FRACTAL SETS IN ANALYSIS

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These notes give a sketch of the lectures; definitions, theorems and perhaps some ideas but not many detailed proofs.

1. Measures and dimensions

The s-dimensional *Hausdorff measure* $\mathcal{H}^s$, $s \geq 0$, is defined by

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A),$$

where, for $0 < \delta \leq \infty$,

$$\mathcal{H}^s_\delta(A) = \inf\{\sum_j d(E_j)^s : A \subset \bigcup_j E_j, d(E_j) < \delta\}.$$

Here $d(E)$ denotes the diameter of the set E.

The *Hausdorff dimension* of $A \subset \mathbb{R}^n$ is

$$\dim A = \inf\{s : \mathcal{H}^s(A) = 0\} = \sup\{s : \mathcal{H}^s(A) = \infty\}.$$

Since (as an easy exercise), $\mathcal{H}^s(A) = 0$ if and only if $\mathcal{H}^s_\infty(A) = 0$, we can replace $\mathcal{H}^s$ in the definition of dim by the simpler $\mathcal{H}^s_\infty$. So, more simply,

$$\dim A = \inf\{s : \forall \varepsilon > 0\ \exists E_1, E_2, \dots \subset X \text{ such that } A \subset \bigcup_j E_j \text{ and } \sum_i d(E_j)^s < \varepsilon\}.$$

For the definition of dimension, the sets E_j above can be restricted to be balls, because each E_j is contained in a ball B_j with $d(B_j) \leq 2d(E_j)$. The spherical measure obtained using balls is not the same as the Hausdorff measure but it is between $\mathcal{H}^s$ and $2^s\mathcal{H}^s$.

$\mathcal{H}^s$ is a Borel regular outer measure: Borel sets are $\mathcal{H}^s$ measurable and for every $A \subset \mathbb{R}^n$ there is a Borel set B such that $A \subset B$ and $\mathcal{H}^s(A) = \mathcal{H}^s(B)$.

Hausdorff dimension is countably stable, that is,

$$\dim \bigcup_{i=1}^\infty A_i = \sup_i \dim A_i.$$

Let

$$A(\delta) = \{x : d(x, A) < \delta\}$$

be the open δ-neighbourhood of A, and let $\mathcal{L}^n$ be the Lebesgue measure in $\mathbb{R}^n$.

Definition 1.1. The *lower Minkowski dimension* of a bounded set $A \subset \mathbb{R}^n$ is

$$\underline{\dim}_M A = \inf\{s > 0 : \liminf_{\delta \to 0} \delta^{s-n}\mathcal{L}^n(A(\delta)) = 0\},$$

and the *upper Minkowski dimension* of A is

$$\overline{\dim}_M A = \inf\{s > 0 : \limsup_{\delta \to 0} \delta^{s-n} \mathcal{L}^n(A(\delta)) = 0\}.$$

Let $N(A, \delta)$ be the smallest number of balls of radius δ needed to cover A. Then

$$\underline{\dim}_M A = \liminf_{\delta \to 0} \frac{\log N(A,\delta)}{\log(1/\delta)},$$

and

$$\overline{\dim}_M A = \limsup_{\delta \to 0} \frac{\log N(A,\delta)}{\log(1/\delta)}.$$

We have also that

$$\dim A \leq \underline{\dim}_M A \leq \overline{\dim}_M A.$$

Minkowski dimensions are also called box counting dimensions. They are not countably stable. For example the countable compact set $\{0, 1, 1/2, 1/3, \dots\}$ has positive Minkowski dimensions although every singleton has 0.

Definition 1.2. The *packing dimension* of $A \subset \mathbb{R}^n$ is

$$\dim_P A = \inf\{\sup_j \overline{\dim}_M A_j : A = \bigcup_{j=1}^{\infty} A_j, A_j \text{ is bounded}\}.$$

Then

$$\dim A \leq \dim_P A \leq \overline{\dim}_M A.$$

Packing dimension is countably stable.

A closed set F is called *Ahlfors-David regular*, or AD-regular if for some positive numbers s and C,

$$r^s/C \leq \mathcal{H}^s(F \cap B(x,r)) \leq Cr^s \quad \text{for } x \in F, 0 < r < d(F).$$

Then all the above dimensions of F agree and equal s.

