shift the subspace M by x

Let $x \in H$ and define $x + M = \{x + y : y \in M\}$

This is a closed set since M is closed, and also convex:

assume $z_1 = x + y_1$ and $z_2 = x + y_2$ are in $x+M$.
let $t \in (0,1)$. Then since M is a subspace we have

$$tz_1 + (1-t)z_2 = x + ty_1 + (1-t)y_2 \in x + M$$

Hence the previous theorem gives the existence of a unique element of smallest norm in $x+M$, define this to be Qx:

$$Qx = \inf\{\|y\| : y \in x + M\}$$

and then define $Px = x - Qx$.

Hence, (iii) is true and (iv) is also valid.

Since $Qx \in x + M$, then we must have $Px \in M$, $x \in H$.

→ We next show that $Qx \in M^\perp$, i.e. $\langle Qx, y\rangle = 0$ $\forall y \in M$.

Without loss of generality we may assume that $y \in M$ and $\|y\| = 1$.

Then the definition of Qx gives

$$\|Qx\|^2 \leq \|Qx - \alpha y\|^2 = \langle Qx - \alpha y, Qx - \alpha y\rangle \text{ for every } \alpha \in \mathbb{C}$$

since $y \in M$. Since $\|y\| = 1$, we can simply this to

$\|Qx\|^2 \leq \|Qx\|^2 - \alpha\langle y, Qx\rangle - \bar{\alpha}\langle Qx, y\rangle + |\alpha|^2\|y\|^2$

$$0 \leq |\alpha|^2 - \alpha\langle y, Qx\rangle - \bar{\alpha}\langle Qx, y\rangle$$

$\langle Qx, y\rangle = \overline{\langle y, Qx\rangle}$ Choose $\alpha = \langle Qx, y\rangle$. Then from above we get

$$0 \leq -|\langle Qx, y\rangle|^2$$

and this is possible only if $\langle Qx, y\rangle = 0$. Hence we have shown that

$$Qx \in M^\perp, \quad x \in H.$$

We want to prove now uniqueness of the decomp. ⊕ and also properties (i) and (ii)

Assume now that

$$x = y + z, \quad y \in M, \quad z \in M^\perp.$$

We show that $y = Px$ and $z = Qx$ (hence proving uniqueness of decomp. (*))

Since $Px + Qx = x = y + z$, we have

$$\underbrace{y - Px}_{\in M} = \underbrace{Qx - z}_{\in M^\perp};$$

{Let $x \in M \cap M^\perp$. Then $\langle x, x\rangle = 0$ and thus $x = 0$}

Thus, both belong to $M \cap M^\perp = \{0\}$, i.e. $y = Px$ and $z = Qx$.

This also implies (i) and (ii)

→ The proof of (iv) is an immediate consequence of orthogonality:

$$\|x\|^2 = \langle Px + Qx, Px + Qx\rangle = \|Px\|^2 + \|Qx\|^2 + \langle Px, Qx\rangle + \langle Qx, Px\rangle =$$
$$= \|Px\|^2 + \|Qx\|^2$$

→ It remains to prove the linearity (v). ~~Let~~

Let $x, y \in H$ and $\alpha, \beta \in \mathbb{C}$ be arbitrary.

Applying ~~the~~ the decomposition ⊕ to x, y and $\alpha x + \beta y$ and adding, we get

$$\alpha Px + \alpha Qx + \beta Py + \beta Qy = P(\alpha x + \beta y) + Q(\alpha x + \beta y),$$

and recombining these terms we get

$$\underbrace{\alpha Px + \beta Py - P(\alpha x + \beta y)}_{\in M} = \underbrace{Q(\alpha x + \beta y) - \alpha Qx - \beta Qy}_{\in M^\perp}.$$

Hence they both vanish and we have

$$P(\alpha x + \beta y) = \alpha Px + \beta Py, \quad Q(\alpha x + \beta y) = \alpha Qx + \beta Qy$$

⇒ P and Q are linear mappings. □.

<u>REMARK</u>: The linear maps P and Q have also the following properties:

$$P^2 x = Px, \quad Q^2 x = Qx, \quad PQ = QP = 0.$$

The map P is the <u>orthogonal projection</u> to M.

• <u>DEFINITION</u>: An orthogonal projection on a Hilbert space H is a linear map $P: H \to H$ that satisfies

$$P^2 = P \quad , \quad \langle Px, y\rangle = \langle x, Py\rangle \quad \text{for all } x, y \in H.$$

→ An orthogonal projection is necessarily bounded.

• <u>PROPOSITION</u>: If P is a nonzero orthogonal projection, then $\|P\| = 1$.

$\boxed{\text{Proof:}}$

If $x \in H$ and $Px \neq 0$, then by Cauchy-Schwarz inequality,

$$\|Px\| = \frac{\langle Px, Px\rangle}{\|Px\|} = \frac{\langle x, P^2x\rangle}{\|Px\|} = \frac{\langle x, Px\rangle}{\|Px\|} \leq \|x\|.$$

Therefore $\|P\| \leq 1$.

If $P \neq 0$, then there is an $x \in H$ with $Px \neq 0$, and $\|P(Px)\| = \|Px\|$, so that $\|P\| \geq 1$.

$\hookrightarrow \|\cancel{Px}\| \leq \|P\| \, \|\cancel{Px}\|$

D.

→ In the ~~previous~~ above theorem we saw that if M is a <u>closed</u> subspace of H, then there exists ~~a projection~~ an orthogonal projection P to M. In fact, there is a one-to-one correspondence between orthogonal projections P and closed subspaces M of H such that $\operatorname{Im} P = M$ and $\ker P = M^{\perp}$. Now we have all the ingredients to prove the following:

• <u>THEOREM</u>: Let H be a Hilbert space.

(i) If P is an orthogonal projection on H, then $\operatorname{Im} P$ is closed, and
$$H = \operatorname{Im} P \oplus \ker P$$

(ii) If M is a closed subspace of H, then there is an orthogonal projection P on H with $\operatorname{Im} P = M$ and $\ker P = M^{\perp}$.

<u>EXERCISE</u>

***REMARK:** If P is an orthogonal projection on H with $\operatorname{Im} P = M$ and associated orthogonal direct sum $H = M \oplus N$, then $I - P$ is the orthogonal projection with $\operatorname{Im}(I-P) = N$.

* • **EXAMPLE 1:** The space $L^2(\mathbb{R})$ is the orthogonal direct sum of the space M of even functions and the space N of odd functions.

↳ The orthogonal projections P and Q of $L^2(\mathbb{R})$ onto M and N are given by

$$Pf(x) = \frac{f(x) + f(-x)}{2}, \qquad Qf(x) = \frac{f(x) - f(-x)}{2}$$

• **EXAMPLE 2:** Suppose that A is a measurable ~~set~~ subset of $\mathbb{R}$ - for ex. an interval - with characteristic function

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Then

$$P_A f(x) = \chi_A(x) f(x)$$

is an orthogonal projection of $L^2(\mathbb{R})$ onto the subspace of functions with support contained in $\bar{A}$.

We will ~~see~~ these orthogonal projections to characterize compact operators, and spectral theorem. I wanted to introduce them here, as a review of bounded operators.

Now we go back to int. equations

* NEUMANN SERIES

For operator equations of the second kind

$$\varphi - A\varphi = f$$

existence and uniqueness of a solution can be established by the Neumann series provided A is a contraction, i.e. $\|A\| < 1$.

• **PROPOSITION:** Let $A : H \to H$ be linear, H Hilbert. If $\|A\| < 1$, then $I - A : H \to H$ has a bounded inverse operator given by the Neumann series

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$$

and which satisfies

$$\|(I-A)^{-1}\| \leq \frac{1}{1-\|A\|}.$$

The iterated operators A^k are defined by $A^0 := I$ and $A^n := A A^{n-1}$ for $n \in \mathbb{N}$.

$\boxed{\text{Proof:}}$

First observe that it holds

$$\|A^n\| \leq \|A\| \|A^{n-1}\| \leq \ldots \leq \|A\|^n$$ product of operators

Since $\|A\| < 1$ we have absolute convergence

$$\sum_{k=0}^{\infty} \|A^k\| \leq \sum_{k=0}^{\infty} \|A\|^k = \frac{1}{1-\|A\|}$$

and hence, we have the convergence of $\sum_{k=0}^{\infty} A^k$.

Let

$$S := \sum_{k=0}^{\infty} A^k$$

with $\quad \|S\| \leq (1-\|A\|)^{-1}$.

Let us see that S is the inverse of $I-A$:

$$(I-A)S = (I-A) \lim_{n\to\infty} \sum_{k=0}^{n} A^k$$

$$= \lim_{n\to\infty} (I-A)[A^0 + \ldots + A^n] = \lim_{n\to\infty} (I - A^{n+1}) = I$$

and

$$S(I-A) = \lim_{n\to\infty} \sum_{k=0}^{n} A^k (I-A) = \lim_{n\to\infty} (I - A^{n+1}) = I,$$

since $\quad \|A^{n+1}\| \leq \|A\|^{n+1} \to 0, \quad n \to \infty.$

$\square$.

→ Obviously, the partial sums

$$\varphi_n := \sum_{k=0}^{n} A^k f$$

of the Neumann series satisfy $\varphi_{n+1} = A\varphi_n + f$ for $n \geq 0$.
Hence, the Neumann series is related to successive approximations by the following:

• <u>THEOREM</u>: Under the assumptions of previous theorem, for all $f \in H$ the successive approximations

$$\varphi_{n+1} := A\varphi_n + f, \quad n = 0, 1, 2, \dots,$$

with arbitrary $\varphi_0 \in H$ converge to the unique solution φ of $\varphi - A\varphi = f$.

Proof:

By induction, it is readily seen that

$$\varphi_n = A^n \varphi_0 + \sum_{k=0}^{n-1} A^k f, \quad n = 1, 2, \dots$$

whence

$$\lim_{n\to\infty} \varphi_n = \sum_{k=0}^{\infty} A^k f = (I - A)^{-1} f$$

$\square$.

We explicitly want to state this result for integral equations:

• <u>COROLLARY</u>: Let K be a continuous kernel satisfying

$$\sup_{x\in\mathbb{R}} \int_{\mathbb{R}} |k(x,y)|\, dy < 1, \quad \sup_{y\in\mathbb{R}} \int_{\mathbb{R}} |k(x,y)|\, dx < 1.$$

Then for all $f \in \overset{L^2(\mathbb{R})}{C(\mathbb{R})}$ the integral equation

$$\varphi(x) - \int_I k(x,y)\varphi(y)\, dy = f(x), \quad x \in I$$

has a unique solution $\varphi \in \overset{\in L^2(\mathbb{R})}{C(\mathbb{R})}$. The successive approximations

$$\varphi_{n+1}(x) := \int_I K(x,y)\, \varphi_n(x) + f(x), \quad n = 0, 1, 2, \dots$$

with arbitrary $\varphi_0 \in \overset{L^2(\mathbb{R})}{C(\mathbb{R})}$ uniformly converges to this solution.

COMPACT OPERATORS

→ In functional analysis, compact operators on Hilbert spaces are a direct extension of matrices: in the Hilbert spaces, they are precisely the closure of finite-rank operators in the uniform operator topology.

→ Lets start with the definition

• **DEFINITION:** Let H_1, H_2 be Hilbertspaces. A bounded linear operator $T: H_1 \to H_2$ is compact if for any bounded set $U \subset H$, then $\overline{T(U)}^{\subset H_2}$ (= closure of $T(U)$ in H_2) is compact set. (if every open covering of $\overline{T(U)}$ contains a finite subcovering)

• **REMARK:**

(i) If H_1, H_2 are finite dimensional (actually it is enough to assume only H_2 to be finite dimensional), then any bounded linear map $T: H_1 \to H_2$ is compact, since in $\mathbb{R}^n$ (or $\mathbb{C}^n$)

$$\text{bounded} + \text{closed} \iff \text{compact}.$$

- $\overline{T(U)}$ closed ✓
- $\overline{T(U)}$ bounded? T bounded ✓ } ⇒ ($\overline{T(U)}$ compact set)

{ If H_2 is finite dimensional ⇓ $Im(T) \subseteq H_2$ finite dimensional; $\dim Im(T) < \infty$ ⟹ The operator T is finite dimensional }

(ii) If H_2 is not finite dimensional (and H_1 also), then

T bounded $\neq \overline{T}$ compact

→ We have the following characterization for compact operators:

• **PROPOSITION:** $T: H_1 \to H_2$ compact $\iff$ any bounded sequence (x_n) in H_1 contains a subsequence (x_{n_j}) such that (Tx_{n_j}) converges.

(if for any bounded seq. x_n in H_1, (Tx_n) has a convergent subsequence).

↓

This follows immediately from the ~~[illegible]~~ definition of compact operators and the following result:

"A bounded subset $A \subset X$ (X Banach or Hilbert) is compact iff every sequence (a_j) of A has a convergent subsequence (a_{i_j})."

• Rule of thumb: Compact operators are the natural generalization of finite dimensional operators and they have many of their useful properties

• **PROPOSITION:** Let H, H_1, H_2 be Hilbert spaces.

(i) If $T: H_1 \to H_2$ is compact, then T is also bounded

(ii) If $T: H_1 \to H_2$ has finite dimensional image ($\dim Im(T) < \infty$), then T is compact.

(iii) Linear combination of compact operators are compact

(iv) Let $T_n : H_1 \to H_2$ be compact for all $n \in \mathbb{N}$ and assume that

$$\lim_{n\to\infty} \|T_n - T\| = 0.$$

then T is compact.

(v) If $T, S : H \to H$ are linear and bounded, then if T (or S) compact $\Rightarrow$ TS and ST are also compact

if one of them is compact, the product is compact

$\boxed{\text{Proof:}}$

(i) Assume that T is not bounded

$\Rightarrow \exists\ \tilde{x}_n \in H_1$ s.t. $\|T\tilde{x}_n\| > n\|\tilde{x}_n\|$

Let $x_n = \dfrac{\tilde{x}_n}{\|\tilde{x}_n\|}$, $\|x_n\| = 1$ (x_n is a bounded sequence in H_1)

$$\|Tx_n\| = \frac{\|T\tilde{x}_n\|}{\|\tilde{x}_n\|} > n$$

Now, since x_n is bounded and T is compact,

$\exists\ x_{n_k}$ subsequence of x_n s.t. $Tx_{n_k} \to z$

$\downarrow$

$\|x_{n_k}\| = 1$, $\|Tx_{n_k}\| > n_k$

$$\underbrace{\lim_{n_k\to\infty} n_k}_{+\infty} < \lim_{n_k\to\infty} \|Tx_{n_k}\| = \|z\| < \infty \quad \#$$

Hence, T has to be bounded.

(ii) $B \subset H_1$ bounded. $\overline{T(B)}$ compact?

$\dim \mathrm{Im}(T) < \infty \Rightarrow \overline{T(B)}$ closed + bounded $\Rightarrow \overline{T(B)}$ compact ✓

(iii) Assume $T_1, T_2 : H_1 \to H_2$ are compact. Let

$(aT_1 + bT_2)x_i$ be a sequence in $\overline{(aT_1 + bT_2)(B)}$
$(a, b \in \mathbb{C})$

$B \subset H$ bounded set

Then $\exists$ subsequence (x_j') s.t. $(T_1 x_j')$ converges; and by assumption, $\exists$ subsequence (x_j'') of (x_j') s.t. $(T_2 x_j'')$ converges; then $(aT_1 + bT_2)x_j''$ converges, i.e. $aT_1 + bT_2$ is compact.

(iv) T_n compact and $\|T_n - T\| \to 0$.

