FUNKTIONAALIANALYYSIN PERUSKURSSI
KEVÄT / SPRING 2015
SOLUTUONS 6

1. Olkoon $\mathbb{K} = \mathbb{R}$. Osoita, että funktiot $\sin(2\pi nx)$, $n \in \mathbb{N}$, ovat keskenään ortogonaaliset avaruudessa $L^2(0,1)$, sisätulona

$$(f|g) = \int_0^1 f(x)g(x)dx \ , \quad f, g \in L^2(0,1).$$

Samoin funktioille $\cos(2\pi nx)$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, ja tutki vielä lopuksi sisätuloja $\big(\sin(2\pi nx)|\cos(2\pi mx)\big)$. Laske normitusvakiot, eli positiiviset luvut, joilla kertominen tekee yllä mainittujen funktioiden normeista ykkösen.

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Let $\mathbb{K} = \mathbb{R}$. Show that the functions $\sin(2\pi nx)$, $n \in \mathbb{N}$, are mutually orthogonal in the space $L^2(0,1)$ with inner product

$$(f|g) = \int_0^1 f(x)g(x)dx \ , \quad f, g \in L^2(0,1).$$

Do the same for the functions $\cos(2\pi nx)$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and then also investigate the inner products $\big(\sin(2\pi nx)|\cos(2\pi mx)\big)$. Find the normalisation constants, i.e. positive real numbers, multiplying by which makes the norms of the above mentioned functions equal to 1.

Solution

i) Let $n, m \in \mathbb{N}_0$, $n \neq m$. Recall the product formulas for trigonometric functions! Then it holds that
(0.1)

$$\int_0^1 \sin(2\pi nx)\sin(2\pi mx)dx = \frac{1}{2}\int_0^1 \cos(2\pi(n-m)x) - \cos(2\pi(n+m)x)dx$$

$$= \frac{1}{2}\left(\left[\frac{1}{2\pi(n-m)}\sin(2\pi(n-m)x)\right]_0^1 - \left[\frac{1}{2\pi(n+m)}\sin(2\pi(n+m)x)\right]_0^1\right) = 0,$$

since $\sin(2\pi k) = 0$ for all $k \in \mathbb{N}$. Thus $(\sin(2\pi n\cdot)|\sin(2\pi m\cdot))_{L^2} = 0$, if $n \neq m$. If $n = m \neq 0$, we see from equation (0.1) that

$$(\sin(2\pi n\cdot)|\sin(2\pi n\cdot))_{L^2} = \frac{1}{2} \text{ and } (\sin(2\pi 0\cdot)|\sin(2\pi 0\cdot))_{L^2} = 0$$

ii) Let $n, m \in \mathbb{N}_0$, $n \neq m$. It holds that

(0.2)

$$\int_0^1 \cos(2\pi nx)\cos(2\pi mx)dx = \frac{1}{2}\int_0^1 \cos(2\pi(n-m)x)+\cos(2\pi(n+m)x)dx = 0.$$

Thus $(\cos(2\pi n\cdot)|\cos(2\pi m\cdot))_{L^2} = 0$, if $n \neq m$. If $n = m \neq 0$, we see similarly as in part i), that

$$(\cos(2\pi n\cdot)|\cos(2\pi n\cdot))_{L^2} = \frac{1}{2} \text{ and } (\cos(2\pi 0\cdot)|\cos(2\pi 0\cdot))_{L^2} = 1$$

iii) Let $n, m \in \mathbb{N}_0$. It holds that

$$\int_0^1 \sin(2\pi nx)\cos(2\pi mx)dx = \frac{1}{2}\int_0^1 \sin(2\pi(n-m)x)+\sin(2\pi(n+m)x)dx = 0.$$