For $0 < d < 1/2$ we define the Cantor set with dissection ratio d by the usual process: Let $I = [0,1]$. Delete from the middle of I an open interval of length $1-2d$ and denote by $I_{1,1}$ and $I_{1,2}$ the two remaining intervals of length d. Next delete from the middle of each $I_{1,j}$ an open interval of length $(1-2d)d$ and denote by $I_{2,i}, i = 1,2,3,4$, all the four remaining intervals of length d^2. Continuing this we have after k steps 2^k closed intervals $I_{k,i}, i = 1, \dots, 2^k$, of length d^k. Define

$$C_d = \bigcap_{k=1}^{\infty} \bigcup_{i=1}^{2^k} I_{k,i}.$$

Let μ_d be the 'natural' probability measure on C_d. This is the unique Borel measure $\mu_d \in \mathcal{M}(C_d)$ which is uniformly distributed in the sense that

$$\mu_d(I_{k,i}) = 2^{-k} \quad \text{for } i = 1, \dots, 2^k, k = 1, 2 \dots. \tag{1.1}$$

The uniqueness follows easily by, for example, checking that this condition fixes the values of integrals of continuous functions. The existence can be verified by showing (easily) that the probability measures

$$(2d)^{-k}\sum_{i=1}^{2^k}\mathcal{L}^1\llcorner I_{k,i}$$

converge weakly as $k\to\infty$ to such a uniformly distributed measure μ_d. Here, and later $\mu\llcorner A$ means the restriction of the measure μ to a set A, which is defined by

$$\mu\llcorner A(B)=\mu(A\cap B).$$

Define

$$s_d=\log 2/\log(1/d),\quad \text{that is,}\quad 2d^{s_d}=1.$$

Notice that then

$$\mu_d(I_{k,i})=d(I_{k,i})^{s_d}\quad \text{for } i=1,\dots,2^k, k=1,2\dots,$$

which easily yields with some positive constants a and b,

$$ar^{s_d}\le\mu_d([x-r,x+r])\le br^{s_d}\quad\text{for } x\in C_d, 0<r<1. \tag{1.2}$$

Using μ_d we can now check that

$$0<\mathcal{H}^{s_d}(C_d)\le 1\quad\text{and}\quad \dim C_d=s_d,$$

and C_d is AD-regular. The upper bound $\mathcal{H}^{s_d}(C_d)\le 1$ is trivial since

$$\sum_{i=1}^{2^k}d(I_{k,i})^{s_d}=2^k(d^k)^{s_d}=1$$

for all k. To prove that $\mathcal{H}^{s_d}(C_d)>0$ it is enough by Frostman's lemma (see below) to show that $\mu_d(J)\lesssim d(J)^{s_d}$ for every open interval $J\subset\mathbb{R}$. To prove this we may assume that $J\subset[0,1]$ and $C_d\cap J\neq\varnothing$. Let $I_{l,j}$ be the largest (or one of them) of all the intervals $I_{k,i}$ contained in J. Then $J\cap C_d$ is contained in four intervals $I_{l,j_1}=I_{l,j_1},\dots,I_{l,j_4}$, whence

$$\mu_d(J)\le 4\mu_d(I_{l,j})=4d(I_{l,j})^{s_d}\le 4d(J)^{s_d},$$

and so $\mathcal{H}^{s_d}(C_d)>0$.

By a modification of the above argument one can show that in fact

$$\mathcal{H}^{s_d}\llcorner C_d=\mu_d\quad\text{and}\quad\mathcal{H}^{s_d}(C_d)=1.$$

For $A\subset\mathbb{R}^n$, let $\mathcal{M}(A)$ be the set of Borel measures μ such that $0<\mu(A)<\infty$ and $\mu(\mathbb{R}^n\setminus A)=0$.