Let (x_n) be a bounded sequence in H_1:

$$\sup_{n \in \mathbb{N}} \|x_n\| \le c.$$

{ since T_n compact, for every $n \in \mathbb{N}$ $\exists$ a subsequence of (x_n), $x_{k,n}$ such that $T_n x_{k,n}$ converges as $k \to \infty$. }

Inductively, let

① $T_1 x_{1,n}$ be a convergent subsequence of $T_1 x_n$ ($x_{1,n}$ is a bounded subs. of x_n)

② $T_{k+1} x_{k+1,n}$ be a convergent subsequence of $T_{k+1} x_n$ ($x_{k+1,n}$ is a bounded subs. of x_n)

Let $y_i = x_{i,i}$ (diagonal sequence)

$$\begin{matrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} & \cdots \\ x_{2,1} & x_{2,2} & & x_{2,n} & \cdots \\ \vdots & & & \vdots & \\ x_{n,1} & x_{n,2} & \cdots & x_{n,n} & \cdots \end{matrix}$$

$\Rightarrow$ (y_m) is a subsequence of (x_m) and also a subsequence of $x_{i,m}$. ($T_i x_{i,n}$ converges as $i \to \infty$)

Then for $i > k$, $(Ty_i)_{i>k}$ is a subsequence of $(Tx_{k,i})$ and on the other hand $(T_k x_{k,i})$ converges as $i \to \infty$.

Now, since $\|T_n - T\| \to 0$,

Let $\varepsilon > 0$. Choose $N > 0$ s.t. $i > N$ and

$$\|T - T_N\| < \frac{\varepsilon}{3c}$$ ⤳ c is the upper bound of $\|x_n\|$.

Then if $j, i > N$

$$\|Ty_i - Ty_j\| \le \overbrace{\|(T - T_N)y_i\|}^{\le \frac{\varepsilon}{3c}\|y_i\|} + \overbrace{\|T_N(y_i - y_j)\|}^{\le} + \overbrace{\|(T - T_N)y_j\|}^{\le \frac{\varepsilon}{3c}\|y_j\|} \rightsquigarrow$$

$$\leq \|T-T_N\|\,\|y_i\| + \|T_N(y_i-y_j)\| + \|T-T_N\|\,\|y_j\|$$

$$\leq \frac{2\varepsilon}{3} + \|T_N(y_i-y_j)\| \qquad \text{since } \|y_j\| \leq C.$$

But $(T_N y_i)_{i>N}$ converges $\Rightarrow \exists N' > N$ s.t. $i,j > N' \Rightarrow$

$$\Rightarrow \|T_N(y_{i'}-y_{j'})\| < \frac{\varepsilon}{3}, \quad i',j' > N' > N.$$

Hence, $i',j' > N \Rightarrow$

$$\|Ty_{i'} - Ty_{j'}\| < \varepsilon \Rightarrow (Ty_i) \text{ is Cauchy in } H_2$$

$\Rightarrow Ty_i$ is convergent $\Rightarrow$ T is compact.

(v) EXERCISE.

$\left.\begin{array}{l} T_i \text{ bounded} \\ \dim \operatorname{Im}(T_i) < \infty \end{array}\right\} \overset{(ii)}{\Rightarrow} \left.\begin{array}{l} T_i \text{ compact} \\ \|T_i - T\| \to 0 \end{array}\right\} \overset{(iv)}{\Rightarrow} T \text{ compact}$

$\square$.

- **CORollary** ⊛: If $\|T-T_i\| \to 0$, $\dim \operatorname{Im} T_i < \infty \Rightarrow T$ compact

(If an operator is approximated by a sequence of finite dimensional operator, then the operator is compact. $\dim \operatorname{Im}(T_i) < \infty$)

PAPELA !!

- **REMARK**: Orthogonal projection in a Hilbert space is compact iff $\dim(\operatorname{Im}(P)) < \infty$.

$P: H \to M$, $M \subset H$ closed

(the subspace is finite dimensional).

- **PROPOSITION**: Let H_0, H_1 be Hilbert spaces, $P_n: H_1 \to H_1$, $\dim \operatorname{Im}(P_n) < \infty$, $P_n \to I$ pointwise ($P_n y \to y \ \forall y \in H_2$).

$\underbrace{P_n: H_1 \to H_1, \dim \operatorname{Im}(P_n)}_{\text{→ } P_n \text{ is compact → bounded}}$

If $T: H_0 \to H_1$ is compact and linear, then $\|P_nT - T\| \to 0$.

For example P_n orthogonal projection over finite dimensional space.

CORollary ⊛

$\left.\begin{array}{l} T_n \text{ bounded} \\ \dim \operatorname{Im}(T_n) < \infty \\ \|T_n - T\| \to 0 \end{array}\right\} \Rightarrow T \text{ compact}$

$\overset{?}{\Leftarrow}$

→ Then we can construct

$T_n = P_nT$ with orthogonal projections (T_n compact)

FREDI

Proof:

P_n is linear, pointwise convergent to I

• We want to use the "Banach-Steinhaus theorem" or the "uniform boundedness principle", one of the fundamental results in functional analysis:

(BS) • Theorem (Banach-Steinhaus)

A sequence of linear mappings (continuous), $A_n: E \to F$, E, F Banach spaces converges pointwise to a linear and continuous map A iff the following holds:

(a) the sequence of $\|A_n\|$ is bounded

(b) the sequence $(A_n x)$ converges for all $x \in M$, $M \subseteq E$ dense.

If we apply (BS) to P_n, we can deduce that

$$\sup_n \|P_n\| < \infty \quad , \quad \sup_n \|P_n - I\| < \infty$$

$A \subseteq \bigcup_{i=1,..,k} B(y_i, \varepsilon)$

$\Uparrow$

every open covering of A contains a finite subcovering

$B(y, y_i) < \varepsilon$

Now, let A be a compact subset of H_2.

Choose $\varepsilon > 0 \Rightarrow \exists k \in \mathbb{N}$ and $y_1, \dots, y_k \in H_2$ s.t. $\forall y \in A$ (bounded)

$$\inf\{\|y - y_i\| : i \in \{1, \dots, k\}\} < \varepsilon$$

(this is a property of compact set $A \subseteq \bigcup_{i=1,..,k} B_\varepsilon(y, y_i)$)

Now for a given y ($\in A$ (bounded)) choose y_j s.t.

$$\|y - y_j\| \le \inf\{\|y - y_i\| \; i \in \{1, .., k\}\} < \varepsilon$$

For each point, we choose the point such that $\|y - y_j\| < \varepsilon$

Then

$$\|(P_n - I)y\| \le \|(P_n - I)y - (P_n - I)y_j\| + \|(P_n - I)y_j\|$$

$$\le \|P_n - I\| \, \|y - y_j\| + \|(P_n - I)y_j\|$$

$$\le \varepsilon \|P_n - I\| + \sup_j \|(P_n - I)y_j\| \quad \text{for each } j \in \{1, .., k\}$$

$P_n y \in BC \; Im P_n$ where is a finite dimensional space

$y \in A \subset H_2$ compact

(BS) Note that since $P_n \to I$ pointwise, then

$$(P_n - I)y_j \xrightarrow{(y_j \in M,\ M \subseteq H_1 \text{ dense})} 0 \quad n \to \infty \text{ and hence,}$$

$$\lim_{n \to \infty} \sup_{y_j \in M \subseteq H_1} \|(P_n - I)y_j\| = 0.$$

y_j can be $\in A$ or otherwise, y_j is a limit point of A

$\downarrow$

(if every neighborhood of y_j contains a point of A)

$\Downarrow$ $\forall y$ ✓

On the other hand,

$$\sup_n \|P_n - I\| < +\infty.$$

$\longrightarrow$

↳ As a consequence, we have proved that $\exists\, n_0 \in \mathbb{N}$ s.t. $\forall n \geq n_0$, then

$$\|(P_n - I)y\| < \overset{\text{some constant}}{C}\,\varepsilon \qquad (\forall \varepsilon > 0)$$

$\Rightarrow$ And we can also prove that

$$\sup_{y \in A} \|(P_n - I)y\| < C\varepsilon . \qquad (\forall \varepsilon > 0)$$

So, we have

$$\lim_{n\to\infty} \sup_{y \in A} \|(P_n - I)y\| = 0 \qquad (*)$$

Now since T is a compact operator, if we take $\mathcal{U}$ the unit ball in H_1 ($\mathcal{U} = \{x \in H_1 : \|x\|_1 < 1\}$), then we know that $\overline{T(\mathcal{U})}$ is compact

$$\Rightarrow \quad 0 \leq \lim_{n\to\infty} \cancel{\sup} \|P_n T - T\|$$

$$\overset{\text{norm def.}}{=} \lim_{n\to\infty} \cancel{\sup} \left[\sup_{x \in \mathcal{U}} \|P_n T x - T x\| \right]$$

$$= \lim_{n\to\infty} \cancel{\sup} \left[\sup_{y \in \overline{T(\mathcal{U})}} \|P_n y - y\| \right]$$

$$\overset{(*)\text{ with } A = \overline{T(\mathcal{U})}}{=} 0$$

Thus we have $\|P_n T - T\| \longrightarrow 0$.

→ Let $\{e_n\}$ be a orthogonal basis of H_2

$\Downarrow$

$$y = \sum_{i=1}^{\infty} \langle y, e_i \rangle e_i$$

$$P_n y = \sum_{i=1}^{n} \langle y, e_i \rangle e_i, \quad \dim \operatorname{Im}(P_n) = n$$

$\Rightarrow P_n \to I$ ✓

$\square$.

• REMARK: • Standard examples for P_n are orthogonal projections on finite dimensional subspaces.

~~• T is compact $\Leftrightarrow$ there exist bounded, finite dimensional operators T_n s.t. $T_n \to T$ in operator norm.~~

($T_n = P_n T$)

• $I : H_1 \to H_1$ is compact $\Leftrightarrow \dim H_1 < \infty$

→ Now we go to study the compactness of integral operators.

• **PROPOSITION:** Let $k \in C(I \times I)$, $I = [a,b] \subset \mathbb{R}$ and

$$K f(x) = \int_I k(x,y) f(y)\, dy, \qquad x \in I$$

Then K is compact as operator between any of the spaces $L^2(I)$ and $C(I)$.

$$K: \begin{cases} L^2(I) \to C(I) \\ C(I) \to C(I) \\ L^2(I) \to L^2(I) \\ C(I) \to L^2(I) \end{cases} \text{ is compact if } k \text{ is continuous.}$$

Proof:

We consider

$K: L^2(I) \to C(I)$

$K_1: C(I) \to L^2(I)$

$K_2: C(I) \to C(I)$

$K_3: L^2(I) \to L^2(I)$

Let us consider the inclusion map

$i: C(I) \to L^2(I)$ (Recall that $C(I)$ is dense in $L^2(I)$)!!

Then, note that

$K_1 = i \circ K \circ i$ $\qquad K_1: C(I) \xrightarrow{i} L^2(I) \xrightarrow{K} C(I) \xrightarrow{i} L^2(I)$

$K_2 = i \circ K$ $\qquad K_2: C(I) \xrightarrow{i} L^2(I) \xrightarrow{K} C(I)$

$K_3 = K \circ i$ $\qquad K_3: L^2(I) \xrightarrow{K} C(I) \xrightarrow{i} L^2(I)$

In addition,

$$\|x\|_2^2 = \int_I |x(t)|^2 dt \leq \sup_{t \in I} |x(t)|^2 |I|$$

$$\Rightarrow \|x\|_2 \leq \sqrt{|I|}\, \|x\|_\infty \Rightarrow i: \underset{\|\cdot\|_\infty}{C(I)} \to \underset{\|\cdot\|_2}{L^2(I)} \text{ is continuous.}$$

$$x \longmapsto i(x) = x$$

$\Longrightarrow$ K_1, K_2, K_3 are compact if K is compact.

by using the fact that the product of compact and bounded is compact

So, let us show that $K: L^2(I) \to C(I)$ is a compact operator.

The idea of the proof will be to construct a sequence K_n of {continuous ~~compact~~} finite dimensional operators which will converge to K.

($\|K_n - K\| \to 0$, compact, $\dim K_n < +\infty$ $\Rightarrow$ K compact)

Let

$$I_k^{(n)} = \left[\frac{k}{n}, \frac{(k+1)}{n} \right], \quad k = 0, 1, \dots, n-1.$$

$(I = [0,1])$

$$Q_k^{(n)} = I \times I_k^{(n)},$$

and choose $x \in I$, $y_k^{(n)} \in I_k^{(n)}$.

We define the kernel

$\leadsto k^{(n)}$ continuous

$$\boxed{k^{(n)}(x,y) := \sum_k k(x, y_k^{(n)}) \, \chi_{Q_k^{(n)}}(x,y)}$$

we discretize the integration variable

(Since the kernel k is continuous, $\forall \varepsilon > 0 \ \exists n(\varepsilon) > 0$ s.t. $n > n(\varepsilon)$

$$\left| k(x,y) - \sum_{k=0}^{n-1} k(x, y_k^{(n)}) \, \chi_{Q_k^{(n)}}(x,y) \right| < \varepsilon \quad \forall (x,y) \in I \times I).$$

Now let

$$(k_n f)(x) := \int_I k^{(n)}(x,y) f(y) \, dy$$

$$= \sum_{k=0}^{n-1} k(x, y_k^{(n)}) \int_I \underbrace{\chi_{Q_k^{(n)}}(x,y)}_{= \chi_I(x) \times \chi_{I_k^{(n)}}(y)} f(y) \, dy$$

$$= \sum_{k=0}^{n-1} k(x, y_k^{(n)}) \, \chi_I(x) \int_I \chi_{I_k^{(n)}}(y) f(y) \, dy$$

Note:

(i) k_n is linear

(ii) $(k_n f) \in C(I)$ (because $k^{(n)}(x,y)$ is continuous)

(iii) $\mathrm{Im}(k_n) \in \mathrm{span}\{ k(\cdot, y_k^{(n)}) : k = 0, 1, \dots, n-1 \}$

$\Rightarrow \dim \mathrm{Im}(k_n) < \infty$

(iv) $\|k_n f\|_\infty \leq C \|f\|_{L^2}$?

$$\|K_n f\|_\infty = \sup_{x\in I} |K_n f(x)|$$

$$= \sup_{x\in I} \left| \sum_{k=0}^{n-1} k(x,y_k^{(n)})\, \chi_I(x) \int_I \chi_{I_k^{(n)}}(y) f(y)\, dy \right|$$

$$\leq \sum_{k=0}^{n-1} \|k(\cdot, y_k^{(n)})\|_\infty \int_{I_k^{(n)}} |f(y)|\, dy$$

$$\leq \|k\|_\infty \sum_{k=0}^{n-1} \int_{I_k^{(n)}} |f(y)|\, dy$$

$$= \|k\|_\infty \int_I |f(y)|\, dy$$

$$\overset{C-S}{\leq} \|k\|_\infty \left(\int_I dy\right)^{1/2} \left(\int_I |f(y)|^2 dy\right)^{1/2}$$

$$\leq \|k\|_\infty \sqrt{|I|}\, \|f\|_{L^2}$$

$\Longrightarrow$ $K_n : L^2 \to C(G)$ is continuous !

So let us finally see that $\|K_n - K\| \to 0$. This will guarantee that K is compact.

$$\|K_n - K\| = \sup_{\|f\|\leq 1} \|K_n f - K f\|_\infty = \sup_{\|f\|\leq 1} \sup_{x\in I} |K_n f(x) - K f(x)|$$

$$|K_n f(x) - K f(x)| = \left| \sum_{k=0}^{n-1} k(x,y_k^{(n)})\, \chi_I(x) \int_{I_k^{(n)}} f(y)\, dy - \sum_{k=0}^{n-1} \int_{I_k^{(n)}} k(x,y) f(y)\, dy \right|$$

$$\leq \sum_{k=0}^{n-1} \left| \int_{I_k^{(n)}} \left(k(x,y_k^{(n)}) - k(x,y)\right) f(y)\, dy \right|$$

$$\overset{C-S}{\leq} \sum_{k=0}^{n-1} \left(\int_{I_k^{(n)}} |k(x,y_k^{(n)}) - k(x,y)|^2 dy \right)^{1/2} \left(\int_{I_k^{(n)}} |f(y)|^2 dy \right)^{1/2}$$

we use C-S in $\mathbb{R}^n$: $\sum_{i=1}^n a_i b_i \leq \left(\sum_{i=1}^n a_i^2\right)^{1/2} \left(\sum_{i=1}^n b_i^2\right)^{1/2}$

$$\leq \left(\sum_{k=0}^{n-1} \int_{I_k^{(n)}} \underbrace{|k(x,y_k^{(n)}) - k(x,y)|^2 dy}_{(*)} \right)^{1/2} \Bigg(\underbrace{\sum_{k=0}^{n-1} \int_{I_k^{(n)}} |f(y)|^2 dy}_{= \|f\|_{L^2(I)}} \Bigg)^{1/2}$$

(*) one can prove that ~~the kernel k is~~

$$|k(x,y) - k(x, y_k^{(n)})| < \frac{\varepsilon}{\sqrt{|I|}} \qquad k=0,\dots,n-1$$

for $n > n_0$ (this comes from the continuity of k on $I\times I$ and the partition $\{I_k^{(n)}\}$).

