

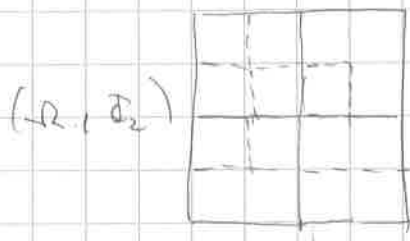
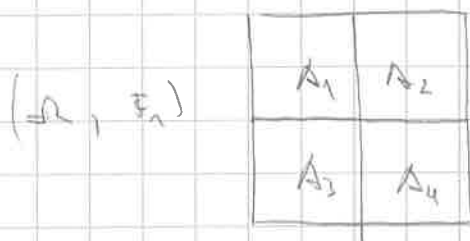
⑦

Lähdöllinen alkuehtona äärellisessä kentässä

12.2-140.

$$\Omega = \{\omega_1, \dots, \omega_m\}, \quad \mathcal{F}_0 = \{\varnothing, \Omega\}, \quad \mathcal{F}_1, \mathcal{F}_2 = 2^\Omega$$

$$\mathcal{F}_n = \mathcal{B}(A_1, \dots, A_n), \quad \Omega = A_1 \cup \dots \cup A_n, \quad \text{osittain}$$



$$\mathbb{E}(S | A_j) = \sum_{\omega \in A_j} S(\omega) \frac{P(\omega)}{P(A_j)}, \quad j=1, \dots, n.$$

Jos  $\eta$   $\mathcal{F}_n$ -määrällinen, niin  $\exists \beta_1, \dots, \beta_n \in \mathbb{R}$ :

$$\eta(\omega) = \sum_{j=1}^n \beta_j \mathbb{1}(\omega \in A_j) = \beta_j, \quad \forall \omega \in A_j.$$

$$\mathbb{E}(\eta S | A_j) = \sum_{\omega \in A_j} \eta(\omega) S(\omega) \frac{P(\omega)}{P(A_j)} = \beta_j \mathbb{E}(S | A_j)$$

Lähdöllinen alkuehtona  $\mathcal{F}_n$ -määrällinen  $= \mathbb{E}(S | \mathcal{F}_n)$ , s.m.

(i)  $\mathbb{E}(S | \mathcal{F}_n)$   $\mathcal{F}_n$ -määrällinen

(ii)  $\mathbb{E}(\mathbb{E}(S | \mathcal{F}_n) \mathbb{1}(A)) = \mathbb{E}(S \mathbb{1}(A)), \quad \forall A \in \mathcal{F}_n$

(iii)  $\mathbb{E}(\mathbb{E}(S | \mathcal{F}_n) \mathbb{1}(A_j)) = \mathbb{E}(S \mathbb{1}(A_j)), \quad \forall j=1, \dots, n$

Wrt. dyasta siis  $\exists \alpha_1, \dots, \alpha_n \in \mathbb{R}$ :

$$\mathbb{E}(S | \mathcal{F}_n)(\omega) = \sum_{j=1}^n \alpha_j \mathbb{1}(\omega \in A_j),$$

$$\mathbb{E}(\mathbb{E}(S | \mathcal{F}_n) \mathbb{1}(A_j)) = \alpha_j P(A_j) = \mathbb{E}(S \mathbb{1}(A_j))$$

$$\Rightarrow \mathbb{E}(S | \mathcal{F}_n)(\omega) = \frac{\mathbb{E}(S \mathbb{1}(A_j))}{P(A_j)} = \mathbb{E}(S | A_j), \quad \forall \omega \in A_j.$$

$$\mathbb{E}(S | \mathcal{F}_n) = \sum_{j=1}^n \mathbb{E}(S | A_j) \mathbb{1}(A_j).$$

⑤) Für  $\eta \in \mathbb{R}$ -Werten, mit

$$\begin{aligned} \mathbb{E}(\eta S | \mathcal{F}_n) &= \sum_{j=1}^n \mathbb{E}(\eta S | A_j) \mathbb{1}(A_j) \\ &= \sum_{j=1}^n \beta_j \mathbb{E}(S | A_j) \mathbb{1}(A_j) \\ &= \eta \mathbb{E}(S | \eta) \end{aligned}$$

Für  $\alpha_1, \alpha_2 \in \mathbb{R}$ , mit

$$\begin{aligned} \mathbb{E}(\alpha_1 S_1 + \alpha_2 S_2 | A_j) &= \alpha_1 \sum_{\omega \in A_j} S_1(\omega) \frac{\mathbb{P}(\omega)}{\mathbb{P}(A_j)} \\ &\quad + \alpha_2 \sum_{\omega \in A_j} S_2(\omega) \frac{\mathbb{P}(\omega)}{\mathbb{P}(A_j)} \\ &= \alpha_1 \mathbb{E}(S_1 | A_j) + \alpha_2 \mathbb{E}(S_2 | A_j). \end{aligned}$$

Sei  $J$  pair

$$\begin{aligned} \mathbb{E}(\alpha_1 S_1 + \alpha_2 S_2 | \mathcal{F}_n) &= \sum_{j=1}^n (\alpha_1 \mathbb{E}(S_1 | A_j) + \alpha_2 \mathbb{E}(S_2 | A_j)) \mathbb{1}(A_j) \\ &= \alpha_1 \mathbb{E}(S_1 | \mathcal{F}_n) + \alpha_2 \mathbb{E}(S_2 | \mathcal{F}_n). \end{aligned}$$

$$\mathbb{E}(\mathbb{E}(S | \mathcal{F}_n)) = \sum_{j=1}^n \mathbb{E}(S | A_j) \mathbb{P}(A_j)$$

$$= \sum_{j=1}^n \sum_{\omega \in A_j} S(\omega) \cdot \frac{\mathbb{P}(\omega)}{\mathbb{P}(A_j)} \cdot \mathbb{P}(A_j)$$

$$= \sum_{\omega \in \Omega} S(\omega) \mathbb{P}(\omega) = \mathbb{E}(S).$$