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Integral Equations **Spring 2015**
Due to 04.02.2015

Exercise List 3

11. Assume that $K\colon H_1 \longrightarrow H_2$ is a compact operator between Hilbert spaces. Given bounded linear maps $A\colon H \longrightarrow H_1$ and $B\colon H_2 \longrightarrow H$, where H is again Hilbert, prove that KA and BK are compact. Also, prove that the sum $K_1 + K_2$ of two compact operators $K_1, K_2\colon H_1 \longrightarrow H_2$ is compact.

12. Assume that $\langle a_n \rangle$ is a sequence of complex numbers converging to zero. Consider the linear map

$$A\colon \ell^2 \longrightarrow \ell^2, \qquad \langle x_n \rangle \longmapsto \langle a_n x_n \rangle .$$

Prove that A is compact. Hint: Use the previous exercise with suitable operators K_n having finite dimensional image spaces.

13. Assume that $K : H \to H$ is a linear operator and that for some positive integer n_0 we know that K^{n_0} is compact. What can you say about $\operatorname{Ker}(I - K)$?

14. Prove that a compact operator $K\colon \ell^2(\mathbb{C}) \longrightarrow \ell^2(\mathbb{C})$ is a norm limit of finite dimensional operators. Hint: Let Q_n be the orthogonal projection to span $\{e_1, \ldots, e_n\}$, where $\langle e_i \rangle$ is the standard orthonormal basis of $\ell^2(\mathbb{C})$. Let $K_n = Q_n K$ and prove that $\|K - K_n\| \longrightarrow 0$ by considering a suitable finite covering of the compact set $\overline{K(B)}$, where B is the closed unit ball of $\ell^2(\mathbb{C})$.

15. Show that the following integral equation

$$\varphi(x) - \frac{1}{2}\int_0^1 \cos(xy)\,\varphi(y)\,dy = f(x), \qquad x \in [0,1]$$

has a unique solution $\varphi \in C([0,1])$ for any $f \in C([0,1])$.
Hint: Let $(K\varphi)(x) := \frac{1}{2}\int_0^1 \cos(xy)\,\varphi(y)\,dy$. Show that for $\varphi \in Ker(I-K)$

$$|\varphi(x)| \leq \frac{1}{2}\|\varphi\|_\infty, \qquad x \in [0,1]$$

holds and implies that $Ker(I-K) = \{0\}$.