Then, we get:

$$|k_n f(x) - k f(x)| \leq \left(\sum_{k=0}^{n-1} \frac{\varepsilon^2}{|I|} |I_k^{(n)}| \right)^{1/2} \|f\|_{L^2}$$

$$\leq \varepsilon \|f\|_{L^2}$$

$\Rightarrow \|k - k_n\| < \varepsilon \quad \Rightarrow \quad \|k - k_n\| \to 0$ and k is compact.

$\square$.

So by discretizing the kernel k, we construct a linear operator k_n with ~~dim~~ $\dim \operatorname{Im}(k_n) < +\infty$ such that $\|k_n - k\| \to 0 \Rightarrow k$ compact.

Hemen kosi ostegmian!

→ We next show that integral operators with Hilbert-Schmidt kernel are compact.

• **PROPOSITION:** let $k \in L^2(I\times I)$ and

$$Kf(x) = \int_I k(x,y) f(y)\, dy.$$

Then $K : L^2(I) \to L^2(I)$ is compact.

Proof:

$C(I\times I)$ is dense in $L^2(I\times I)$.

$\Rightarrow \exists\, k_n \in C(I\times I)$ s.t. $\|k_n - k\|_{L^2} \to 0$ as $n\to\infty$.

Let us consider the integral operator K_n generated by k_n:

$$(K_n f)(x) = \int_I k_n(x,y) f(y)\, dy$$

And let us check that $\|K_n - K\|^2_{L^2\to L^2} \to 0$.

$$\|k_n - k\|^2 = \sup_{\|f\|=1} \|k_n f - k f\|_{L^2}^2$$

$$= \sup_{\|f\|=1} \int_I |k_n f(x) - k f(x)|^2 \, dx$$

$$= \sup_{\|f\|=1} \int_I \left| \int_I k_n(x,y) f(y)\, dy - \int_I k(x,y) f(y)\, dy \right|^2 dx$$

$$= \sup_{\|f\|=1} \int_I \left| \int_I (k_n(x,y) - k(x,y)) f(y)\, dy \right|^2 dx$$

$$\overset{C-S}{\leq} \sup_{\|f\|=1} \int_I \int_I |k_n(x,y) - k(x,y)|^2 \, dx\, dy \int |f(y)|^2 dy$$

$$= \int_I \int_I |k_n(x,y) - k(x,y)|^2 \, dx\, dy \longrightarrow 0$$

K_n is generated by $k_n \in C(I \times I) \underset{\substack{\uparrow \\ \text{previous result}}}{\Longrightarrow} K_n$ compact

$\Longrightarrow K$ compact.

$\square$.

* Can we extend this for another kind of kernel?

For example, kernel with singularities ...

we could do it for weakly singular kernels.

The integral operator generated by this kernel $K : C(I) \to C(I)$ is compact! $\Longleftarrow$

$$k(x,y) = \frac{M(x,y)}{|x-y|^\alpha}, \quad 0 < \alpha < 1$$

and $M(x,y)$ is a ~~bounded~~ continuous function

$k : I \times I \setminus \{(s,s) : s \in I\} \to \mathbb{R}$

- $k|_{I \times I \setminus \{(s,s) : s \in I\}}$ cont.
- $|k(x,y)| \leq m |x-y|^{-\alpha}$

RIESZ THEORY FOR COMPACT OPERATORS

→ Our next goal is to present the basic theory for an operator equation

$$\varphi - K\varphi = f$$ → we want to solve this equation without using the adjoint operator (Fredholm theory)

of the second kind with a compact linear operator $K : X \to X$ with H a Hilbert space. This theory was developed from Riesz and originated by Fredholm's work on integral equations of the second kind.

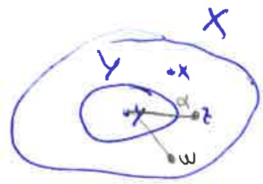
→ We start with a very useful ~~[illegible]~~ result which has some applications when dealing with infinite-dimensional spaces.

$\boxed{X \text{ normed space}}$

• **PROPOSITION** (Riesz Lemma)

Let X be a normed space, $Y \subseteq X$, $Y \neq X$. If Y is a closed subspace of X, $\varepsilon \in (0,1)$, then there exists $x \in X$ with $\|x\| = 1$ such that

$$\|y - x\| \geq \varepsilon, \quad \forall y \in Y.$$



$\boxed{\text{Proof}}$

Let $z \in X \setminus Y$

Set $\alpha := \inf \{ \|z - y\| : y \in Y \}$

As Y is closed, then $z \notin \partial Y \Rightarrow \underline{\alpha > 0}$

Choose $w \in Y$ s.t. $\alpha \leq \|z - w\| \leq \frac{\alpha}{\varepsilon}$ $\quad$ ε small

Set
$$x = \frac{z - w}{\|z - w\|} \in X \quad (\Rightarrow \|x\| = 1)$$

Then

$$\|y - x\| = \left\| y - \frac{z-w}{\|z-w\|} \right\| = \left\| \frac{\|z-w\| y}{\|z-w\|} - \frac{z-w}{\|z-w\|} \right\|$$

$$= \frac{1}{\|z-w\|} \left\| \|z-w\| y - (z-w) \right\|$$

$$= \frac{1}{\|z-w\|} \underbrace{\Big\| \overbrace{z}^{\in X \setminus Y} - (\overbrace{w}^{\in Y} + \underbrace{\|z-w\| y}_{\in Y}) \Big\|}_{\geq \alpha} \geq \alpha \frac{1}{\|z-w\|}$$

Hence,

$$\|y - x\| \geq \frac{1}{\|z-w\|} \alpha \geq \frac{\alpha}{\frac{\alpha}{\varepsilon}} = \varepsilon$$

$\square$.

~~the~~

→ We can use that to prove the following:

• <u>LEMMA</u>: The identity operator $I: X \to X$ is compact iff $\dim X < \infty$.

$\boxed{\text{Proof:}}$

$\Leftarrow |$ Since $I : X \to X$ is a bounded linear operator, if $\dim X < \infty$, we saw that then I is compact. ✓

$\Rightarrow |$ Let $\dim X = \infty$

Choose an arbitrary $\varphi_1 \in X$ with $\|\varphi_1\| = 1$.

Then $Y_1 = \operatorname{span}\{\varphi_1\}$ is a finite dimensional and closed subspace of X

$\Rightarrow \exists\ \varphi_2 \in X$ with $\|\varphi_2\| = 1$ and $\|\varphi_2 - \varphi_1\| \geq \frac{1}{2}$.

Now consider $Y_2 = \operatorname{span}\{\varphi_1, \varphi_2\}$

↳ $\exists\ \varphi_3 \in X$ with $\|\varphi_3\| = 1$ and $\|\varphi_3 - \varphi_1\| \geq \frac{1}{2}$, $\|\varphi_3 - \varphi_2\| \geq \frac{1}{2}$.

Repeating this procedure, we obtain a sequence (φ_n) with

$\|\varphi_n\| = 1$ and $\|\varphi_n - \varphi_m\| \geq \frac{1}{2}$, $n \neq m$.

If I is compact, then $I\varphi_n = \varphi_n$ has a convergent subsequence, but

$\|\varphi_n - \varphi_m\| \geq \frac{1}{2}$ $\Rightarrow$ no convergent subsequent! #

$\Rightarrow \dim X < +\infty$.

□.

• <u>REMARK</u>: This result justifies the distinction between operator equations of the first and second kind, since obviously the operators K and $I - K$, where K is a compact operator, have different properties.

Pentsam! ↳ Note that the compact operator K cannot have a bounded inverse unless its range has finite dimension!

If $\exists K^{-1}$ bounded for K compact $\Leftrightarrow KK^{-1} = K^{-1}K = I \Rightarrow I$ compact $\Rightarrow \dim \operatorname{Im}(K) < \infty$

→ Let us define the operator

$$L := I - K,$$

where $K: X \to X$ (X normed space) is a compact linear operator, and $I: X \to X$ denotes the identity operator.

→ We will study properties of L that will allow us to tell something about the solvability of integral equations of the second kind.

• **THEOREM : (First Riesz Theorem)**

The kernel of the operator L, $\ker L = \{\varphi \in X : L\varphi = 0\}$, is a finite dimensional subspace.

$\boxed{\text{Proof :}}$

Let $\varphi_n \in \ker(L)$ with $\varphi_n \to \varphi$ as $n \to \infty$.

$$L\varphi_n = 0 \underset{L = I - K}{\Longrightarrow} 0 = \lim_{n\to\infty} L\varphi_n \underset{L\text{ cont.}}{=} L(\lim \varphi_n) = L\varphi$$

$\Rightarrow \varphi \in \ker(L) \Rightarrow \ker(L)$ is a closed subspace of X

$\downarrow$

$K\varphi = \varphi$ for all $\varphi \in \ker(L)$

Set $\tilde{K} = K\big|_{\ker(L)}$ (The restriction of K to $\ker(L)$)

Let $\varphi \in \ker(L)$, then

• Remark : A restriction of a compact operator to a closed subspace is compact.

$$\tilde{K}\varphi = \varphi = I\big|_{\ker(L)}.$$

Since K is compact on X, then $\tilde{K} : \ker(L) \to \ker(L)$ is compact. Hence, by the previous result, $\dim \ker(L) < \infty$.

$\square$.

• **THEOREM (Second Riesz Theorem)**

The image of the operator L, $\operatorname{Im} L = \{L\varphi : \varphi \in X\}$, is a closed linear subspace.

$\boxed{\text{Proof:}}$

$\operatorname{Im}(L)$ is closed $\Updownarrow$ we want to see that $f \in \overline{\operatorname{Im}(L)} \Rightarrow \overline{\operatorname{Im}(L)} = \operatorname{Im}(L)$

Let $f \in \overline{\operatorname{Im}(L)} \Longrightarrow$ there exists a sequence (φ_n) in X such that

$$L\varphi_n \to f, \quad n \to \infty.$$

this is the best approximation with respect to $\ker(L)$

Let $\chi_n \in \ker(L)$ such that

$$\|\varphi_n - \chi_n\| = \inf_{\chi \in \ker(L)} \|\varphi_n - \chi\|$$

$\left\{ \inf_{\chi \in \ker(L)} \|\varphi_n - \chi\| = d(\varphi_n, \ker(L)) \quad \not\Downarrow \quad \exists \chi_n \in \ker(L) : \|\chi_n - \varphi_n\| \to d(\varphi_n, \ker(L)) \right\}$

we define the sequence

$$\overline{\varphi_n} := \varphi_n - \chi_n. \quad \text{(our original sequence with some influence of the kernel of the operator)}$$

Let us check that $\overline{\varphi_n}$ is bounded.

$\hookrightarrow$

↳ We assume that it is not bounded

Then there exists a subsequence $(\tilde{\varphi}_{n_k})$ s.t. $\|\tilde{\varphi}_{n_k}\| \geq k$ for all $k \in \mathbb{N}$.

Set $\psi_k = \dfrac{\tilde{\varphi}_{n_k}}{\|\tilde{\varphi}_{n_k}\|}$, $k \in \mathbb{N}$.

Since $\|\psi_k\| = 1$ and K compact,

$\exists\, \psi_{k_j}$ subsequence of ψ_k s.t. $K\psi_{k_j} \to \psi \in X$, $j \to \infty$.

In addition,

$$\|L\psi_k\| = \frac{\|L\tilde{\varphi}_{n_k}\|}{\|\tilde{\varphi}_{n_k}\|} \leq \frac{\|L\tilde{\varphi}_{n_k}\|}{k} \longrightarrow 0 \quad \text{as } k \to \infty$$

since the sequence $L\varphi_n$ is convergent and therefore, bounded.

Hence,

$$L\psi_{k_j} \longrightarrow 0, \quad j \to \infty.$$

Now, since

$$\psi_{k_j} \overset{L=I-A}{=} \underbrace{L\psi_{k_j}}_{\to 0} + \underbrace{K\psi_{k_j}}_{\to \psi} \xrightarrow{j\to\infty} \psi$$

$$\left.\begin{array}{lll} L \text{ bounded} \Rightarrow & L(\psi_{k_j}) \longrightarrow L\psi \\ \text{But also} & L(\psi_{k_j}) \longrightarrow 0 \end{array}\right\} \Rightarrow \underline{\underline{L\psi = 0}}$$

$\Rightarrow \psi \in \ker(L)$

Hence,
$$\underbrace{\chi_{k_j}}_{\in \ker(L)} + \|\tilde{\varphi}_{n_k}\| \underbrace{\psi}_{\in \ker(L)} \in \ker(L) \quad \text{for all } k.$$

$$\Rightarrow \|\psi_k - \psi\| = \frac{1}{\|\tilde{\varphi}_{n_k}\|} \left\| \tilde{\varphi}_{n_k} - \|\tilde{\varphi}_{n_k}\|\psi \right\|$$

$$= \frac{1}{\|\tilde{\varphi}_{n_k}\|} \left\| \varphi_{n,k} - \{\chi_{n_k} + \|\tilde{\varphi}_{n,k}\|\psi\} \right\|$$

$$\geq \frac{1}{\|\tilde{\varphi}_{n_k}\|} \inf_{\chi \in \ker(L)} \|\varphi_{n,k} - \chi\| \geq \frac{1}{\|\tilde{\varphi}_{n,k}\|} \|\varphi_{n_k} - \chi_{n_k}\| = 1,$$

which is a contradiction, since $\psi_{k_j} \to \psi$ as $j \to \infty$.

Therefore, $(\tilde{\varphi}_n)$ is bounded. $\Rightarrow$

$\Rightarrow$ As K is compact, $K\overline{\varphi}_n$ has a convergent subsequence $K\overline{\varphi}_{n_k}$.

$$\overline{\varphi}_{n_k} = L\overline{\varphi}_{n_k} + \underbrace{K\overline{\varphi}_{n_k}}_{\text{conv.}} = L(\varphi_{n_k} - \underbrace{x_{n_k}}_{\in \ker(L)}) + K\varphi_{n_k} = \underbrace{L\varphi_{n_k}}_{\to f} + \underbrace{K\varphi_{n_k}}_{\text{conv}}$$

$$\Rightarrow \lim_{k\to\infty} \overline{\varphi}_{n_k} = \varphi \qquad (L\varphi = f\ ?)$$

$$L\varphi = L\left(\lim_{k\to\infty} \overline{\varphi}_{n_k}\right) = \lim_{k\to\infty} L\overline{\varphi}_{n_k} = \lim_{k\to\infty} L(\varphi_{n_k} - x_{n_k}) = f \quad ✓$$

$\Rightarrow f \in \mathrm{Im}(L) \Rightarrow \mathrm{Im}(L)$ is closed.

$\square$.

- REMARK: Note that if $f \in \mathrm{Im}(L)$, then we constructed the solution of $L\varphi = f$. (as the limit of φ_n s.t. $L\varphi_n \to f$)

- $\mathrm{Im}(L)$ closed $\Rightarrow \exists$ a convergent sequence φ_n on X s.t. $L\varphi_n \to f$.
- $\overline{\varphi}_n = \varphi_n - x_n$, $x_n \in \ker L$ (we add some "influence" of the kernel space)
- Using the compactness of K, we get a convergent subsequence $\overline{\varphi}_{n_k}$ and the boundedness of $\overline{\varphi}_n$. $(\varphi = \lim_{n\to\infty} \overline{\varphi}_{n_k})$

- Finally, $\left.\begin{array}{l} L\varphi_n \to f \\ \downarrow \\ L\varphi \end{array}\right\} \Rightarrow L\varphi = f$.