Thus

$$(\cos(2\pi m\cdot)|\sin(2\pi n\cdot))_{L^2} = 0.$$

iv) We note that by parts i) and ii) we have found that

$$\|\cos(2\pi m\cdot)\|_{L^2} = \sqrt{1/2} = \|\sin(2\pi m\cdot)\|_{L^2},\ \forall\, m \in \mathbb{N}$$

and also

$$\|\cos(2\pi 0\cdot)\|_{L^2} = 1 \text{ and } \|\sin(2\pi 0\cdot)\|_{L^2} = 0$$

In order to make sets

$$\{\cos(2\pi m\cdot) : n \in \mathbb{N}_0\} \text{ and } \{\sin(2\pi m\cdot) : n \in \mathbb{N}_0\}$$

orthonormal with respect to L^2-norm the normalization constants is $\sqrt{2}$ for case $n \neq 0$.

2.(∗) Tarkastellaan Hilbert–avaruutta $L^2(0, 2)$. Olkoon M sen 3-ulotteinen aliavaruus, jonka virittävät polynomit 1, t ja t^2. Etsi aliavaruuden M jokin ortonormaali kanta.

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In the Hilbert space $L^2(0, 2)$ let M be the 3-dimensional subspace spanned by the polynomials 1, t and t^2. Find an orthonormal basis for M.

Solution

First we note that any polynomials, $p_1(t) = 1, p_2(t) = t + b, p_3(t) = t^2 + ct + d$ span the same vector subspace as $1, t, t^2$ do. There are many ways to find the right coefficients for these polynomials, for instance the Gram-Smith method. We do it by solving the following system of integral equations.

$$\begin{cases} \int_0^2 t + b \; dt = 0 \\ \int_0^2 (t+b)(t^2 + ct + d) \; dt = 0 \\ \int_0^2 t^2 + ct + d \; dt = 0 \end{cases}$$

These equation tell, that all the solution polynomials p_1, p_2, p_3 are orthogonal in L^2-sence. Solving this system is a straight forward calculation and we skip it. Then we get that

$$p_1(t) = 1, \; p_2(t) = t - 1, \; p_2(t) = t^2 - 2t + \frac{2}{3}.$$

We still need to normalize the polynomials $p_1, \; p_2, \; p_2$. To do so we calculate the Ł^2 norms.

$$\|p_1\|_{L^2} = \left(\int_0^2 1^2 \; dt \right)^{1/2} = \sqrt{2},$$

$$\|p_2\|_{L^2} = \left(\int_0^2 (t-1)^2 \; dt \right)^{1/2} = \left(\int_0^2 t^2 - 2t + 1 \; dt \right)^{1/2} = \left(\left[1/3t^3 - t^2 + t \right]_0^2 \right)^{1/2}$$

$$= \left(8/3 - 4 + 2 \right)^{1/2} = \sqrt{2/3}$$

$$\|p_3\|_{L^2} = \left(\int_0^2 (t^2 - 2t + 2/3)^2 \; dt \right)^{1/2} = \left(\int_0^2 t^4 - 4t^3 + 16/3t^2 - 8/3t + 4/9 \; dt \right)^{1/2}$$

$$= \left(\left[1/5t^5 - t^4 + 16/9t^3 - 4/3t^2 + 4/9 \right]_0^2 \right)^{1/2} = \sqrt{8/45} = \frac{2}{3}\sqrt{2/5}.$$

Thus we get that an orthonormal basis for vector subspace $\text{span}(1, t, t^2)$ is

$$\frac{1}{\sqrt{2}}, \; \sqrt{3/2}(t-1) \text{ and } \frac{3}{2}\sqrt{5/2}(t^2 - 2t + 2/3)$$

3. Olkoot E ja F Banach-avaruuksia. Tiedetään, että kaikkien lineaarikuvausten $S : E \to F$ joukko $L(E, F)$ on vektoriavaruus. Muistutetaan mieleen, että lineaarinen $T : E \to F$ on jatkuva, jos on olemassa vakio $C > 0$, jolle $\|Tx\|_F \le C\|x\|_E$ kaikilla $x \in E$. Jatkuvalle T lauseke

$$\sup_{\|x\|_E \le 1} \|Tx\|_F := \|T\| \tag{0.3}$$

on hyvin määritelty (äärellinen).
a) Osoita, että myös jatkuvat lineaarikuvaukset $T : E \to F$ muodostavat vektoriavaruuden (jota merkitään esim. $\mathcal{L}(E, F)$).
b) Osoita, että $\|T\|$ on normi avaruudessa $\mathcal{L}(E, F)$; sitä kutsutaan operaattorinormiksi.