Theorem 1.3. [Frostman's lemma]
Let $0\le s\le n$. For a closed set $A\subset\mathbb{R}^n$, $\mathcal{H}^s(A)>0$ if and only there is $\mu\in\mathcal{M}(A)$ such that

$$\mu(B(x,r))\le r^s\quad\text{for all } x\in\mathbb{R}^n, r>0. \tag{1.3}$$

In particular,

$$\dim A = \sup\{s : \text{there is } \mu \in \mathcal{M}(A) \text{ such that (1.3) holds}\}.$$

The *s-energy*, $s > 0$, of a Borel measure μ is

$$I_s(\mu) = \iint |x-y|^{-s}\, d\mu x\, d\mu y = \int k_s * \mu\, d\mu,$$

where k_s is the *Riesz kernel*:

$$k_s(x) = |x|^{-s}, \quad x \in \mathbb{R}^n.$$

If μ has compact support, $\operatorname{spt}\mu$, we have trivially,

$$I_s(\mu) < \infty \text{ implies } I_t(\mu) < \infty \quad \text{for } 0 < t < s.$$

We can quite easily relate the energies to the Frostman condition using the standard formula

$$\int |x-y|^{-s}\, d\mu y = s\int_0^\infty \frac{\mu(B(x,r))}{r^{s+1}}\, dr.$$

This immediately gives that if $\mu \in \mathcal{M}(\mathbb{R}^n)$ satisfies (1.3), then for $0 < t < s$,

$$I_t(\mu) \leq t \iint_0^{d(\operatorname{spt}\mu)} \frac{\mu(B(x,r))}{r^{t+1}}\, dr\, d\mu x \leq t\mu(\mathbb{R}^n)\int_0^{d(\operatorname{spt}\mu)} r^{s-t-1}\, dr < \infty.$$

On the other hand, if $I_s(\mu) < \infty$, then $\int |x-y|^{-s}\, d\mu x < \infty$ for μ almost all $x \in \mathbb{R}^n$ and we can find $0 < M < \infty$ such that the set $A = \{x : \int |x-y|^{-s}\, d\mu x < M\}$ has positive μ measure. Then one checks easily that $(\mu \llcorner A)(B(x,r)) \leq 2^s M r^s$ for all $x \in \mathbb{R}^n, r > 0$. This gives

Theorem 1.4. For a closed set $A \subset \mathbb{R}^n$,

$$\dim A = \sup\{s : \text{there is } \mu \in \mathcal{M}(A) \text{ such that } I_s(\mu) < \infty\}.$$

These two theorems hold also for Borel sets A, but the proof for closed sets is easier.

Let us look at a few easy examples:

Example 1.5.
(i) Let $\mu = \mathcal{L}^1 \llcorner [0,1]$. Then $\dim[0,1] = 1, \mu \in \mathcal{M}([0,1])$ and $I_s(\mu) < \infty$ if and only $s < 1$. Similarly, if $A \subset \mathbb{R}^n$ is Lebesgue measurable and bounded with $\mathcal{L}^n(A) > 0$ and $\mu = \mathcal{L}^n \llcorner A$, then $I_s(\mu) < \infty$ if and only $s < n$.
(ii) Let $\mu = \mathcal{H}^1 \llcorner \Gamma$ where Γ is a rectifiable curve. Again $I_s(\mu) < \infty$ if and only $s < 1$.
(iii) Let μ_d as above be the natural measure on the Cantor set C_d, that is, $\mu_d = \mathcal{H}^{s_d} \llcorner C$ where $s_0 = \log 2/\log(1/d)$ is the Hausdorff dimension of C. Then $I_s(\mu) < \infty$ if and only $s < s_d$.

The proof of Frostman's lemma is based on weak convergence of measures:

Definition 1.6. The sequence (μ_j) of Borel measures on $\mathbb{R}^n$ converges weakly to a Borel measure μ if for all $\varphi \in C_0(\mathbb{R}^n)$,

$$\int \varphi \, d\mu_j \to \int \varphi \, d\mu.$$

Here $C_0(\mathbb{R}^n)$ is the space of continuous functions $\varphi : \mathbb{R}^n \to \mathbb{R}$ which vanish outside some compact set, that is their support

$$\operatorname{spt} f = closure(\{x : f(x) \neq 0\})$$

is compact.

The weak convergence does not imply that $\mu_j(A)$ would convergence to $\mu(A)$ for all sets A, not even for all closed sets A. To see this consider the Dirac measures at points a: $\delta_a(A) = 1$, if $a \in A$, and $\delta_a(A) = 0$, if $a \notin A$. Then $\delta_{a_j} \to \delta_a$ if $a_j \to a$, but $\delta_{a_j}(\{a\}) = 0 \neq 1 = \delta_a(\{a\})$ for all j if $a_j \neq a$ for all j.