$\rightarrow$ While working with orthogonal projections, we saw that if H is a Hilbert space, we can always decompose it as a direct sum of a closed subspace $M \subseteq X$ and its orthogonal $M^\perp$.

↳ In addition, we saw that we can do this by using (orth.) projections:

If $P: H \to H$ is an orthogonal projection, (remember that $\mathrm{Im}\, P$ is closed!!) then $H = \mathrm{Im}\, P \oplus \ker P$.

$\rightarrow$ Next question would be:

- Can we do that for a general normed space X?
- Does there exist some linear map T such that

$$X = \mathrm{Im}\, T \oplus \ker T\ ?$$

- ~~In the case that H is a Hilbert space, is this the unique decomposition? Or is it some other way to represent the elements?~~

→ The answer of this general question will be given by "Third Riesz Theorem" and the operator L plays a fundamental role there.

More precisely, we will work with iterated operators L^n, $n \geq 1$ given by:

$$L^n = (I-K)^n = I - K_n \,, \qquad n \geq 1$$

where

$$K_n = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} K^k .$$

- K compact $\Rightarrow$ K_n compact (linear combination of compact operators is compact and product of compact and bounded op. is compact too)

$\Downarrow$

Riesz 1: $\ker(L^n)$ is finite dimensional $\quad \forall n \geq 1$

Riesz 2: $\mathrm{Im}(L^n)$ is closed $\quad \forall n \geq 1$

GOAL: $L = I - k$, K compact. There exists $\nu \in \mathbb{N}$ such that

$$X = \ker(L^\nu) \oplus \mathrm{Im}(L^\nu)$$

ν is called the Riesz index of L.

→ We will now see several results as the ingredients to achieve our goal.

→ We start with the following chain of kernels: (Hemen basi!!) 2014.02.03

- **LEMMA A1:** Let X be a normed space, $k: X \to X$ compact, $L = I - k$. Then there exists a $\nu_1 \in \mathbb{N}_0$ such that

$$\{0\} = \ker(L^0) \subsetneq \ker(L) \subsetneq \ker(L^2) \subsetneq \dots \subsetneq \ker(L^{\nu_1}) = \ker(L^{\nu_1+1}) = \ker(L^{\nu_1+2}) = \dots$$

Note that $L^0 = I$ and $\ker(I) = \{0\}$ $\Downarrow$ so if we want to have this chain, it gives us the intuition that it has to be that the smallest set is the kernel of the smallest exponent of L.

Proof:

Obviously, $\ker(L^k) \subseteq \ker(L^{k+1})$, $k \geq 0$

$\varphi \in \ker(L^k) \Rightarrow L^k\varphi = 0 \Rightarrow L(L^k\varphi) = L^{k+1}\varphi = L0 = 0 \Rightarrow \varphi \in \ker(L^{k+1})$

Now assume $\ker(L^k) = \ker(L^{k+1})$

For $p > k$, if $\varphi \in \ker(L^{p+1}) \Rightarrow \varphi \in \ker(L^p)$? → This will mean that $\ker(L^p) = \ker(L^{p+1})$ $\forall p > k$

$L^{p+1}\varphi = 0 \Rightarrow 0 = L^{p+1}\varphi = L^{k+1}(L^{p-k}\varphi) \Rightarrow L^{p-k}\varphi \in \ker(L^{k+1}) = \ker(L^k)$

$\Rightarrow L^p\varphi = L^k(L^{p-k}\varphi) = 0 \Rightarrow \varphi \in \ker(L^p)$ ✓

→ The answer of this general question will be given by "Third Riesz Theorem" and the operator L plays a fundamental role there.

More precisely, we will work with iterated operators L^n, $n\geq 1$ given by:

$$L^n = (I-K)^n = I - K_n, \qquad n\geq 1$$

where

$$K_n = \sum_{k=1}^{n} (-1)^{k-1}\binom{n}{k} K^k.$$

- K compact $\Rightarrow$ K_n compact (linear combination of compact operators is compact and product of compact and bounded op. is compact too)

$\Downarrow$

Riesz 1: · $\ker(L^n)$ is finite dimensional $\forall n\geq 1$

Riesz 2: $\operatorname{Im}(L^n)$ is closed $\forall n\geq 1$

GOAL: $L = I - k$, K compact. There exists $\nu\in\mathbb{N}$ such that

$$X = \ker(L^\nu) \oplus \operatorname{Im}(L^\nu)$$

ν is called the Riesz index of L.

→ We will now see several results as the ingredients to achieve our goal.

→ We start with the following chain of kernels:

LECTURE 7

2014.02.03

- **LEMMA A1:** Let X be a normed space, $k: X\to X$ compact, $L = I - k$. Then there exists a $\nu_1\in\mathbb{N}_0$ such that

Note that $L^0 = I$ and $\ker(I) = \{0\}$ ⇓ so if we want to prove this chain, it gives us the intuition that it has to be that the smallest set is the kernel of the smallest exponent of L.

$$\{0\} = \ker(L^0) \subsetneq \ker(L) \subsetneq \ker(L^2) \subsetneq \dots \subsetneq \ker(L^{\nu_1}) = \ker(L^{\nu_1+1}) = \ker(L^{\nu_1+2}) = \dots$$

Proof:

Obviously, $\ker(L^k) \subseteq \ker(L^{k+1})$, $k\geq 0$

$\{\varphi\in\ker(L^k) \Rightarrow L^k\varphi = 0 \Rightarrow L(L^k\varphi) = L^{k+1}\varphi = L0 = 0 \Rightarrow \varphi\in\ker(L^{k+1})\}$

Now assume $\ker(L^k) = \ker(L^{k+1})$

For $p>k$, if $\varphi\in\ker(L^{p+1}) \Rightarrow \varphi\in\ker(L^p)$? → This will mean that $\ker(L^p) = \ker(L^{p+1})$ $\forall p>k$.

$L^{p+1}\varphi = 0 \Rightarrow 0 = L^{p+1}\varphi = L^{k+1}(L^{p-k}\varphi) \Rightarrow L^{p-k}\varphi\in\ker(L^{k+1}) = \ker(L^k)$

$\Rightarrow L^p\varphi = L^k(L^{p-k}\varphi) = 0 \Rightarrow \varphi\in\ker(L^p)$ ✓

Now we want to show that $\exists \nu_1 \in \mathbb{N}$ s.t. $\ker(L^{\nu_1}) = \ker(L^{\nu_1+1})$.
↳ And then we are done

Assume such ν_1 does not exist.

Then $\ker(L^k) \subsetneq \ker(L^{k+1})$ (Riesz lemma)

$\ker(L^k)$ is a closed subspace of $\ker(L^{k+1})$ $\Longrightarrow$

$\forall k \in \mathbb{N} \ \exists \varphi_k : \|\varphi_k\| = 1$ and $\|\varphi - \varphi_k\| \geq \frac{1}{2}$, $\forall \varphi \in \ker(L^k)$, $\varphi_k \in \ker(L^{k+1})$

Consider the sequence (φ_k).

For $n > m$, we have

$\varphi_m \in \ker(L^{m+1})$
$\varphi_n \in \ker(L^{n+1})$

$$\underbrace{\varphi_m + L\varphi_n - L\varphi_m \in \ker(L^n)}$$

$$L^n(\varphi_m + L\varphi_n - L\varphi_m) = \underbrace{L^{n-m-1}\underbrace{L^{m+1}\varphi_m}_{=0}}_{=0} + \underbrace{L^{n+1}\varphi_n}_{=0} - \underbrace{L^{n-m}\underbrace{L^{m+1}\varphi_m}_{=0}}_{=0}$$

$\varphi_m \in L^{m+1}$

$$\Rightarrow \underbrace{\|\underset{\in \ker(L^{n+1})}{\varphi_n} - (\overbrace{\varphi_m + L\varphi_n - L\varphi_m}^{\in \ker(L^n)})\|}_{=\ \varphi_n - \varphi_m - (I-A)\varphi_n + (I-A)\varphi_m \ =\ K\varphi_n - K\varphi_m} \geq \frac{1}{2}$$

$L = I - K$

$$\Rightarrow \|K\varphi_n - K\varphi_m\| \geq \frac{1}{2}$$

But K is compact! and (φ_n) is bounded $\Rightarrow$ there exists a convergent subsequence of $K\varphi_n$ #

$$\Rightarrow \exists \nu_1 \in \mathbb{N} \text{ s.t. } \ker(L^{\nu_1}) = \ker(L^{\nu_1+1})$$

□.

→ Now we study the chain of images.

• **LEMMA A2**: Under the assumptions of Lemma A1, there exists $\nu_2 \in \mathbb{N}_0$ such that

Here, $Im(I) = X$ → So it gives us the intuition that it goes the other way around

$$X = Im(L^0) \supsetneq Im(L) \supsetneq Im(L^2) \supsetneq \dots \supsetneq Im(L^{\nu_2}) = Im(L^{\nu_2+1}) = \dots$$

Proof:

Obviously, $\underbrace{Im(L^k) \supseteq Im(L^{k+1})}$

$\varphi \in Im(L^{k+1}) \Rightarrow \varphi = L^{k+1}\psi = L^k(\underbrace{L\psi}_{\in X}) \Rightarrow \varphi \in Im(L^k)$

If we have $Im(L^k) = Im(L^{k+1})$, then for all $p>k$ we have:

$$\text{if } \varphi \in Im(L^p) \Rightarrow \varphi = L^p \phi = L^{p-k}(L^k \phi) \Rightarrow \varphi \in Im(L^k) = Im(L^{k+1})$$

$$\Rightarrow \exists \psi \text{ s.t. } L^{k+1}\psi = L^k \phi$$

$$\Rightarrow \varphi = L^{p-k} L^k \phi = L^{p-k} L^{k+1} \psi = L^{p+1}\psi$$

$$\Rightarrow \varphi \in Im(L^{p+1}) \Rightarrow Im(L^{p+1}) \supseteq Im(L^p) \Rightarrow$$

$$\overset{\oplus}{\Rightarrow} \underline{Im(L^p) = Im(L^{p+1}) \quad \forall p > k}$$

Now we show existence of ν_2.

We assume the opposite: $Im(L^k) \supsetneq Im(L^{k+1})$

$$L^n = (I - A)^n = I - A_n, \quad A_n \text{ compact}$$

$\Rightarrow$ first and second Riesz theorems hold for L^n

$\Rightarrow$ $Im(L^{k+1})$ is closed subspace of $Im(L^k)$.

$$\underset{\text{Riesz lemma}}{\Rightarrow} \forall k \in \mathbb{N}_0 \ \exists \psi_k \in Im(L^k) \text{ s.t. } \|\psi_k\| = 1 \text{ and}$$

$$\|\psi_k - \psi\| \geq \frac{1}{2} \quad \forall \psi \in Im(L^{k+1})$$

Consider $\psi_k \in Im(L^k) \Rightarrow \exists \varphi_k \in X: \psi_k = L^k \varphi_k$

For $m>n$,

$$\psi_m + L\psi_n - L\psi_m = L^{n+1}(L^{m-n-1}\varphi_m + \varphi_n - L^{m-n}\varphi_m)$$

$$\Rightarrow \psi_m + L\psi_n - L\psi_m \in Im(L^{n+1})$$

$$\Rightarrow \underbrace{\| \underset{\in Im(L^n)}{\psi_n} - (\underbrace{\psi_m + L\psi_n - L\psi_m}_{\in Im(L^{n+1})}) \|}_{= \|A\psi_n - A\psi_m\|} \geq \frac{1}{2}$$

$$\|A\psi_n - A\psi_m\| \geq \frac{1}{2} \quad \forall n > m \qquad \# \text{ (since } A \text{ is compact)}$$

$$\Rightarrow \exists \nu_2 \text{ s.t. } Im(L^{\nu_2}) = Im(L^{\nu_2+1}).$$

$\square$.

$\Rightarrow$ So our next goal is to show that in fact, the exponents ν_1 and ν_2 related to the chains of the kernels and images of L^n are <u>equal</u>!

• **LEMMA A3**: Let $\nu_1, \nu_2 \in \mathbb{N}_0$ be given as in Lemmas A1 and A2, and

$$\nu_1 = \min_{\nu} \{ \ker(L^{\nu}) = \ker(L^{\nu+1}) \}$$

$$\nu_2 = \min_{\nu} \{ \mathrm{Im}(L^{\nu+1}) = \mathrm{Im}(L^{\nu}) \}$$

Then, $\underline{\nu_1 = \nu_2}$.

Proof:

Assume that $\nu_1 > \nu_2$

Then according to the definition of ν_2, we have

$$\mathrm{Im}(L^{\nu_1}) = \mathrm{Im}(L^{\nu_1 - 1}) = \dots = \mathrm{Im}(L^{\nu_2})$$

Let $\varphi \in \ker(L^{\nu_1}) \Rightarrow L^{\nu_1 - 1}\varphi \in \mathrm{Im}(L^{\nu_1 - 1}) = \mathrm{Im}(L^{\nu_1})$

in particular

$$\Rightarrow \exists \bar{\varphi} \text{ s.t. } \boxed{L^{\nu_1 - 1}\varphi = L^{\nu_1}\bar{\varphi}} \quad \oplus$$

$$0 = L^{\nu_1}\varphi = L(L^{\nu_1 - 1}\varphi) = L L^{\nu_1}\bar{\varphi} = L^{\nu_1 + 1}\bar{\varphi}$$

$$\Rightarrow \bar{\varphi} \in \ker(L^{\nu_1 + 1}) = \ker(L^{\nu_1}) \Rightarrow \bar{\varphi} \in \ker(L^{\nu_1})$$

$$\Rightarrow 0 = L^{\nu_1}\bar{\varphi} \overset{\oplus}{=} L^{\nu_1 - 1}\varphi \Rightarrow \varphi \in \ker(L^{\nu_1 - 1})$$

$$\Rightarrow \ker(L^{\nu_1}) \subseteq \ker(L^{\nu_1 - 1}) \Rightarrow \boxed{\ker(L^{\nu_1}) = \ker(L^{\nu_1 - 1})}$$

(because of Lemma A1 we have 2)

But this is a contradiction to the minimality of ν_1 #

Hence $\underline{\nu_1 \leq \nu_2}$.

Now assume $\nu_1 < \nu_2$

Let $\psi \in \mathrm{Im}(L^{\nu_2 - 1})$, $\psi = L^{\nu_2 - 1}\varphi \Rightarrow L\psi = L^{\nu_2}\varphi \in \mathrm{Im}(L^{\nu_2}) = \mathrm{Im}(L^{\nu_2 + 1})$

$$\Rightarrow \exists \bar{\varphi} : L\psi = L^{\nu_2 + 1}\bar{\varphi}$$

$\left\{ L\psi = L^{\nu_2}\varphi \right\}$

$$\Rightarrow L^{\nu_2}(\varphi - L\bar{\varphi}) = 0 \Rightarrow \varphi - L\bar{\varphi} \in \ker(L^{\nu_2})$$

As $\nu_2 > \nu_1$,

$$\ker(L^{\nu_1}) = \ker(L^{\nu_1 + 1}) = \dots = \ker(L^{\nu_2 - 1}) = \ker(L^{\nu_2})$$

$$\Rightarrow \varphi - L\bar{\varphi} \in \ker(L^{\nu_2 - 1}) \Rightarrow L^{\nu_2 - 1}\varphi = L^{\nu_2}\bar{\varphi} = \psi \in \mathrm{Im}(L^{\nu_2}) \Rightarrow$$

$\Rightarrow \mathrm{Im}(L^{\nu_2-1}) \subseteq \mathrm{Im}(L^{\nu_2}) \overset{\uparrow}{\Rightarrow} \mathrm{Im}(L^{\nu_2-1}) = \mathrm{Im}(L^{\nu_2})$ #

(Lemma A2) contradiction with the minimality of ν_2

Hence, $\underline{\underline{\nu_1 = \nu_2}}$.