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Let E and F be Banach spaces. It is known that the set $L(E, F)$ of all linear mappings $S : E \to F$ is a vector space. We also recall that a linear $T : E \to F$ is continuous, if there exists a constant $C > 0$ such that $\|Tx\|_F \le C\|x\|_E$ for all $x \in E$. The expression

$$\sup_{\|x\|_E \le 1} \|Tx\|_F := \|T\| \tag{0.4}$$

is well defined (finite) for a continuous T.
a) Show that the set $\mathcal{L}(E, F)$ of all continuous linear mappings $S : E \to F$ is a also vector space.
b) Prove that the expression $\|T\|$ is a norm in $\mathcal{L}(E, F)$; it is called the operator norm.

Solution

a) Let $T, S \in \mathcal{L}(E, F)$ and $a, b \in \mathbb{R}$. We have to show that $\|aT + bS\| < \infty$. This proves that $aT + bT \in \mathcal{L}(E, F)$. Let $x \in \overline{B_E(0, 1)} \subset E$ and calculate

$$\|(aT + bS)x\|_F \le |a|\|Tx\|_F + |b|\|Sx\|_F \le |a|\|T\| + |b|\|S\| < \infty. \tag{0.5}$$

Thus $\|aT + bS\| < \infty$.

b) Let $T, S \in \mathcal{L}(E, F)$ and $a \in \mathbb{R}$. Let $x \in \overline{B_E(0, 1)} \subset E$. Note that the equation (0.5) already proved, that the operator norm satisfies the triangle inequality. We still have to check two other properties of norm. We start with scaling invariance.

$$\|aTx\|_F = |a|\|Tx\|_F \leq |a|\|T\| \Rightarrow \|aT\| \leq |a|\|T\|$$

Let $\epsilon > 0$ and choose a sequence $(x_n)_{n\in\mathbb{N}} \subset \overline{B_E(0,1)}$ such that $\|Tx_n\|_F \longrightarrow \|T\|$. Then there exists $n_\epsilon \in \mathbb{N}$ such that for all $n \geq n_\epsilon$ holds

$$\|Tx_n\|_F + \epsilon > \|T\|.$$

Since

$$\|aT\| = \sup_{\|x\|_E \leq 1} \|aTx\|_F = |a|\|T\|,$$

it holds also

$$|a|\|T\| + \epsilon \geq |a|\|Tx_n\|_F + \epsilon > |a|\|T\| = \|aT\|.$$

Since ϵ was arbitrary, we have also proved that $|a|\|T\| \geq \|aT\|$.

Suppose that $T \in \mathcal{L}(E,F)$ is such that $\|T\| = 0$. Then it holds that $T(\overline{B_E(0_E,1)}) = \{0_F\}$. Let $x \in E$. By linearity of T it holds that

$$Tx = \|x\|_E T\left(\frac{1}{\|x\|_E}x\right) = 0_F.$$

Thus T is the constant mapping 0_F.

