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Integral Equations **Spring 2015**
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Exercise List 2

6. Let K be a continuous integral kernel. Let us consider the iterated kernels

$$K^{(1)}(s,t) = K(s,t), \quad K^{(n)}(s,t) = \int_t^s K(s,r)\, K^{(n-1)}(r,t)\, \mathrm{d}r,$$

which were defined in the lectures.

   (i) Show that

$$K^{(n)}(s,t) = \int_t^s K^{(n-1)}(s,r)\, K(r,t)\, \mathrm{d}r.$$

   Hint: Use induction on n.

   (ii) Give a detailed proof for the uniform convergence of the series defining the resolvent kernel

$$\Gamma(x,y;\lambda) = K(x,y) + \lambda K^{(2)}(x,y) + \ldots + \lambda^{n-1} K^{(n)}(x,y) + \ldots$$

   for $|\lambda| < \infty$.

7. Consider the integral equation

$$f(x) + \frac{1}{20} \int_0^1 e^{-|xy|^2} \sin\big(x^2 + y^2\big)\, f(y)\, \mathrm{d}y = \sin x.$$

Prove that this has a unique solution $L^2\big([0,1]\big)$, and that in fact this solution is also continuous.

8. Let H be a Hilbert space, and $A\colon H \longrightarrow H$ a bounded linear map for which $\|A^{n_0}\| < 1$ for some positive integer n_0. Prove that $I - A$ is invertible and determine its inverse.

9. Convolution is one of the most important examples of integral operators. Given functions f, g on $\mathbb{R}$, their *convolution* is the function $f * g$ defined by

$$(f * g)(x) = \int_{\mathbb{R}} g(x-y) f(y)\, dy,$$

provided that the integral makes sense. Note that with g fixed, the mapping $f \mapsto f * g$ is an integral operator with kernel $k(x, y) = g(x - y)$.

(i) Let $g \in L^1(\mathbb{R})$ be fixed. Show that $Kf = f * g$ is a bounded mapping of $L^2(\mathbb{R})$ into itself.
Hint: Use Young's Inequality.

(ii) Prove that convolution is commutative, i.e., that $f * g = g * f$.

(iii) Let $f, g \in L^1(\mathbb{R})$. Prove that $f * g \in L^1(\mathbb{R})$ and $\|f * g\|_{L^1} \leq \|f\|_{L^1} \|g\|_{L^1}$.

(iv) Show that if $g \in L^1(\mathbb{R})$ and $g \in C([a, b])$, then $f * g \in C([a, b])$.

(v) Show that if $g \in L^1(\mathbb{R})$ and f is differentiable, then $f * g$ is also differentiable and $\frac{d}{dx}(f * g) = \frac{df}{dx} * g$.

10. (Numerics) Let $\mathcal{F}$ be the Fourier transform on the circle, i.e., it is the isomorphism $\mathcal{F} : L^2([0, 1]) \to l^2(\mathbb{Z})$ given by $\mathcal{F}f = \widehat{f} = \{\widehat{f}(n)\}_{n \in \mathbb{Z}}$, where

$$\widehat{f}(n) = \langle f, e_n \rangle = \int_0^1 e^{2\pi inx} f(x)\, dx, \qquad e_n(x) = e^{2\pi inx}.$$

Note that it is an integral operator with kernel $k(n, x) = e^{2\pi inx}$.

(i) Prove that the Fourier transform converts convolution into multiplication. That is, prove that if $f, g \in L^2([0, 1])$, then

$$\widehat{(f * g)}\ (n) = \widehat{f}(n)\widehat{g}(n), \qquad n \in \mathbb{Z}.$$

(ii) In Matlab investigate the discrete Fourier transform and its inverse

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>> help fft
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(ii) Consider the following functions in the interval $[-2, 2]$

$$f(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & \text{elsewhere} \end{cases} \tag{1}$$

$$g(x) = \begin{cases} 1 - |x| & \text{if } |x| \leq a \\ 0 & \text{elsewhere} \end{cases} \tag{2}$$

Create an approximation of these functions and calculate the time-domain convolution using the builtin function as follows:

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>> conv(f,f)
>> conv(g,g)
>> conv(f,g)
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(iii) Now that we know what output to expect, compute using the convolution property proved in 10.(i)