$\square$.

→ Recall that we are trying to find an index ν such that
$$X = \ker(L^\nu) \oplus \mathrm{Im}(L^\nu) \quad !$$

• **LEMMA A4:** Under the previous assumptions, we have

$$(\dim \ker(L^\nu) < \infty)$$ → we already saw it $L^\nu = (I-K)^\nu = I - K_n$, K_n compact

$$X = \ker(L^\nu) \oplus \mathrm{Im}(L^\nu),$$

where ν is the minimal index (as in Lemma A3)

$\boxed{\text{Proof:}}$

· Let $\psi \in \ker(L^\nu) \cap \mathrm{Im}(L^\nu)$

$\Leftrightarrow$ $L^\nu \psi = 0$ and $\psi = L^\nu \varphi$ for some $\varphi \in X$.

$\overset{\text{apply } L^\nu}{\Rightarrow} L^{2\nu}\varphi = L^\nu(L^\nu \varphi) = L^\nu \psi = 0$

$\Rightarrow \varphi \in \ker(L^{2\nu}) = \ker(L^\nu)$

$\Rightarrow \underset{\psi}{\underset{\parallel}{L^\nu \varphi}} = 0 \quad \Rightarrow \psi = 0$ whenever $\psi \in \ker(L^\nu) \cap \mathrm{Im}(L^\nu)$

$\Rightarrow \underline{\ker(L^\nu) \cap \mathrm{Im}(L^\nu) = \{0\}}$

· Next, show $X = \ker(L^\nu) \oplus \mathrm{Im}(L^\nu)$

Let $\varphi \in X \Rightarrow L^\nu \varphi \in \mathrm{Im}(L^\nu) = \mathrm{Im}(L^{2\nu})$

$\Leftrightarrow \exists \tilde{\varphi}: \quad L^\nu \varphi = L^{2\nu}\tilde{\varphi}$

Set $\psi := L^\nu \tilde{\varphi}$ and $\chi := \varphi - \psi \in X$

$\underset{(L^\nu\varphi = L^{2\nu}\tilde{\varphi})}{\overset{\text{apply } L^\nu}{\Rightarrow}} L^\nu \chi = L^\nu \varphi - L^{2\nu}\tilde{\varphi} = 0 \quad \Rightarrow \underline{\chi \in \ker(L^\nu)}$

and hence,

$$\underline{\underset{\ker(L^\nu)}{\underset{\cap}{\chi}} + \underset{\mathrm{Im}(L^\nu)}{\underset{\cap}{\psi}}} = \varphi \quad \Rightarrow \quad \underline{\underline{X = \ker(L^\nu) \oplus \mathrm{Im}(L^\nu)}}$$

$\square$.

→ Hence, we have proved the Third Riesz Theorem.

• **THEOREM** (Third Riesz Theorem)

Let $K: X \to X$ be a compact operator, $L = I - K$. Then there exists a $\nu \in \mathbb{N}_0$ (nonnegative integer), called the Riesz number of the operator K, such that

$$\{0\} = \ker(L^0) \subsetneq \ker(L) \subsetneq \dots \subsetneq \ker(L^\nu) = \ker(L^{\nu+1}) = \dots$$

$$X = \mathrm{Im}(L^0) \supsetneq \mathrm{Im}(L) \supsetneq \dots \supsetneq \mathrm{Im}(L^\nu) = \mathrm{Im}(L^{\nu+1}) = \dots$$

and

$$X = \ker(L^\nu) \oplus \mathrm{Im}(L^\nu)$$

Proof: It follows from Lemmas A1 – A4. □.

→ Now we have the tools to derive the fundamental results of the Riesz theory concerning with the solvability of the equation $\varphi - K\varphi = f$, by distinguishing the two cases $\nu = 0$ and $\nu > 0$.

• **THEOREM R1**: Let X be a normed space, $K: X \to X$ a compact operator, and $I - K$ injective. Then $(I-K)^{-1}: X \to X$ exists and is bounded.

Proof.

$L = I - K$ injective $\Rightarrow \ker(L) = \{0\} \Rightarrow \nu = 0 \Rightarrow \mathrm{Im}(L) = X$

$\Rightarrow L$ surjective

$\Rightarrow \exists L^{-1}: X \to X$.

Assume L^{-1} is not bounded. $(\|\varphi_n\| \geq n\|f_n\|)$

$\Rightarrow \exists (f_n)$ with $\|f_n\| = 1$ s.t. $\varphi_n := L^{-1} f_n$ is not bounded.

Define

$$g_n := \frac{f_n}{\|\varphi_n\|}, \qquad \psi_n := \frac{\varphi_n}{\|\varphi_n\|}, \quad n \in \mathbb{N}.$$

NOTE: $\left[L\psi_n = \frac{L\varphi_n}{\|\varphi_n\|} = \frac{f_n}{\|\varphi_n\|} = g_n \right]$

Then $\left(\|g_n\| = \frac{\|f_n\|}{\|\varphi_n\|} \leq \frac{1}{n}\right)$ $g_n \to 0$, $n \to \infty$ and $\|\psi_n\| = 1$.

K compact $\Rightarrow$ we can choose a subsequence (ψ_{n_k}) s.t. $K\psi_{n_k} \xrightarrow{k \to \infty} \psi \in X$.

Since $\psi_n - K\psi_n = g_n$ $(L\psi_n = g_n)$ $\Rightarrow \psi_{n_k} \to \psi \Rightarrow \psi \in \ker(L)$

$\Rightarrow \psi = 0$ # (since $\|\psi_n\| = 1 \ \forall n \in \mathbb{N}$).

$$\left.\begin{array}{l} L\psi_n \to L\psi \\ L\psi_n \to 0 \end{array}\right\} \Rightarrow L\psi = 0.$$

□.

→ The idea now is to rewrite Theorem R1 in terms of the solvability of the operator equation of the second kind

• THEOREM: X normed space, $K: X \to X$ compact.
If the homogeneous equation
$$\varphi - K\varphi = 0$$
only has the trivial solution $\varphi = 0$, then for all $f \in X$ the inhomogeneous equation
$$\varphi - K\varphi = f$$
has a unique solution $\varphi \in X$ and this solution depends continuously on f.

UNIQUENESS OF SOLUTION IMPLIES EXISTENCE OF SOLUTION

→ If $I - K$ is not injective ??

• THEOREM R2: X normed space, $K: X \to X$ compact and assume that $I-K$ is not injective. Then the kernel $\ker(I-K)$ is finite dimensional and the image $\boxed{Im(I-K) \neq X}$ is a proper closed subspace.

Im L closed
by second Riesz theorem

Proof:

$I-K$ is not injective $\Rightarrow \ker(L) \neq \{0\} \Rightarrow \nu > 0 \Rightarrow Im(L) \neq X$

$\square$.

• COROLLARY: If the homogeneous equation
$$\varphi - K\varphi = 0 \qquad (1)$$
has a nontrivial solution, then the inhomogeneous equation
(finite solutions)
$$\varphi - K\varphi = f \qquad (2)$$

This is the best we can say with this theory
But we don't know if (2) is solvable or not
↳ for that → we need the Fredholm theory

i) is unsolvable
or
ii) its general solution is of the form
$$\varphi = \tilde{\varphi} + \sum_{k=1}^{m} \alpha_k \varphi_k ,$$

where • $\varphi_1, \dots, \varphi_m$ are l.i. solutions of (1)
• $\alpha_1, \dots, \alpha_m$ are arbitrary complex numbers
• $\tilde{\varphi}$ is a particular sol. of (2)

***REMARK:** The main importance of the results of the Riesz theory for compact operators lies in the fact that we can conclude existence from uniqueness as in the case of finite dimensional linear equations.

***REMARK:** Theorems R1 and R2 and their corollaries are still true when we replace $I-k$ by $S-k$, where S is a bounded operator that has a bounded inverse S^{-1}:

$$S\varphi - k\varphi = f \Rightarrow \varphi - \underbrace{S^{-1}K}_{\text{compact}}\varphi = S^{-1}f \Leftrightarrow \varphi - \underset{\text{compact}}{\tilde{K}}\varphi = \tilde{f}.$$

→ We conclude this section with the following result related to projections.

• **THEOREM:** The projection operator $P: X \to \ker(L^\nu)$ defined by the decomposition

$$X = \ker(L^\nu) \oplus Im(L^\nu)$$

is compact. The operator $L - P = I - (K + P)$ is bijective.

↳ {Existence and uniqueness of the equation $(L-P)\varphi = f$.}

$\boxed{\text{Proof.}}$

We know that $\ker(L^\nu)$ is finite dimensional.
On $\ker(L^\nu)$ we can define the following norm:

$$\|\psi\|_\nu := \inf_{\chi \in Im(L^\nu)} \|\psi + \chi\|$$ → check that defines a norm!

Since on a finite dimensional linear space all norms are equivalent, we can deduce that for all $\psi \in \ker(L^\nu)$,

$\varphi \in X$ ⇏

$$\|\psi\| \leq C\|\psi\|_\nu = C \inf_{\chi \in Im(L^\nu)} \|\psi + \chi\|$$

Then for $\varphi \in X \underset{\text{by def of } P}{\Rightarrow} P\varphi \in \ker(L^\nu)$ and hence,

$$\|P\varphi\| \leq \inf_{\chi \in Im(L^\nu)} \|P\varphi + \chi\| \leq C\|\varphi\| \Rightarrow P \text{ is bounded}$$

$(\varphi - P\varphi \in Im(L^\nu))$
$X = Im(L^\nu) \oplus \ker(L^\nu)$
$\varphi \in X,\ P\varphi \in \ker(L^\nu)$

and it has finite dimension:

$Im(P) = \ker(L^\nu)$ (by def)

Therefore, P is compact.

• $\left.\begin{array}{l} P \text{ compact} \\ k \text{ compact} \end{array}\right\} \Rightarrow P + k$ compact.

L-P injec?
↳ Let $\varphi \in \ker(L-P) \Rightarrow L\varphi = P\varphi \Rightarrow L^{\nu+1}\varphi = L^\nu(\underset{\in \ker(L^\nu)}{P\varphi}) = 0 \Rightarrow$

$\mathrm{Ker}(L^{\nu+1}) = \ker(L^{\nu})$

$\Rightarrow \varphi \in$ ~~[illegible]~~

$P\varphi = \varphi$

$\Rightarrow L\varphi = \varphi \Rightarrow L\varphi \in \ker(L-P)$

$\Rightarrow L^2\varphi - P(L\varphi) = 0$

$L^2\varphi = P(L\varphi) = L\varphi = \varphi$

$\Downarrow$

$L^2\varphi \in \ker(L-P)$

$L^3\varphi = P(L^2\varphi) = L^2\varphi = \varphi$

$\Downarrow$

$\dots \quad L^{\nu}\varphi = \varphi$

$\Downarrow$

$\varphi = 0 \Leftarrow$ because $\varphi \in \ker(L^{\nu})$

$\square$.

By iteration, we get $\varphi = L^{\nu}\varphi = 0$

$\Rightarrow \ker(L-P) = \{0\}$ ($L-P$ is injective) $\xRightarrow{\text{Thm R.1}}$ $L-P$ is surjective.

SPECTRAL THEORY FOR COMPACT OPERATORS

The idea of this last section related to the Riesz theory, is to formulate a result of the Riesz theory in terms of spectral analysis.

→ Let us start with some definitions.

DEFINITION: X normed space, $K: X \to X$ bounded linear operator.

(a) A complex number λ is called an eigenvalue of K if there exists $\varphi \in X$, $\varphi \neq 0$ s.t. $K\varphi = \lambda\varphi$.
φ is called an eigenelement of K. {↳ This means that $\lambda I - K$ is not injective!!}

(b) The spectrum of an operator K is the set of all complex numbers λ such that $\lambda I - K$ does not have a bounded inverse.
We denote the spectrum of K by $\sigma(K)$ → $r(K) := \sup_{\lambda \in \sigma(K)} |\lambda|$ spectral radius of K

{Note that every eigenvalue of K is in the spectrum of K.
~~Also, if K is compact operator on an infinite-dimensional space $\Rightarrow 0 \in \sigma(K)$~~}
~~$\Rightarrow K$ does not have a bounded inverse!!~~ zero

(c) The complement (in $\mathbb{C}$) of $\sigma(K)$, that is,
↳ the set of all complex numbers such that $(\lambda I - K)^{-1}$ exists and is bounded, is called the resolvent set of K and is denoted by $\rho(K)$.
In this case, λ is called a regular value of K.
$R(\lambda, K) := (\lambda I - K)^{-1}$ is called the resolvent of K.

(*) REMARK: If $\lambda I - K$ is either non-injective or non-surjective, then $\lambda \in \sigma(K)$

non-injective $\to$ λ is called an eigenvalue of K; the set $\sigma_p(K)$ of eigenvalues is called the point spectrum

non-surjective $\to$ λ is called a point of the continuous spectrum $\sigma_c(K)$ or the residual spectrum $\sigma_r(K)$, depending whether the subspace $(\lambda I - K)X$ is dense in X or not.

→ For the spectrum of a compact operator we have the following properties:

- THEOREM: X {infinite dimensional normed space}, $K: X \to X$ compact.

  (i) $0 \in \sigma(K)$ { If $\lambda \neq 0 \Rightarrow$ either λ is an eigenvalue of K or $\lambda \in \rho(K)$ } $\exists (\lambda I - K)^{-1}$ bounded

  (ii) $\sigma(K) \setminus \{0\}$ consists of at most a countable set of eigenvalues.
  The only possible point of accumulation of $\sigma(K)$ is 0.
  { The eigenvalues of K are countable and whenever they are put into a sequence (λ_n), we have that $\lim_{n\to\infty} \lambda_n = 0$ }

$\boxed{\text{Proof:}}$

(i) Suppose $\lambda = 0$ is a regular value of K, $\lambda \in \rho(K)$.

$\Leftrightarrow$ K^{-1} exists and is bounded

$\Rightarrow I = K^{-1}K$ is compact $\Rightarrow$ X finite dimensional #

$\Rightarrow 0 \in \sigma(K)$

(ii) $\lambda \neq 0 \to$ we apply the Riesz theory to $\lambda I - K$

$\Rightarrow$ Either • $\mathrm{Ker}(\lambda I - K) = \{0\}$ and $(\lambda I - K)^{-1}$ exists and is bounded (Thm. R1) $\to \lambda \in \rho(K)$

or • $\mathrm{Ker}(\lambda I - K) \neq \{0\}$ which means that λ is an eigenvalue. (And we know that $\mathrm{Ker}(\lambda I - K)$ is finite dimensional $\Rightarrow$ so at most we have a countable set of eigen.)

→ It remains to show that for each $R > 0$ there exists only a finite number of eigenvalues λ with $|\lambda| \geq R$.
(This is to show that $\sigma(K)$ has no points of accumulation other than (possibly) 0) ~~we want~~ to show that $\{\lambda \in \mathbb{C} : |\lambda| \geq R\} \cap \sigma(K)$ is finite.

Assume, on the contrary, that we have a sequence (λ_n) of distinct eigenvalues satisfying $|\lambda_n| \geq R$.

↳ Choose eigenelements φ_n s.t. $K\varphi_n = \lambda_n \varphi_n$

↳ and define finite dimensional subspaces

$$U_n := \text{span}\{\varphi_1, \dots, \varphi_n\}$$

Eigenelements corresponding to distinct eigenvalues are linearly ind.

$\Longrightarrow$ $\underset{\text{closed}}{U_{n-1}} \subsetneq U_n \xRightarrow[\text{Riesz lemma}]{}$ we choose a sequence (ψ_n) of elements $\psi_n \in U_n$ with $\|\psi_n\| = 1$ and $\|\psi_n - \psi\| \geq \frac{1}{2}$ for all $\psi \in U_{n-1}$.