4.(∗) Jos Tehtävän 3 tilanteessa $F = \mathbb{K}$, avaruutta $\mathcal{L}(E,\mathbb{K})$ sanotaan E:n duaaliavaruudeksi ja merkitään $\mathcal{L}(E,\mathbb{K}) = E^*$. Operaattorinormia (0.3) sanotaan duaalinormiksi, merk, $\|y\|_{E^*}$, kun $y \in E^*$.
Kun $1 < p < \infty$ ja $1/p + 1/q = 1$, osoita, että jokainen avaruuden ℓ^q alkio $y = (y_n)_{n=1}^\infty$ määrää avaruuden $(\ell^p)^*$ alkion kaavalla

$$y : x \mapsto \sum_{n=1}^{\infty} x_n y_n, \tag{0.6}$$

missä $x = (x_n)_{n=1}^\infty \in \ell^p$. Osoita, että $\|y\|_{(\ell^p)^*} \leq \|y\|_q$. Kun $y = (1,-1,0,0,0,0\ldots)$, osoita myös $\|y\|_{(\ell^p)^*} \geq \|y\|_q$. Taikasana: Hölder.

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If $F = \mathbb{K}$ in the situation of Problem 3, the space $\mathcal{L}(E,\mathbb{K})$ is called the dual space of E and it is denoted by $\mathcal{L}(E,\mathbb{K}) = E^*$. The operator norm (0.4) is called the dual norm, with notation $\|y\|_{E^*}$ for $y \in E^*$.
When $1 < p < \infty$ ja $1/p+1/q = 1$, show that every element $y = (y_n)_{n=1}^\infty$ of ℓ^q defines an element of the space $(\ell^p)^*$ by the formula

$$y : x \mapsto \sum_{n=1}^{\infty} x_n y_n, \tag{0.7}$$

where $x = (x_n)_{n=1}^{\infty} \in \ell^p$. Prove that $\|y\|_{(\ell^p)^*} \leq \|y\|_q$. When $y = (1, -1, 0, 0, 0, 0 \ldots)$, prove also that $\|y\|_{(\ell^p)^*} \geq \|y\|_q$. Magic word: Hölder.

Solution

i) Let $y \in \ell^q$. We have to prove that mapping $L_y : \ell^p \to \mathbb{K}$,

$$L_y(x) = \sum_{n=1}^{\infty} x_n y_n$$

is well defined, linear and continuous. Let $x \in \ell^p$. Then

(0.8)

$$|L_y(x)| = \left| \sum_{n=1}^{\infty} x_n y_n \right| \leq \sum_{n=1}^{\infty} |x_n y_n| \overset{Hölder}{\leq} \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} |y_n|^q \right)^{1/q} = \|x\|_p \|y\|_q < \infty.$$

This proves that L_y is well defined and bounded. Let $x, z \in \ell^p$ and $a, b \in \mathbb{K}$. Then

$$L_y(ax+by) = \sum_{n=1}^{\infty} (ax_n + bz_n) y_n = a \sum_{n=1}^{\infty} x_n y_n + b \sum_{n=1}^{\infty} z_n y_n = aL_y x + bL_y z.$$

This proves that L_y is linear. Since it is bounded, it is also continuous.

ii) Claim $\|y\|_{(\ell^p)^*} \leq \|y\|_q$ follows from inequality (0.8). Let $y = (1, -1, 0, 0, \ldots) \in \ell^q$ and $x \in \overline{B_{\ell^p}(0,1)}$. Then

$$|L_y(x)| = |\sum_{n=1}^{\infty} x_n y_n| = |x_1 - x_2| \text{ and } \|y\|_q = (2)^{1/q}.$$

Consider a vector $x = (\frac{1}{2^{1/p}}, -\frac{1}{2^{1/p}}, 0, 0, \ldots)$. Then it holds that

$$\|x\|_p^p = 2\left(\frac{1}{2^{1/p}}\right)^p = 1 \text{ and } |x_1 - x_2| = 2\frac{1}{2^{1/p}} = 2^{1-\frac{1}{p}} = 2^{\frac{1}{q}}.$$

Thus $x \in \overline{B_{\ell^p}(0,1)}$ and therefore $\|y\|_{(\ell^p)^*} = \|y\|_q = 2^{1/q}$, due inequality $\|y\|_{(\ell^p)^*} \leq \|y\|_q$.