The following weak compactness theorem is very important. It follows from the separability of the space $C_0(\mathbb{R}^n)$, which means that there is a sequence $\varphi_j, j = 1, 2, \dots$ which is dense under the norm $\|\varphi\| = \sup_{x \in \mathbb{R}^n} |\varphi(x)|$. In addition one needs the Riesz represention theorem which identifies positive linear functionals with Borel measures. More precisely, if $L : C_0(\mathbb{R}^n) \to \mathbb{R}$ is linear and it is positive in the sense that

$$Lf \geq 0 \text{ whenever } f \in C_0(\mathbb{R}^n) \text{ with } f \geq 0,$$

then there is a locally finite (finite for compact sets) Borel measure μ such that

$$Lf = \int f \, d\mu \quad \text{for } f \in C_0(\mathbb{R}^n).$$

Theorem 1.7. Any sequence (μ_j) of Borel measures on $\mathbb{R}^n$ such that $\sup_j \mu_j(\mathbb{R}^n) < \infty$ has a weakly converging subsequence.

We shall not prove these results in the course but the proofs can be found in many measure theory and functional analysis books.

Here is a sketch of the proof of the more essential direction of Frostman's lemma. Assume $\mathcal{H}^s(A) > 0$. We may assume that A is compact. Then there is $c > 0$ such that

$$\sum_j d(E_j)^s \geq c \tag{1.4}$$

for all coverings $E_j, j = 1, 2, \dots,$ of A. We construct the measure μ as a weak limit of measures μ_k. To define μ_k look at the dyadic cubes of side-length 2^{-k} in a standard cubical partitioning of $\mathbb{R}^n$. First we define a measure $\mu_{k,1}$ which is a constant multiple of Lebesgue measure on each such cube Q. For Q such that $A \cap Q \neq \varnothing$, we normalize Lebesgue measure on Q so that $\mu_{k,1}(Q) = d(Q)^s$ and for the cubes Q such that $A \cap Q = \varnothing$ we let $\mu_{k,1}$ be the zero measure on Q. This measure would be fine for balls with diameter $< 2^{-k}$ but not necessarily for the bigger balls. Thus we modify it to a measure $\mu_{k,2}$ by investigating the dyadic cubes of side-length 2^{1-k}. On each such cube Q we let $\mu_{k,2}$ be $\mu_{k,1}$ if $\mu_{k,1}(Q) \leq d(Q)^s$, otherwise we make it smaller by normalizing $\mu_{k,1}$ on Q so that $\mu_{k,2}(Q) = d(Q)^s$. We continue this until we come to a single cube Q_0 which contains our compact set A (we may assume to begin with that the dyadic partioning is chosen so that A is inside some cube belonging to it). Let μ_k be the final measure obtained in this way. Then, since we never increased the measure along the process, $\mu_k(Q) \leq d(Q)^s$ for all dyadic cubes with side-length at least 2^{-k}. In fact, this holds for all dyadic cubes by the first step of the construction. This implies easily that $\mu_k(B) \lesssim_n d(B)^s$ for all balls B. The construction yields that every $x \in A$ is contained in some dyadic subcube Q of Q_0 with side-length at least 2^{-k} such that

$$\mu_k(Q) = d(Q)^s.$$

Choosing maximal, and hence disjoint, such cubes Q_j, they cover A and thus by (1.4),

$$\mu_k(\mathbb{R}^n) = \sum_j \mu_k(Q_j) = \sum_j d(Q_j)^s \geq c. \tag{1.5}$$

We can now take some weakly converging subsequence of (μ_k) and consider the limit measure μ. Then it is immediate from the construction that $\operatorname{spt}\mu \subset A$ (here we use that A is compact). It is also clear that $\mu(B) \lesssim_n d(B)^s$ for all balls B. The only danger is that μ might be the zero measure, but (1.5) shows that this cannot happen.

When studying measures in $\mathbb{R}^n$, or in more general metric spaces, a very useful tool is the following $5r$-covering lemma.

Its proof can be found in [Ma], Chapter 2. We denote by tB the ball $B(x, tr)$ when $B = B(x, r)$ and $t > 0$.