We write

$$\psi_n = \sum_{k=1}^{n} \alpha_{nk} \varphi_k .$$

Then

$$\lambda_n \psi_n - \underbrace{K\psi_n}_{\substack{\| \\ \sum_{k=1}^{n} \alpha_{nk} \underbrace{K\varphi_k}_{\substack{\| \\ \lambda_k \varphi_k}}}} = \sum_{k=1}^{n} (\lambda_n - \lambda_k) \alpha_{nk} \varphi_k \in U_{n-1}$$

Therefore, for $m < n$ we have $\qquad \lambda_m \psi_m = \sum_{k=1}^{m} \lambda_m \alpha_{mk} \varphi_k \in U_m \in U_{n-1}$ $(m<n)$

$$K\psi_n - K\psi_m = \lambda_n \psi_n - \underbrace{(\overbrace{\lambda_n \psi_n - K\psi_n}^{\in U_{n-1}} + \overbrace{K\psi_m}^{= \lambda_m \psi_m})}_{= \lambda_n \psi} = \lambda_n (\psi_n - \psi)$$

$$\Rightarrow \psi := \psi_n - \frac{(K\psi_n - K\psi_m)}{\lambda_n}$$

$$\Rightarrow \|K\psi_n - K\psi_m\| = |\lambda_n| \, \|\psi_n - \overset{\in U_{n-1}}{\psi}\| \geq \frac{|\lambda_n|}{2} \geq \frac{R}{2}, \quad m < n .$$

$\Rightarrow (K\psi_n)$ does not contain a convergent subsequence # because K is compact.

$\square$.

*REMARK: We have seen that: For any scalar λ,

i) either $(\lambda I - K)^{-1}$ exists $\qquad (\ker(\lambda I - K) = \{0\})$ uniqueness

or

ii) $(\lambda I - K)\varphi = 0$ has a finite number of linearly independent solutions.

✓ Fredholm formulated this result for the specific operator $(K\varphi)(x) = \int_I k(x,y)\varphi(y)\,dy$.

It is known as a <u>Fredholm alternative</u>. $\rightarrow$ lets pass next to study the Fredholm theory.

LECTURE 9 (2014.02.10)

FREDHOLM THEORY FOR COMPACT OPERATORS

→ Our next goal will be to develope the Fredholm theory.

- In the narrowest sense, Fredholm theory concerns itself with the solution of the Fredholm integral equation
- In a broader sense, the abstract structure of Fredholm's theory is given in terms of the
  - ↳ spectral theory of Fredholm operators
  - ↳ Fredholm kernels on Hilbert space (the kernels of the integral operator)

→ The Fredholm alternative is one of Fredholm's theorems. It may be expressed in several ways: as a theorem of linear algebra, as a theorem of integral equations or as a theorem of Fredholm operators. It is also related to the spectrum of a compact operator, as we have already seen.

↳ Fredholm alternative only applies when K is a compact operator.

→ In linear algebra it can be read as follows:

$$\begin{cases} \text{Either } Ax=b \text{ has a solution } x \\ \text{or} \\ A^T y = 0 \text{ has a nontrivial solution } y \text{ with } y^T b \neq 0 \end{cases}$$

In other words,

$$\underset{(b\in \mathrm{Im}(A))}{Ax=b \text{ has solution}} \iff \underset{(b\in(\ker(A^T))^\perp)}{\text{for any } y \text{ s.t. } A^Ty=0, \text{ then } y^Tb=0}$$

That is, we are saying that

$$\mathrm{Im}(A) = (\ker(A^T))^\perp \qquad (A \text{ matrix}, A^T \text{ its transpose})$$

→ In our case, we are working with operators instead of with matrices. Hence, we need to work with the analogue of transpose matrix, which is the adjoint of an operator.

↳ In order to define the adjoint of an operator, we need a dual system.

↓

- A dual system can be defined via bilinear forms, via sesquilinear forms... In fact, we can define it via inner product.

↓

To make life simpler, we will work with Hilbert spaces, because in this case we have an inner product and we can easily define an adjoint operator.

DUALITY

In this section we define and study two new objects: the dual of a Hilbert space and the adjoint of a linear bounded operator $A: H \to H$.

→ We start by defining the dual.

• DEFINITION: (i) Let H be Hilbert. A bounded linear map $\lambda: H \to \mathbb{C}$ is called a linear functional.

(ii) The dual of H, denoted by H^*, is the ~~vector~~ space of all (continuous) linear functionals of H.

FUNCTIONAL ANALYSIS

• EXAMPLE: Let $w \in H$ be fixed. Then

$$\begin{aligned} \lambda: H &\to \mathbb{C} \\ x &\mapsto \lambda(x) = \langle x, w \rangle \end{aligned}$$

is a linear functional. Linearity is clear. And since

$$|\lambda(x)| \overset{C\text{-}S}{\leq} \|x\| \|w\| \Rightarrow \lambda \text{ bounded}$$

→ We can give a description of the dual space of a Hilbert space via the Riesz representation theorem.

• PROPOSITION (Riesz representation theorem) → connection between Hilbert space and its dual

If λ is a linear functional on H^* (H Hilbert), for every element $z \in H$ there is a unique element $w \in H$ s.t.

$$\lambda(z) = \langle z, w \rangle. \qquad (\lambda \in H^*)$$

→ other notation: $\lambda_w(z) = \langle z, w \rangle$

links $w \to \lambda_w$

think of the mapping

$$\Phi: H \to H^*$$
$$w \mapsto \Phi(w) = \lambda_w$$

Proof.

Existence of w: If $\lambda \equiv 0$, we may take $w = 0$.

Assume $\lambda \neq 0 \Rightarrow M := \ker(\lambda)$ is a closed subspace and $\lambda \neq 0$

$\Rightarrow M^\perp \neq \{0\}$ i.e. $\exists v \in M^\perp$, $\|v\| = 1$

Remark: λ cont.
$\Rightarrow \ker(\lambda)$ is closed subspace
$\Rightarrow H = \ker(\lambda) \oplus (\ker(\lambda))^\perp$

Let $z \in H$ and set

$$u = \lambda(z)\, v - \lambda(v)\, z$$

Then $\langle u, z \rangle = \langle z, w \rangle \langle u, v \rangle - \langle u, v \rangle \langle z, w \rangle$

linear functional: $\lambda(u) = \underbrace{\lambda(z)}_{\text{cte}} \lambda(v) - \underbrace{\lambda(v)}_{\text{cte}} \lambda(z) = 0 \Leftrightarrow u \in M = \ker(\lambda) \to$

But then $u \perp v$ i.e.

$$0 = \langle u, v\rangle = d(z)\underbrace{\|v\|^2}_{1} - d(v)\langle z, v\rangle = d(z) - d(v)\langle z, v\rangle$$

$$\Rightarrow d(z) = d(v)\langle z, v\rangle = \langle z, \overline{d(v)}\, v \rangle$$

because $d(\,) \in \mathbb{C}$ or $\mathbb{R}$

need adjoint

That is, we can choose $w = \overline{d(v)}\, v$

This proves existence.

uniqueness

It remains to prove uniqueness.

If $\langle x, w\rangle = \langle x, w'\rangle \ \forall x \in H \iff \langle x, w - w'\rangle = 0 \ \forall x \in H$

$\Rightarrow \|w - w'\| = 0 \Rightarrow w = w'$.

$\square$.

→ So the dual of H "is" H (they are isomorphic vector spaces via Riesz then).

→ We can also evaluate norms via duality.

• PROPOSITION: $\forall z \in H$, $\|z\| = \sup_{\|v\| = 1} |\langle z, v\rangle|$

linear functional

$\left(\|z\| = \sup_{\substack{z \in H \\ \|z\| \leq 1}} |d(z)|\right)$

$(H, \langle\cdot,\cdot\rangle_H)$, $(K, \langle\cdot,\cdot\rangle_K)$

map $U: H \to K$

① U linear

② U surjective

③ $\forall g, h \in H$, $\langle g, h\rangle_H = \langle Ug, Uh\rangle_K$

Proof:

"$\geq$" Cauchy-Schwarz:

$$|\langle z, v\rangle| \leq \|z\|\,\|v\| \underset{\|v\| = 1}{=} \|z\|$$

$$\Rightarrow \sup_{\|v\| = 1} |\langle z, v\rangle| \leq \|z\|.$$

"$\leq$" On the other hand, if $z = 0$ the opposite inequality is clear.

Assume $z \neq 0$. Then if $v = \frac{z}{\|z\|}$, we have

$$\|v\| = 1, \quad |\langle z, v\rangle| = \frac{\|z\|^2}{\|z\|} = \|z\| \quad \geq \|z\| \text{ also}$$

$$\Rightarrow \|z\| \leq \sup_{\|v\| = 1} |\langle z, v\rangle|$$

$\square$.

→ So we can define now the adjoint operator:

• DEFINITION: Let $A: H_1 \to H_2$ bounded. The adjoint of A, A^* is defined by

$$\langle Az, w\rangle_2 = \langle z, A^* w\rangle_1. \qquad \text{Note } A^*: H_2 \to H_1$$

• REMARK: The existence of A^* follows from the Riesz rep. theorem;

namely consider a bounded linear functional (w fixed!)

$$\alpha : z \longmapsto \langle Az, w \rangle_2 .$$

$\alpha \in H^*$

$\alpha : H_1 \to \mathbb{C}$ or $\mathbb{R}$

Riesz: $\Rightarrow \exists ! \, \tilde{z}$ s.t. $\alpha(z) = \langle z, \tilde{z} \rangle_1$

$A : H_1 \to H_2$

$z \mapsto Az$

Define: $A^* w = \tilde{z} \in H_1$ $\qquad \langle z, A^* w \rangle_1$

It can easily check that A^* is bounded and linear and $\|A\| = \|A^*\|$.

"isometric" from Riesz

• Also note that the definition implies that

$$(A^*)^* = A \quad , \quad (AB)^* = B^* A^* .$$

• EXAMPLE 1: The matrix of the adjoint of a linear map on $\mathbb{R}^n$ with matrix A is A^T, since

$$\boxed{\langle x, (Ay) \rangle = \langle (A^T x), y \rangle}$$

$$\sum_{i=1}^n x_i \left(\sum_{j=1}^n a_{ij} y_j \right) = \sum_{j=1}^n \left(\sum_{i=1}^n a_{ij} x_i \right) y_j$$

On $\mathbb{C}^n$, $\underline{A^* = \overline{A^T}}$.

• EXAMPLE 2: Let $k \in L^2(\mathbb{R} \times \mathbb{R})$ or $k \in C(\mathbb{R} \times \mathbb{R})$ s.t. Schur's lemma is valid.

boundedness for x $\wedge$ y

Then the adjoint of the integral operator

$$(Kf)(x) = \int_{\mathbb{R}} k(x,y) f(y) \, dy$$

is

$$(K^* f)(x) = \int_{\mathbb{R}} \overline{k(y,x)} f(y) \, dy .$$

$$\langle Kf, g \rangle_{L^2} = \iint f(y) k(x,y) \overline{g(x)} \, dy \, dx = \int f(y) \overline{\int \overline{k(x,y)} g(x) dx} \, dy$$

$$= \langle f, K^* g \rangle_{L^2}$$

$\Rightarrow$ The adjoint of K, K^*, has integral kernel $\overline{k(y,x)}$, if $k(x,y)$ is the kernel of K.

→ The adjoint plays a crucial role in studying the solvability of a linear equation

$$A\varphi = f \quad ; \qquad A : H \to H \text{ bounded.}$$

Let $\psi \in H$ be any solution of the homogeneous adjoint equation,

$$A^* \psi = 0 .$$

Solution of homogeneous equation ∧ Kernel / Null space are connected!
→ linear algebra ∧ Fredholm I.E

We take inner product

$$\langle A\varphi, \psi \rangle \overset{\text{def.}}{=} \langle \varphi, \boxed{A^*\psi} \rangle \overset{A^*\psi=0}{=} 0$$
$$\overset{\shortparallel}{\langle f, \psi \rangle}$$

Hence, a necessary condition for a solution φ of $A\varphi = f$ to exist is that $\langle f, \psi \rangle = 0$ for all $\psi \in \ker A^*$, i.e.

$f \in (\ker A^*)^\perp$

"if ψ is a solution, then…"

↓

This condition on f is not always sufficient to guarantee the solvability of $A\varphi = f$; the most we can say for general bounded operators is the following:

- THEOREM*: let $A : H \to H$ be bounded and linear, H Hilbert. The following are equivalent:

  (i) $\mathrm{Im}(A)$ is closed

  (ii) $\mathrm{Im}(A^*)$ is closed

  (iii) $\boxed{\mathrm{Im}(A) = (\ker(A^*))^\perp}$ → As in the matrix case

  (iv) $\mathrm{Im}(A^*) = (\ker(A))^\perp$

And if one of these holds, then we have the isomorphisms

$$\mathrm{Coker}(A) \cong \ker(A^*)$$
$$\mathrm{Coker}(A^*) \cong \ker(A)$$

$\boxed{\text{Proof:}}$

coker

$$\begin{tikzcd} & H \arrow[d] \\ H \arrow[ur, "A"] \arrow[r] & Q \end{tikzcd}$$

$\mathrm{coker}(A) = H / \mathrm{Im}(A)$
↓
modulo

to show $(iii) \Leftarrow (i)$ Since $M^\perp$ is always closed. Then $(iii) \Rightarrow (i)$, $(iv) \Rightarrow (ii)$.

other direction

"$\subset$" let now $w \in \ker(A^*)$: then $\forall z \in H$

$$(Az, w) = (z, A^* w) = 0$$

$$\Rightarrow \operatorname{Im}(A) \subset (\ker(A^*))^\perp .$$

Assume that $\operatorname{Im}(A)$ (i) is closed. Then $\operatorname{Im}(A) = (\operatorname{Im}(A)^\perp)^\perp$

and (iii) is then equivalent with ($\ker(A^*)$ is always closed)

to show (iii)' $\quad \ker(A^*) = \operatorname{Im}(A)^\perp$.

"$\supset$" But if $w \in (\operatorname{Im}(A)^\perp)$ then $\forall z \in H$

$$0 = (Az, w) = (z, A^* w) = 0$$

$\Rightarrow A^* w = 0 \Rightarrow w \in \ker(A^*)$, i.e. we have shown

$$(\operatorname{Im}(A)^\perp) \subset \ker(A^*)$$

This proves (iii)' since we already know $\operatorname{Im}(A) \subset \ker(A^*)^\perp$.

$$\Downarrow$$

$$\underbrace{(\ker(A^*)^\perp)^\perp}_{\ker(A^*)} \subset (\operatorname{Im}(A)^\perp).$$

So we proved $(i) \Rightarrow (iii)$

One can prove in a similar way that $(ii) \Rightarrow (iv)$ Exercise!

So we have $(i) \Leftrightarrow (iii)$, $(ii) \Leftrightarrow (iv)$.

Let us finally show that $(i) \Leftrightarrow (ii)$.

$\Downarrow$

For that, we need the following lemma:

- **LEMMA 1:** $\operatorname{Im}(A)$ closed $\Leftrightarrow$ the induced map

$$\bar{A} : H/\ker(A) \to \operatorname{Im}(A)$$
$$[x] \longmapsto Ax$$

co Im has a bounded inverse.

quotient of domain and kernel

Proof of ~~the~~ lemma 1

Sketch of the proof.

$\Rightarrow$ | assume $\mathrm{Im}(A)$ closed $\Rightarrow$ $\mathrm{Im}(A)$ is also Hilbert

(a closed subspace in a Hilbert space is also Hilbert)

EX: Also, since $\ker(A)$ is closed, $H/\ker(A)$ is complete.

By definition $\left\{ \begin{array}{l} \bar{A} \text{ is bounded} \\ \bar{A} \text{ is one-to-one} \end{array} \right\}$ and by using the open mapping theorem $\Rightarrow$ $\bar{A}$ has a bounded inverse.

X, Y Banach spaces (i.e. complete normed spaces) and $L: X \to Y$ bounded, one-to-one. Then the inverse map $L^{-1}: Y \to X$ is also bounded.

$\Leftarrow$ | Assume that $\bar{A}$ has a bounded inverse $\bar{A}^{-1}$.

