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Integral Equations **Spring 2015**
Due to 11.02.2015

Exercise List 4

16. Let X be a linear space and $A, B : X \longrightarrow X$ be linear operators with $AB = BA$. If AB is invertible, what can you say about the invertibility of A and B?

17. Find a normed space X and a compact operator $K : X \longrightarrow X$ for which
$$X \neq Ker\,(I - K) \oplus Im\,(I - K).$$

18. Let
$$k(x, y) := \chi_{[0,1]}(x)\chi_{[0,1]}(y) + \chi_{[0,2]}(x)\chi_{[1,2]}(y)$$
for $x, y \in [0, 2]$ and let $K : L^2([0, 2]) \longrightarrow L^2([0, 2])$ be an integral operator induced by the kernel K. Please show that the Riesz index of $(I - K)$ is $\nu = 2$.

19. Consider the following integral equation
$$f(x) - \int_0^1 k(x, y)f(y)\,dy = 1, \qquad x \in [0, 1]$$
with the Volterra kernel
$$k(x, y) := \begin{cases} 1 & \text{if } y \leq x \\ 0 & \text{if } y > x. \end{cases}$$
We approximate the kernel k with degenerate kernels
$$k_n(x, y) := \sum_{i=1}^{n} \chi_{\left(\frac{i-1}{n}, \frac{i}{n}\right)}(x)\chi_{\left[0, \frac{i-1}{n}\right]}(y),$$
where χ_I are the characteristic functions on the interval I.
Compute the solutions f_n, $n \in \mathbb{N}$, of
$$f_n(x) - \int_0^1 k_n(x, y)f_n(y)\,dy = 1, \qquad x \in [0, 1]$$
and the limit as $n \to \infty$, if it exists.

20. (Numerics) Let $I = [0,1] \subset \mathbb{R}$, $k : I \times I \longrightarrow \mathbb{C}$, $g \in C(I)$. Consider the integral equation

$$f(x) - \int_I k(x,y) f(y)\, dy = g(x), \qquad x \in I. \tag{1}$$

Our goal is to solve equation (1) numerically.

(i) First discretize the integral using the standard discretization method (i.e., rectangle method) for given n points ($n-1$ equally spaced intervals).
In fact, let $y_i = (i-1)\frac{1}{n-1}$, $i = 1, \ldots, n$, and show that the integral operator can be written as

$$(Kf)(x) := \int_I k(x,y) f(y)\, dy \approx \frac{1}{n-1} \sum_{i=1}^{n} k(x, y_i) f(y_i).$$

(ii) We next discretize on the variable x, using the same number of points. Indeed, for $x_j = (j-1)\frac{1}{n-1}$, $j = 1, \ldots n$, we consider

$$f(x_j) - \frac{1}{n-1} \sum_{i=1}^{n} k(x_j, y_i) f(y_i) = g(x_j). \tag{2}$$

Thus, if we denote

$$f_n = \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix}, \qquad g_n = \begin{bmatrix} g(x_1) \\ \vdots \\ g(x_n) \end{bmatrix}$$

and the matrix $K = (k_{ji})_{i,j=1,\ldots,n}$ with $k_{ji} = k(x_j, y_i)$ and $\lambda = \frac{1}{n-1}$ then (2) can be written as the following linear system

$$(I - \lambda K) f_n = g_n.$$

(iii) Let $g(x) = 2x$ and

$$k(x,y) := \begin{cases} x - y & \text{if } y \leq x \\ 0 & \text{if } y > x, \end{cases}$$

(as in **Exercise 1** of the ExList1).
Create the corresponding matrix K and vector g_n as above, and solve the linear system $(I - \lambda K) f_n = g_n$ with some linear algebra techniques or simply inverting it using some Matlab command

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>> x = A \ b
>> x = inv(A)*b
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(iv) Compare the above solution f_n with the analytical solution given previously in ExList1. How many points did you choose? Compare the quality of the solution to the number of points n.