Theorem 1.8. [$5r$ covering theorem] Let $\mathcal{B}$ be a family of closed balls in $\mathbb{R}^n$ with

$$\sup\{d(B) : B \in \mathcal{B}\} < \infty.$$

Then there are disjoint balls $B_i \in \mathcal{B}$ (countably or finitely many) such that

$$\bigcup_{B \in \mathcal{B}} B \subset \bigcup_i 5B_i.$$

For instance this can be used to prove the equivalence of the definitions for Minkowski dimensions in terms of $N(A, \delta)$ and Lebesgue measures of $A(\delta)$, and for the equality for different dimensions for AD-regular sets.

2. Projections and dimension

Let us begin by considering Cantor sets in the plane. Set

$$C(d) = C_d \times C_d, \quad 0 < d < 1/2,$$

where $C_d \subset [0, 1]$ is the Cantor set of the previous chapter.

This is often called a four-corner Cantor set because of the geometric construction:

$$C(d) = \bigcap_{k=1}^{\infty} U_k^d, \quad U_k^d = \bigcup_{i=1}^{4^k} Q_{k,i}. \tag{2.1}$$

Here each $Q_{k,i}$ is a closed square of side-length d^k, and they are defined as follows. First the $Q_{1,i}$ are the four squares in the four corners of the unit square $[0,1]\times[0,1]$, that is, $[0,d]\times[0,d]$, $[0,d]\times[1-d,1]$, $[1-d,1]\times[0,d]$ and $[1-d,1]\times[1-d,1]$. If the squares $Q_{k,i}$, $i=1,\dots,4^k$, have been constructed, the $Q_{k+1,j}$ are obtained in the same way inside and in the corners of the $Q_{k,i}$.

Defining s_d by

$$4d^{s_d} = 1, \quad \text{i.e.,} \quad s_d = \frac{\log 4}{\log(\frac{1}{d})},$$

we have

$$0 < \mathcal{H}^{s_d}(C(d)) < \infty \quad \text{and} \quad \dim C(d) = s_d.$$

This is derived directly from (2.1), for example as in Chapter 1 for the linear Cantor sets C_d.

We shall consider how the projections

$$p_\theta(x,y) = x\cos\theta + y\sin\theta, \quad (x,y)\in\mathbb{R}^2, \theta\in[0,\pi),$$

affect the dimension of these and more general sets. Notice that p_θ is essentially the orthogonal projection onto line making angle θ with the x-axis. We notice immediately that when $\theta = 0$ or $\theta = \frac{\pi}{2}$, that is, when we project into the coordinate axis, we get the Cantor sets C_d whose dimension is $\frac{\log 2}{\log(\frac{1}{d})} = \frac{1}{2}s_d$. Looking more carefully at these projections with different angles θ we easily find a countable dense set of angles θ for which $p_\theta(C(d))$ is a Cantor set in $\mathbb{R}$ with dimension strictly less than s_d. This happens always when p_θ maps two different squares $Q_{k,i}$ exactly onto the same interval. However, this behaviour is exceptional due to Marstrand's general projection theorem which we shall soon prove.

The first observation is that projections cannot increase dimensions, simply because they do not increase distances. More generally,

Theorem 2.1. Let $f:\mathbb{R}^n\to\mathbb{R}^m$ be a *Lipschitz mapping*. This means that there is $L<\infty$ such that

$$|f(x)-f(y)| \le L|x-y| \quad \text{for all } x,y\in\mathbb{R}^n.$$

Then for any $A\subset\mathbb{R}^n$ and $s\ge 0$,

$$\mathcal{H}^s(f(A)) \le L^s\mathcal{H}^s(A) \text{ and } \dim f(A) \le \dim A. \tag{2.2}$$

In addition to sets we also need to project measures. More generally, the *image* or push-forward of a measure μ under a map $f:\mathbb{R}^n\to\mathbb{R}^m$ is defined by

$$f_\sharp\mu(B) = \mu(f^{-1}(B)) \quad \text{for } B\subset\mathbb{R}^m.$$

It is a Borel measure if μ is a Borel measure and f is a Borel function. The definition is equivalent to saying that

$$\int g\, df_\sharp\mu = \int g \circ f\, d\mu$$

for all non-negative Borel functions g on $\mathbb{R}^n$.