If $Ax_n \to y \in H$, we also have

$$\bar{A}[x_n] \xrightarrow{H} y$$

show the limit $y \in \mathrm{Im}(A)$

$$\overset{\bar{A}^{-1}\text{ bound.}}{\Longrightarrow} \quad [x_n] \xrightarrow{H/\ker(A)} \bar{A}^{-1} y \in H/\ker(A)$$

let $[z] = \bar{A}^{-1} y$ i.e.

$$Az = \bar{A}[z] = y \quad \Rightarrow \quad y \in \mathrm{Im}(A)$$

$\Rightarrow$ $\mathrm{Im}(A)$ closed.

$\square$ lemma 1

· inclusion map · **LEMMA 2**: $(\bar{A})^* = \overline{A^*}$ if $\mathrm{Im}(A)$ closed.

Note: In a Hilbert space, we have $H/M \cong M^{\perp}$, if M closed

natural injection

$i: A \to B$

$A \hookrightarrow B$

(A)B

Proof of lemma 2.

co im = co-image

$$(\bar{A})^*: \mathrm{Im}(A) \to H/\ker(A) = \ker(A)^{\perp}$$

$$\langle (\bar{A})^* y, [x] \rangle = \langle y, \bar{A}[x] \rangle = \langle y, Ax \rangle = \langle A^* y, x \rangle$$

On the other hand,

$$(\overline{A^*}): H/\ker(A^*) \to \mathrm{Im}(A^*) \hookrightarrow \ker(A)^{\perp} = H/\ker(A)$$

$\parallel$ $\mathrm{Im}(A)$ closed

$\mathrm{Im}(A)$

imbedding "hooked arrow"

and we have also, $x \in \mathrm{Im}(A^*)$ and identify x with $[x]$

$$\langle (\overline{A^*}) y, [x] \rangle = \langle A^* y, x \rangle.$$

$\square$ lemma 2

befriahi!!

Now if we continue with the proof of the theorem...

(i)

Assume $Im(A)$ closed $\underset{\text{lemma 1}}{\Longleftrightarrow}$ $\bar{A} : H/\ker(A) \to Im(A)$ has bounded inverse

$\underset{Im(A)\text{ closed}}{\Longleftrightarrow}$ $(\bar{A})^*$ has a bounded mapping inverse (open mapping theorem again!)

$\underset{\text{lemma 2}}{\Longleftrightarrow}$ $\overline{(A^*)}$ " " " " "

$\underset{\text{lemma 1}}{\Longleftrightarrow}$ $Im(A^*)$ closed.

So i) $\Leftrightarrow$ ii) $\Leftrightarrow$ iii) $\Leftrightarrow$ iv).

Finally, if $Im(A)$ and $Im(A^*)$ are closed, then

$$\mathrm{Coker}(A) = H/Im(A) \cong Im(A)^\perp \cong \ker(A^*)$$
$$\mathrm{Coker}(A^*) = H/Im(A^*) \cong Im(A^*)^\perp \cong \ker(A)$$

→ This result has the following important consequences: $\square$ Theorem

Lecture 11

- **COROLLARY 1**: Assume $Im(A)$ is closed. Then

  A is injective $\Leftrightarrow$ A^* surjective

  A is surject. $\Leftrightarrow$ A^* injective

- **COROLLARY 2**: If $Im(A)$ closed and A self-adjoint i.e. $A = A^*$, then ($Im(I-K)$ closed!)

  A injective $\Leftrightarrow$ A surjective $\Leftrightarrow$ A one-to-one

$\square$.

→ So we are saying that under the assumption that A has a closed image, the existence of a solution x to

$$Ax = y$$

is equivalent to

$$A^* z = 0 \Rightarrow z = 0,$$

which is often much easier.

Now if we continue with the proof of the theorem...

(i)

Assume $\mathrm{Im}(A)$ closed $\underset{\text{Lemma 1}}{\Longleftrightarrow}$ $\bar{A}: H/\ker(A) \to \mathrm{Im}(A)$ has bounded inverse

$\underset{\mathrm{Im}(A)\text{ closed}}{\Longleftrightarrow}$ $(\bar{A})^*$ has a bounded mapping inverse (open mapping theorem again!)

$\underset{\text{Lemma 2}}{\Longleftrightarrow}$ $\overline{(A^*)}$ " " " " "

$\underset{\text{Lemma 1}}{\Longleftrightarrow}$ $\mathrm{Im}(A^*)$ closed.

So i) $\Leftrightarrow$ ii) $\Leftrightarrow$ iii) $\Leftrightarrow$ iv).

Finally, if $\mathrm{Im}(A)$ and $\mathrm{Im}(A^*)$ are closed, then

$$\mathrm{Coker}(A) = H/\mathrm{Im}(A) \cong \mathrm{Im}(A)^\perp \cong \ker(A^*)$$
$$\mathrm{Coker}(A^*) = H/\mathrm{Im}(A^*) \cong \mathrm{Im}(A^*)^\perp \cong \ker(A)$$

→ This result has the following important consequences: $\square$ Theorem

---

Lecture 11

• **COROLLARY 1**: Assume $\mathrm{Im}(A)$ is closed. Then

A is injective $\Leftrightarrow$ A^* surjective

A is surject. $\Leftrightarrow$ A^* injective

• **COROLLARY 2**: If $\mathrm{Im}(A)$ closed and A self-adjoint i.e. $A = A^*$, then

$\mathrm{Im}(I-K)$ closed!

A injective $\Leftrightarrow$ A surjective $\Leftrightarrow$ A one-to-one

$\square$.

→ So we are saying that under the assumption that A has a closed image, the existence of a solution x to

$$Ax = y$$

is equivalent to

$$A^* z = 0 \Rightarrow z = 0,$$

which is often much easier.

→

→ An equivalent formulation could be the following:

If a bounded linear operator $A: H \to H$ on a Hilbert space H has closed image, then the equation

$A\varphi = \psi$ has a solution for φ $\iff$ ψ is orthogonal to $\ker(A^*)$.

✓

- So we obtained a necessary and sufficient condition for the solvability of $A\varphi = \psi$.

- This result also provides a useful general method of proving existence from uniqueness.

- The condition that A has closed image is implied by an estimate of the form $c\|x\| \leq \|Ax\|$. EXERCISE?

*REMARK: Recall that if $K: H \to H$ is a compact operator, then we proved that $\mathrm{Im}(I-K)$ is closed. Hence, we can apply the above results to $L = I - K$.

→ A commonly occurring dichotomy for the solvability of a linear equation is summarized in the following Fredholm alternative:

- DEFINITION: A bounded linear operator $A: H \to H$ on a Hilbert space H satisfies the Fredholm alternative if one of the following two alternatives holds:

  (a) either $A\varphi = 0$, $A^*\varphi = 0$ have only the zero solution, and the equations $A\varphi = f$, $A^*\varphi = f$ have a unique solution $\varphi \in H$ for every $f \in H$.

  (b) or $A\varphi = 0$, $A^*\varphi = 0$ have nontrivial, finite dimensional solution spaces of the same dimension, $A\varphi = f$ has a (nonunique) solution iff $f \perp \psi$ for every solution ψ of $A^*\psi = 0$ and $A^*\varphi = f$ has a (nonunique) solution iff $f \perp \psi$ for every sol. ψ of $A\psi = 0$.

→ In the int. equations setting, the Fredholm alternative is given as follows:

• COROLLARY: Let $I \subset \mathbb{R}$, $k: I \times I \to \mathbb{C}$ be a continuous, Hilbert-Schmidt or weakly singular kernel ($\Rightarrow$ the int. op. $\mathcal{K}$ is compact). Then,

(a) either the homogeneous integral equations

$$\varphi(x) - \int_I k(x,y)\varphi(y)\,dy = 0 \qquad x \in I$$

and

$$\psi(x) - \int_I \overline{k(y,x)}\,\psi(y)\,dy = 0 \qquad x \in I,$$

only have the trivial solutions $\varphi = 0$ and $\psi = 0$ and the inhomogeneous integral equations

$$\varphi(x) - \int_I k(x,y)\varphi(y)\,dy = f(x) \qquad x \in I$$

and

$$\psi(x) - \int_I \overline{k(y,x)}\,\psi(y)\,dy = g(x) \qquad x \in I,$$

for each right hand side $f \in C(I)\,(L^2(I))$ and $g \in C(I)\,(L^2(I))$ have a unique solution $\varphi \in C(I)\,(L^2(I))$ and $\psi \in C(I)\,(L^2(I))$

or

(b) the homogeneous integral equations have the same finite number of linearly independent solutions and the inhomogeneous integral equations are solvable iff the right hand sides satisfy

$$\int_I f(x)\psi(x)\,dx = 0$$

for all solutions ψ of the homogeneous adjoint equation or

$$\int_I \varphi(x) g(x)\,dx = 0$$

for all solutions φ of the homogeneous equation, respectively.

• REMARK: Any linear operator $A: \mathbb{C}^n \to \mathbb{C}^n$ on a finite dimensional space, associated with an $n \times n$ system of linear equations $A\varphi = f$, satisfies the Fredholm alternative.
The images of A and A^* are closed because they are finite dimensional.
From linear algebra, $\operatorname{Im} A = \operatorname{Im}(A^*)$, and therefore $\ker A = \ker A^*$.
The Fredholm alternative then follows.

→ On an infinite dimensional space, two things can go wrong with the Fredholm alternative:

1. $\operatorname{Im} A$ need not be closed.
2. Even if $\operatorname{Im} A$ is closed, it is not true, in general, that $\ker A$ and $\ker A^*$ have the same dimension.

↓

As a result, the equation $A\varphi = f$ may be solvable for all $f \in H$ even though A is not one-to-one, or $A\varphi = f$ may not be solvable for all $f \in H$ even though A is one-to-one.

✓

• EXAMPLE: Suppose that S and T are the right and left shift operators on the sequence space $\ell^2(\mathbb{N})$, defined by:

$$S(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots)$$

$$T(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$$

First note that $T = S^*$: $\langle x, Sy \rangle = \overline{x_2} y_1 + \overline{x_3} y_2 + \overline{x_4} y_3 + \dots = \langle Tx, y \rangle$

Next, $\operatorname{Im}(S)$ is closed since it consists of $y = (y_1, y_2, y_3 \dots) \in \ell^2(\mathbb{N})$ s.t. $y_1 = 0$.

Now, the equation $Sx = y \Leftrightarrow (0, x_1, x_2, x_3 \dots) = (y_1, y_2, y_3, \dots)$ is solvable iff $y_1 = 0$ or $y \perp \ker T$. So, if a solution exists, then it is unique.

On the other hand, the equation $Tx = y$ is solvable for every $y \in \ell^2(\mathbb{N})$, even though T is not one-to-one, and the solution is not unique.

Et dolit emmegsiten... Pentban!

This fact motivates the ~~def~~ introduction of the notion Fredholm operators.

* FREDHOLM OPERATORS

• DEFINITION: A bounded linear operator $A: H \to H$, H Hilbert, is a Fredholm operator if:

(a) $Im(A)$ is closed

(b) $Ker(A)$ and $coker(A) \cong ker(A^*)$ are finite dimensional.

The index of a Fredholm operator is the integer

$$ind(A) = \dim ker(A) - \dim coker(A)$$
$$= \dim ker(A) - \dim ker(A^*)$$

→ let us see some properties of Fredholm operators:

• LEMMA: $A: H \to H$ bounded linear operator, H Hilbert.

A Fredholm $\Longleftrightarrow$ A^* Fredholm.

In this case: $index(A^*) = -index(A)$.

$\boxed{\text{Proof:}}$ This is a direct consequence of Theorem*, since:

$$coker(A) \cong ker(A^*)$$
$$coker(A^*) \cong ker(A)$$

Then

$$\dim coker(A^*) = \dim ker(A)$$
$$\dim coker(A) = \dim ker(A^*)$$

□.

• EXAMPLES:

1. If H_1, H_2 finite dimensional $\Rightarrow$ Every linear operator $A: H_1 \to H_2$ is ~~finite dimensional~~ Fredholm and
$$index(A) = \dim H_1 - \dim H_2.$$

$\boxed{\text{Proof}}$ $\dim H_1 = \dim(ker(A)) + \dim(Im(A))$ (by linear algebra)

$index(A) = \dim ker(A) - \dim coker(A) = \dim H_1 - \dim Im(A) - \underbrace{\dim coker(A)}_{\dim H_2 - \dim Im(A)} = \dim H_1 - \dim H_2.$ □.

2. $A: H \to H$ bijective $\Rightarrow$ A Fredholm and $\text{index}(A) = 0$.
bounded lin. op.

$$\left\{\begin{array}{l} A \text{ inj.} \Rightarrow \ker(A) = \{0\} \to \dim \ker(A) = 0 \\ A \text{ surj.} \quad \operatorname{Im}(A) = H \to \operatorname{coker}(A) = H/H \Rightarrow \dim \operatorname{coker}(A) = 0 \end{array}\right\}$$

→ We already saw that one of the versions of the Fredholm alternative says that, if $K: H \to H$ is a compact operator, then the associated equation

$$\varphi - K\varphi = f \qquad (*)$$

behaves like in the finite dimensional case:

if the hom. eq. $\varphi = K\varphi$ has only the trivial solution $\varphi = 0$, then the inhom. eq. $(*)$ has a unique solution $\varphi \in H$, for every $f \in H$.

→ We can state a more general version of the Fredholm alternative in terms of Fredholm operators.

• THEOREM: For any compact operator K on H (Hilbert space), then $I - K$ is a Fredholm operator of index zero.

Proof

By First Riesz Theorem, we know that $\ker(I-K)$ is finite dimensional.
By Second " " , we have that $\operatorname{Im}(I-K)$ is closed.
If we apply this to K^*, since K^* is also compact,
EXERCISE: K compact $\Leftrightarrow$ K^* compact
we deduce that $\ker(I-K^*)$ is also finite dimensional.
$\qquad\qquad\qquad\qquad\qquad\qquad \simeq \operatorname{Coker}(I-K)$
Hence, $I - K$ is Fredholm.
We then have a continuous family $\{I - \alpha K, \alpha \in \mathbb{R}\}$ of Fredholm operators.

↳ By the properties of the index, the index at $\alpha = 1$ coincides with the index at $\alpha = 0$, which is zero.
BeiTu!

$\square$.

*REMARK: Compact perturbations do not change Fredholmness and do not change the index.

If K compact, A Fredholm $\Rightarrow$ $A + K$ Fredholm and $\text{index}(A+K) = \text{index}(A)$
one can prove that using the next result.

→ There is another relation between Fredholm and compact operators, known as the Atkinson characterization of Fredholm operators:

• THEOREM: Fredholmness = invertible modulo compact operators

More precisely, given a bounded operator $A: H_1 \to H_2$, the following are equivalent:

(i) A is Fredholm

(ii) A is invertible modulo compact operators, i.e. there exists a bounded linear operator $B: H_2 \to H_1$ and compact operators K_1 and K_2 such that

• $BA: H_1 \to H_1$ ⟹ $K_1: H_1 \to H_1$

• $AB: H_2 \to H_2$ ⟹ $K_2: H_2 \to H_2$

$$BA = I + K_1$$
$$AB = I + K_2$$

{ From the proof we will see that one can actually choose K_1 and K_2 to be finite dimensional operators, and B so that $ABA = A$, $BAB = B$ }

Proof:

(ii) ⟹ (i)

Assume first the existence of B, K_1 and K_2.
Since identity plus compact is Fredholm and since

$$\ker(A) \subset \begin{Bmatrix} \ker(BA) \\ \ker(AB) \end{Bmatrix} \in \begin{matrix} \ker(I+K_1) \\ \ker(I+K_2) \\ \text{finit. dim} \end{matrix}.$$

then $\dim \ker(A) < \infty$.
Similarly, the cokernel of A is finite dimensional.
Hence A is Fredholm.

