Dr. Miren Zubeldia, C414
E-mail: miren.zubeldia@helsinki.fi

Dr. Rodrigo Bleyer, A412
E-mail: ismael.bleyer@helsinki.fi

Integral Equations **Spring 2015**
Due to 18.02.2015

Exercise List 5

21. Let H be a Hilbert space and $K : H \longrightarrow H$ be a bounded linear operator. Show that:
$$K \quad \text{compact} \Leftrightarrow K^*K \quad \text{compact}$$

22. Let H be a Hilbert space. Assume that $B\colon H \times H \longrightarrow \mathbb{R}$ is a real bilinear map for which there exist constants $M > 0$ and $m > 0$ such that
$$|B(u,v)| \leqslant M \|u\| \|v\|, \quad u, v \in H,$$
and
$$m \|u\|^2 \leqslant B(u,u), \quad u \in H.$$
Prove that there is a unique bounded linear operator $A\colon H \longrightarrow H$ such that
$$B(u,v) = \langle Au, v \rangle, \quad u, v \in H.$$

23. Prove now the *Lax–Milgram theorem*: If B is as above and $\lambda\colon H \longrightarrow B$ is a bounded linear functional, then there exists a unique element $u \in H$ such that for all $v \in H$ we have
$$B(u,v) = \lambda(v).$$

24. Let $(\cdot,\cdot) : C([a,b]) \times C([a,b]) \longrightarrow \mathbb{R}$ be a degenerate bilinear form, i.e., there exists $\varphi \in C([a,b])$, $\varphi \neq 0$, such that for all $f \in C([a,b])$, $(\varphi, f) = 0$. Without loss of generality, we may assume that $\varphi(a) = 1$. Consider the operators $A, B : C([a,b]) \longrightarrow C([a,b])$ given by
$$A\phi = \phi(a)\varphi, \qquad B\psi = 0 \qquad \phi, \psi \in C([a,b]).$$
   (i) Show that A and B are compact and adjoint with respect to $(\cdot,\cdot)$ i.e. $(A\phi, \psi) = (\phi, B\psi)$.
   (ii) Compute the nullspaces $Ker\,(I - A)$ and $Ker\,(I - B)$.

   Hint: For (i), show and use: A is bounded and $dim\, Im(A) < \infty$ to conclude that A is compact.

25. Let H_1, H_2 be Hilbert spaces. Let $S : H_1 \to H_1$ and $T : H_2 \to H_2$ be linear continuously invertible operators on H_1 and H_2, respectively and let $A : H_1 \to H_1$ and $B : H_2 \to H_2$ be compact operators such that S is adjoint to T and A is adjoint to B.

Show that: the homogeneous equations

$$S\varphi - A\varphi = 0$$

and

$$T\psi - B\psi = 0$$

have the same number of linearly independent solutions.
Hint: You may use Fredholm alternative. Use and show that S^{-1} is adjoint to T^{-1}, $S^{-1}A$ is adjoint to BT^{-1}, $Ker\,(S-A) = Ker\,(I - S^{-1}A)$ and $dim\,Ker(T-B) = dim\,Ker(I - BT^{-1})$.