Finally, we need something from the differentation theory of measures. First, we say that a measure μ is *absolutely continuous* with respect to a measure ν if $\nu(A) = 0$ implies $\mu(A) = 0$. We denote this by $\mu \ll \nu$.

For $\mu \in \mathcal{M}(\mathbb{R}^n)$ define the *lower derivative* and *derivative* of μ at $x \in \mathbb{R}^n$ by

$$\underline{D}(\mu, x) = \liminf_{r\to 0} \frac{\mu(B(x,r))}{\mathcal{L}^n(B(x,r))}$$

and

$$D(\mu, x) = \lim_{r\to 0} \frac{\mu(B(x,r))}{\mathcal{L}^n(B(x,r))},$$

the latter if the limit exists. We shall make use of the following basic differentiation theorem of measures, for a proof, see, e.g., [Ma], Theorem 2.12:

Theorem 2.2. Let $\mu \in \mathcal{M}(\mathbb{R}^n)$. Then

(a) the derivative $D(\mu, x)$ exists and is finite for $\mathcal{L}^n$ almost all $x \in \mathbb{R}^n$,

(b) $\int_B D(\mu, x)\, dx \leq \mu(B)$ for all Borel sets $B \subset \mathbb{R}^n$ with equality if $\mu \ll \mathcal{L}^n$,

(c) $\mu \ll \mathcal{L}^n$ if and only if $\underline{D}(\mu, x) < \infty$ for μ almost all $x \in \mathbb{R}^n$.

Actually we shall only need part (c) and only the "if" part there. The book [BP] gives a simple proof for it in Section 3.5.

Part (b) is the Radon-Nikodym theorem in this case: when $\mu \ll \mathcal{L}^n$, we have $\int_B D(\mu, x)\, dx = \mu(B)$ for all Borel sets $B \subset \mathbb{R}^n$, and so μ can be identified with the function $D(\mu, \cdot)$.

Now we come to the projection theorem which John Marstrand proved in 1954, I state it for Borel sets, but stating only for closed sets would be about the same.

Theorem 2.3. Let $A \subset \mathbb{R}^2$ be a Borel set. If $\dim A \leq 1$, then

$$\dim p_\theta(A) = \dim A \quad \text{for almost all } \theta \in [0, \pi). \tag{2.3}$$

If $\dim A > 1$, then

$$\mathcal{L}^1(p_\theta(A)) > 0 \quad \text{for almost all } \theta \in [0, \pi). \tag{2.4}$$

Proof. To prove (2.3) let $0 < s < \dim A \leq 1$ and choose by Theorem 1.4 a measure $\mu \in \mathcal{M}(A)$ such that $I_s(\mu) < \infty$. Let $\mu_\theta \in \mathcal{M}(p_\theta(A))$ be the image of μ under p_θ: $\mu_\theta(B) = \mu(p_\theta^{-1}(B))$. Then

$$\begin{aligned}\int_0^\pi I_s(\mu_\theta)\, d\theta &= \int_0^\pi \iint |p_\theta(x-y)|^{-s}\, d\mu x\, d\mu y\, d\theta \\ &= \iint\int_0^\pi |p_\theta(\tfrac{x-y}{|x-y|})|^{-s}\, d\theta |x-y|^{-s}\, d\mu x\, d\mu y = c(s) I_s(\mu) < \infty,\end{aligned}$$

where for $v \in S^1$, $c(s) = \int_0^\pi |p_\theta v|^{-s}\, d\theta < \infty$ as $s < 1$. Referring again to Theorem 1.4 we see that $\dim p_\theta(A) \geq s$ for almost all $\theta \in [0, \pi)$. By the arbitrariness of $s, 0 < s < \dim A$, we obtain $\dim p_\theta(A) \geq \dim A$ for almost all $\theta \in [0, \pi)$. The opposite inequality follows from the fact that the projections are Lipschitz mappings.