(i) ⟹ (ii)

Assume that A is Fredholm.
Choose a complement A_1 of $\ker(A)$ in H_1 and a complement A_2 of $\mathrm{Im}(A)$ in H_2.
Then $A_3 = A_1|_{\ker(A)}$ is an isomorphism from H_1 into $\mathrm{Im}(A)$ and we define B such that

$$\begin{cases} B = (A_3)^{-1} & \text{on } \mathrm{Im}(A) \\ B = 0 & \text{on } H_2. \end{cases}$$

↳ Then the resulting k_1 will be a projection onto $\ker(A)$ and $I+k_2$ will be a projection onto $\operatorname{Im}(A)$; hence k_1 and k_2 will have the desired properties.

| Prove it !! | □.

$k_1 : H_1 \to \ker(A)$ $f \mapsto k_1 f = Pf$ (proj.)	$I+k_2 : H_2 \to \operatorname{Im}(A)$ (closed) $f \mapsto (I+k_2)f = Pf$ $\Downarrow$ $k_2 f = Pf - f \to P : H_2 \to \operatorname{Im}(A) = (\ker(A^*))$ orth. projection.

→ With this characterization in hands, it is very easy to check the following property related to Fredholm operators.

• <u>THEOREM</u>: If $A : H_1 \to H_2$, $B : H_2 \to H_3$ Fredholm operators, H_1, H_2, H_3 Hilbert spaces, then

$BA : H_1 \to H_3$ is a Fredholm operator and

$\operatorname{index}(BA) = \operatorname{index}(A) + \operatorname{index}(B)$

Proof

By Atkinson characterization, $\exists\, F : H_2 \to H_1,\ G : H_3 \to H_2$ bounded lin. op. s.t.

$$\left.\begin{matrix} I - FA,\ I - AF \\ I - GB,\ I - BG \end{matrix}\right\} \text{ are compact } \Rightarrow$$

$$\Rightarrow \cdot\ I - FGBA = \underbrace{I - FA}_{\text{comp.}} + F(\underbrace{I - GB}_{\text{comp.}})A \quad \text{compact}$$

$$\cdot\ I - BAFG = I - BG - B(I - GB)A \quad \text{compact}$$

$\Rightarrow$ BA Fredholm.

Let us now check the index formula.

$$A_0 : \ker BA / \ker A \longrightarrow \ker B$$
$$[x] \longmapsto Ax$$

$$B_0 : H_2 / \operatorname{Im} A \longrightarrow \operatorname{Im} B / \operatorname{Im} BA$$
$$[y] \longmapsto By$$

$\Rightarrow$ A_0 injective $(x \in \ker A_0 \Leftrightarrow A_0 x = 0 \Leftrightarrow Ax = 0 \overset{x \in \ker A}{\Leftrightarrow} x = 0)$

B_0 surjective (

↳ Now:

- $\operatorname{Im}(A_0) = \operatorname{Im} A \cap \ker B$
- $\operatorname{coker}(A_0) = \ker B / (\operatorname{Im} A \cap \ker B)$
- $\ker(B_0) = \{ [y] \in H_2 / \operatorname{Im} A \text{ s.t. } By \in \operatorname{Im} BA \}$

$$= \{ [y] \in H_2/\operatorname{Im} A : \exists x \in H_1 : By = BAx \}$$

$$= \{ [y] \in H_2 / \operatorname{Im} A : \exists x \in H_1 \text{ s.t. } y - Ax \in \ker B \}$$

$$= (\operatorname{Im} A + \ker B) / \operatorname{Im} A$$

$$\cong \ker B / (\operatorname{Im} A \cap \ker B) = \operatorname{coker}(A_0)$$

$$\Rightarrow 0 = \dim \ker(B_0) - \dim \operatorname{coker} A_0 - \dim \operatorname{coker} B_0 + \dim \ker A_0$$

$$= \operatorname{index} A_0 + \operatorname{index} B_0$$

$$= \dim (\ker(BA)/\ker A) - \dim \ker(B) + \dim \operatorname{coker}(A) - \dim \operatorname{Im} B / \ker BA$$

$$= \dim \ker(BA) - \dim \ker A - \dim \ker(B) + \dim \operatorname{coker} A$$

$$- \dim \ker H_2/\operatorname{Im} BA + \dim H_2 / \operatorname{Im} B$$

$$= \operatorname{index}(BA) - \operatorname{index}(A) - \operatorname{index}(B)$$

$\square$

* THE SPECTRAL THEOREM FOR LINEAR COMPACT SELF-ADJOINT OPERATORS

* THE SPECTRAL THEOREM FOR LINEAR COMPACT SELF-ADJOINT OPERATORS

→ As in finite-dimensional linear algebra, the adjoint operation gives the sort of symmetry which admits a simpler structure theory.
The type of operator we shall consider then is that which is equal to its own adjoint; i.e. self-adjoint operators.

→ Note that in the integral equations setting, if the kernel k is continuous, or is an L^2-kernel, the corresponding integral operator on $L^2([a,b])$ is self-adjoint if and only if $k(x,y) = \overline{k(y,x)}$ for almost all x and y.

hermitian kernel

If k is real-valued, this corresponds to $k(x,y)$ being symmetric kernel. ($k(x,y)=k(y,x)$).

→ The spectral theorem provides a canonical decomposition of the underlying vector space on which the operator acts. → called spectral decomposition

- As a motivation, let us consider the finite dimensional case.

  Let H be a Hilbert space, $\dim H = n < \infty$.

  $A : H \longrightarrow H$ self-adjoint

  $\Leftrightarrow \forall x,y \in H \quad \langle Ax, y\rangle = \langle x, Ay\rangle$

  Observe that this condition implies that all eigenvalues of A are real:

  $$\left\{ \begin{array}{l} \text{If } \lambda \text{ is an eigenvalue of } A \Rightarrow Ax = \lambda x \ \ \forall x \in H \\ \text{Then } \left.\begin{array}{l} \langle Ax, y\rangle = \langle x, Ay\rangle = \langle x, \lambda y\rangle \\ \quad \shortparallel \\ \langle \lambda x, y\rangle = \langle x, \bar{\lambda} y\rangle \end{array}\right\} \Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda \in \mathbb{R} \end{array} \right\}$$

  Then the spectral theorem says that "there exists an orthonormal basis $\{x_k\}_{k=1}^{n}$ of H consisting of eigenvectors of A, i.e. $Ax_k = \lambda_k x_k$, $k=1,\dots,n$".

- If we go now to the infinite-dimensional case, ~~when~~ can we decompose H via the ~~eigenvalues~~ eigenvectors of A? What can we say about the spectrum of A?

• **THEOREM** (Spectral Theorem)

Let $K: H \longrightarrow H$ be a non-zero self-adjoint compact operator. Then

(i) there exists at least one eigenvalue $\lambda \in \{\pm \|K\|\}$.

(ii) There are at most many non-zero eigenvalues, $\{\lambda_n\}_{n=1}^{N}$, where $N = \infty$ is allowed (unless $\dim \operatorname{Im}(K) < \infty$, N will be infinite)

(iii) The λ_n's (including multiplicities) may be arranged so that $|\lambda_n| \geq |\lambda_{n+1}|$ for all N. If $N = \infty$ then $\lim_{n\to\infty} |\lambda_n| = 0$.

(iv) The eigenvectors $\{\phi_n\}_{n=1}^{N}$ can be chosen to be an orthonormal set such that

$$H = \overline{\operatorname{span}\{\phi_n\}} \oplus \ker(K)$$

(v) Using the $\{\phi_n\}_{n=1}^{N}$ above,

$$K\psi = \sum_{n=1}^{N} \lambda_n \langle \psi, \phi_n \rangle \phi_n \,, \qquad \forall \psi \in H.$$

(vi) The spectrum of K is

$$\sigma(K) = \{0\} \cup \bigcup_{n=1}^{\infty} \{\lambda_n\}.$$

***REMARK:** Note that (ii), (iii) and (vi) ~~[illegible]~~ were already proved by a general compact operator K.

So the goal of this section will be to prove (i), (iv) and (v) properties of the spectral theorem. We will give a sequence of lemmas, ~~that~~ which in particular they prove the above result.

→ we start with the following result of linear algebra.

• <u>LEMMA 1</u>: Let H be a Hilbert space and $K: H \to H$ self-adjoint. Then the eigenvalues of K are real and eigenvectors corresponding to distinct eigenvalues are orthogonal.

$\boxed{\text{Proof:}}$

• $K = K^*$, $K\phi = \lambda\phi$ where $\phi \neq 0$.

$\lambda\langle\phi,\phi\rangle = \langle\lambda\phi,\phi\rangle = \langle K\phi,\phi\rangle = \langle\phi,K^*\phi\rangle = \langle\phi,K\phi\rangle = \bar{\lambda}\langle\phi,\phi\rangle$

$\Rightarrow$ since $\langle\phi,\phi\rangle \neq 0$, then $\lambda = \bar{\lambda} \Rightarrow \lambda \in \mathbb{R}$.

• If we now suppose that $K\phi_i = \lambda_i\phi_i$ with $\phi_i \neq 0$ $(i=1,2)$, then

$\lambda_1\langle\phi_1,\phi_2\rangle = \langle K\phi_1,\phi_2\rangle = \langle\phi_1,K\phi_2\rangle = \bar{\lambda}_2\langle\phi_1,\phi_2\rangle = \lambda_2\langle\phi_1,\phi_2\rangle$

$\Rightarrow$ so, if $\lambda_1 \neq \lambda_2 \Rightarrow \langle\phi_1,\phi_2\rangle = 0$.

$\square$.

→ This proves that the spectrum of a compact self-adjoint operator is real. Although we are concerned with compact operators, we will later consider non-compact examples; so we insert the more general case here.

• <u>LEMMA 2</u>: Let H be a Hilbert space and $K: H \to H$ self-adjoint bounded linear operator. Then $\sigma(K) \subset \mathbb{R}$.

$\boxed{\text{Proof:}}$

We need to prove that if $\lambda \in \mathbb{C}\setminus\mathbb{R}$, then $\lambda I - K$ is invertible and $(\lambda I - K)^{-1}$ is bounded.

Let $\lambda = \lambda_1 + i\lambda_2$, $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_2 \neq 0$.

By Lemma 1 λ is not an eigenvalue of K so $\lambda I - K$ is injective.

By the same reasoning, $\bar{\lambda} I - K$ is injective.

Let us now show that $\lambda I - K$ is surjective.

We first show that $\operatorname{Im}(\lambda I - K)$ is closed.

Since $K = K^*$, if $\phi \in H$ then

$\langle K\phi,\phi\rangle = \langle\phi,K\phi\rangle = \overline{\langle K\phi,\phi\rangle} \Rightarrow \langle K\phi,\phi\rangle \in \mathbb{R}$

Hence,

$|\langle(\lambda I - K)\phi,\phi\rangle| = |\langle(\lambda_1 I - K)\phi,\phi\rangle + i\lambda_2\langle\phi,\phi\rangle|$

$\underset{(|\operatorname{Im} z| \leq |z|)}{\geq} |\lambda_2|\,\|\phi\|^2$

and by Cauchy-Schwarz inequality, we get

$$|\lambda_2| \, \|\phi\|^2 \le \|(\lambda I - k)\phi\| \, \|\phi\|$$

$$\Rightarrow \underline{|\lambda_2| \, \|\phi\| \le \|(\lambda I - k)\phi\| \quad \forall \phi \in H} . \quad (*)$$

Now suppose $\psi \in \overline{Im(\lambda I - k)}$, so there is a sequence (ϕ_n) in H for which $(\lambda I - k)\phi_n \to \psi$ as $n \to \infty$.

Then $((\lambda I - k)\phi_n)$, and hence (ϕ_n), is a Cauchy sequence and therefore, for some $\phi \in H$, $\phi_n \to \phi$ as $n \to \infty$ and we see that $(\lambda I - k)\phi = \psi \Rightarrow \underline{\psi \in Im(\lambda I - k)}$.

$$\left.\begin{array}{l} \phi_n \to \phi \\ (\lambda I - k)\phi_n \to \psi \\ \downarrow \\ (\lambda I - k)\phi \end{array}\right] \Rightarrow (\lambda I - k)\phi = \psi$$

~~Now, since~~

So $Im(\lambda I - k)$ is closed, and by Theorem*, we have that

$$Im(\lambda I - k) = Ker(\bar{\lambda} I - k^*)^{\perp}$$

$$\overset{k = k^*}{=} Ker(\bar{\lambda} I - k)^{\perp}$$

$\bar{\lambda} I - k$ injective

$\Rightarrow Im(\lambda I - k)$ is a dense subset of H

$\Rightarrow \underset{H}{\underset{\shortparallel}{\overline{Im(\lambda I - k)}}} = Im(\lambda I - k) \Rightarrow Im(\lambda I - k)$ is surjective.

Therefore, $\lambda I - k$ is bijective, so that it has an inverse linear map.

And by $(*)$, we get

$$\|(\lambda I - k)^{-1}\phi\| \le |\lambda_2|^{-1} \|\phi\| \quad (\phi \in H),$$

that is, $(\lambda I - k)^{-1}$ is bounded and $\lambda \notin \sigma(k)$.

$\square$.

→ The next lemma relates the norm of a self-adjoint operator to its spectrum.

↳ This is a result relating the analysis (the norm) to the algebraic properties of the operator (the spectrum) and will allow us to trade off the two sorts of properties against each other.

Notice that the following lemma guarantees that the spectrum of a self-adjoint operator is non-empty.

• <u>LEMMA 3</u>: Suppose that H is a non-zero Hilbert space and $k: H \to H$ is a bounded self-adjoint operator.
Then at least one of $\pm\|k\|$ belongs to $\sigma(k)$ and

$$\|k\| = \sup_{\lambda \in \sigma(k)} |\lambda| .$$

$\boxed{\text{Proof:}}$

Since $\lambda \in \sigma(k) \Rightarrow |\lambda| \leq \|k\|$.
The second part will follow if we can show that one of $\pm\|k\|$ is in $\sigma(k)$.

Let $\alpha = \|k\|$.
Then there is a sequence (ϕ_n) in H with $\|\phi_n\| = 1$ such that $\|k\phi_n\| \to \alpha$ as $n \to \infty$.

Then, noting that $\langle k^2\phi_n, \phi_n\rangle$ and α are real,

$$\cancel{\|(k^2-\alpha^2}\ \|(\alpha^2 I - k^2)\phi_n\|^2 = \alpha^4\langle\phi_n,\phi_n\rangle - 2\alpha^2\langle k^2\phi_n,\phi_n\rangle + \langle k^2\phi_n, k^2\phi_n\rangle$$

$$\overset{\|\phi_n\|=1,\ k=k^*}{=} \|k^2\phi_n\|^2 - 2\alpha^2\|k\phi_n\|^2 + \alpha^4$$

$$\leq \underbrace{\|k\|^2}_{=\alpha^2}\ \underbrace{\|k\phi_n\|^2}_{\downarrow\ \alpha^2} - 2\alpha^2\underbrace{\|k\phi_n\|^2}_{\downarrow\ \alpha^2} + \alpha^4 \xrightarrow{n\to\infty} 0$$

Therefore, since for all $n \in \mathbb{N}$ $\|\phi_n\| = 1$,

$\alpha^2 I - k^2$ cannot have an inverse (bounded) in H.

Since $\alpha^2 I - k^2 = (\alpha I - k)(\alpha I + k)$ at least one of $\alpha I \pm k$ is not invertible $\Rightarrow$ at least one of $\pm\alpha$ is in $\sigma(k)$ □.

3

- **LEMMA 4** (Theorem 4.15 of Porter & Stirling)

Let H be a Hilbert space and $K \in B(H)$ be compact and self-adjoint. Then there is a, possibly finite, sequence (μ_n) of non-zero eigenvalues of K and a correspondent orthonormal sequence (Φ_n) of eigenvectors such that for each $\Phi \in H$, $K\Phi = \sum_{n=1}^{\infty} \mu_n (\Phi, \phi_n)\phi_n$, where the sum is a finite sum if there are only finitely many eigenvalues. Moreover if we define $K_N \in B(H)$ by $K_N \Phi = \sum_{n=1}^{N} \mu_n (\Phi, \phi_n)\phi_n$ then in the case where there are infinitely many eigenvalues $\|K - K_N\| \to 0$ as $N \to \infty$.

Proof: p. 110 – 111, Integral Eq. (David Porter)