To prove (2.4) let $\dim A > 1$ and choose by Theorem 1.4 a measure $\mu \in \mathcal{M}(A)$ such that $I_1(\mu) < \infty$. Let $\mu_\theta \in \mathcal{M}(p_\theta(A))$ be as above. Then by Fatou's lemma and Fubini's theorem

$$\begin{aligned}
&\int_0^\pi \int \underline{D}(\mu_\theta, u)\, d\mu_\theta u\, d\theta \\
&\leq \liminf_{r\to 0} \int_0^\pi \int \frac{\mu_\theta(B(u,r))}{2r}\, d\mu_\theta u\, d\theta \\
&= \liminf_{r\to 0} \frac{1}{2r} \int_0^\pi \int \mu(\{y : |p_\theta(x-y)| \leq r\})\, d\mu x\, d\theta \\
&= \liminf_{r\to 0} \frac{1}{2r} \iint \mathcal{L}^1(\{\theta : |p_\theta(x-y)| \leq r\})\, d\mu y\, d\mu x \\
&\leq I_1(\mu).
\end{aligned}$$

Here used here the elementary geometric (or analytic) fact $\mathcal{L}^1(\{\theta : |p_\theta(v)| \leq r\}) \leq 2r$ when $|v| = 1$, which implies $\mathcal{L}^1(\{\theta : |p_\theta(x-y)| \leq 2r\}) \leq 2/|x-y|$ for all $x, y \in \mathbb{R}^2$. It follows that for almost all $\theta \in [0, \pi)$, $\underline{D}(\mu_\theta, u) < \infty$ for μ_θ almost all u. For such θ, $\mu \ll \mathcal{L}^1$ by Theorem 2.2. (2.4) follows from this. □

We return to to the Cantor sets $C(d)$. We now know the dimension of $p_\theta(C(d))$ for almost all $\theta \in [0, \pi)$. In the case $d = 1/4$, $\dim p_\theta(C(1/4)) = 1$ for almost all $\theta \in [0, \pi)$. But what can we say about the Lebesgue one-dimensional measures of these one-dimensional (in the sense of Hausdorff) subsets of $\mathbb{R}$? Here is the answer:

Theorem 2.4.

$$\mathcal{L}^1(p_\theta(C(1/4))) = 0 \quad \text{for almost all } \theta \in [0, \pi).$$

The following elementary proof is due to Peres, Simon and Solomyak. It is also given in the book [BP] of Bishop and Peres.

R. Kenyon has proven a sharper result, which in particular implies that there are only countably many directions θ for which $\mathcal{L}^1(p_\theta(C(d))) > 0$ and that the set of such directions is countably infinite and dense.

Set now $C = C(1/4)$. We can write

$$C = \bigcup_{i=1}^4 \left(\frac{1}{4}C + c_i\right)$$

where $c_1 = (0,0)$, $c_2 = \left(\frac{3}{4}, 0\right)$, $c_3 = \left(0, \frac{3}{4}\right)$, $c_4 = \left(\frac{3}{4}, \frac{3}{4}\right)$. Hence, writing again $\boldsymbol{\theta} = (\cos\theta, \sin\theta)$,

$$p_\theta(C) = \bigcup_{i=1}^4 \left(\frac{1}{4}p_\theta(C) + \boldsymbol{\theta} \cdot c_i\right) \subset \mathbb{R}.$$

Let us first look more generally at this type of self-similar subsets of $\mathbb{R}$. Let $K \subset \mathbb{R}$ be compact such that for some integer $m \geq 2$ and some $d_1, \dots, d_m \in \mathbb{R}$ ($d_i \neq d_j$ for $i \neq j$),

$$K = \bigcup_{i=1}^{m} K_i \quad \text{with} \quad K_i = \frac{1}{m}K + d_i.$$

Lemma 2.5. (1) $\mathcal{L}^1(K_i \cap K_j) = 0$ for $i \neq j$.
(2) $K_i \cap K_j \neq \varnothing$ for some $i \neq j$.

Proof. (1) follows easily from

$$\mathcal{L}^1(K) \leq \sum_{i=1}^{m} \mathcal{L}^1(K_i) = \sum_{i=1}^{m} \frac{1}{m}\mathcal{L}^1(K) = \mathcal{L}^1(K).$$

If $K_i \cap K_j = \varnothing$ for all $i \neq j$, then for some $\varepsilon > 0$ the open ε-neighbourhoods $K_i(\varepsilon)$ of the K_i are also disjoint. The ε-neighbourhood of $K_i = \frac{1}{m}K + d_i$ is $(\frac{1}{m}K)(\varepsilon) + d_i = \frac{1}{m}K(m\varepsilon) + d_i$, whence

$$\mathcal{L}^1(K_i(\varepsilon)) = \mathcal{L}^1(\frac{1}{m}K(m\varepsilon)) = \frac{1}{m}\mathcal{L}^1(K(m\varepsilon)).$$

It follows that

$$\mathcal{L}^1(K(\varepsilon)) = \sum_{i=1}^{m} \mathcal{L}^1(K_i(\varepsilon)) = \sum_{i=1}^{m} \frac{1}{m}\mathcal{L}^1(K(m\varepsilon)) = \mathcal{L}^1(K(m\varepsilon)).$$

This is a contradiction, since $K(\varepsilon)$ is a strict subset of $K(m\varepsilon)$ and both are bounded open sets. □

Since

$$K_i = \frac{1}{m}K + d_i = \frac{1}{m}\left(\bigcup_{j=1}^{m}\left(\frac{1}{m}K + d_j\right)\right) + d_i = \bigcup_{j=1}^{m} K_{i,j},$$

where $K_{ij} = \frac{1}{m^2}K + \frac{1}{m}d_j + d_i$, we can write K also as the union of the m^2 sets K_{ij}. Set

$$\mathcal{I} = \{1, \dots, m\},$$

$$\mathcal{I}^k = \{u : u = (i_1, \dots, i_k),\, i_j \in I\}, \quad k = 1, 2, \dots .$$

Then for each k,

$$K = \bigcup_{u \in \mathcal{I}^k} K_u, \quad \text{where} \quad K_u = m^{-k}K + d_u.$$

The the translation numbers K_u were defined above for $k = 1, 2$, and the general case should be clear from this.

The following notion is due to C. Bandt and S. Graf.

Definition 2.6. Let $\varepsilon > 0$. We say that K_u and K_v are ε-relatively close if $u, v \in \mathcal{I}^k$ for some k, $u \neq v$, and

$$|d_u - d_v| \leq \varepsilon d(K_u) = \varepsilon d(K)m^{-k}.$$

Observe that this means that

$$K_v = K_u + x$$

with $x = d_v - d_u$ and $|x| \le \varepsilon d(K_u)$.

Lemma 2.7. If for every $\varepsilon > 0$ there are k and $u, v \in \mathcal{I}^k$ with $u \neq v$ such that K_u and K_v are ε-relatively close, then $\mathcal{L}^1(K) = 0$.

Proof. To prove this suppose $\mathcal{L}^1(K) > 0$ and let $1/2 < t < 1$. Then there is some interval I such that $\mathcal{L}^1(K \cap I) > t\mathcal{L}^1(I)$. Pick small $\varepsilon > 0$ and K_u and K_v, $u, v \in I^k$, $u \neq v$, which are ε-relatively close. By an iteration of Lemma 2.5(1) $\mathcal{L}^1(K_u \cap K_v) = 0$. Setting $I_u = m^{-k}I + d_u$ and $I_v = m^{-k}I + d_v$, $\mathcal{L}^1(K_u \cap I_u) > t\mathcal{L}^1(I_u)$, $\mathcal{L}^1(K_v \cap I_v) > t\mathcal{L}^1(I_v)$ and $\mathcal{L}^1(I_v \setminus I_u) \le \varepsilon d(K)m^{-k}$. It follows that

$$\begin{aligned}
2tm^{-k}\mathcal{L}^1(I) &= t\mathcal{L}^1(I_u) + t\mathcal{L}^1(I_v) \\
&\le \mathcal{L}^1(K_u \cap I_u) + \mathcal{L}^1(K_v \cap I_v) = \mathcal{L}^1((K_u \cap I_u) \cup (K_v \cap I_v)) \\
&\le \mathcal{L}^1(I_u) + \mathcal{L}^1(I_v \setminus I_u) \le (\mathcal{L}^1(I) + \varepsilon d(K))m^{-k}.
\end{aligned}$$

This is a contradiction if ε is sufficiently small. □

Proof of Theorem 2.4. We now return to the proof that $\mathcal{L}^1(p_\theta(C)) = 0$ for almost all θ. Let $p_\theta(C) = C^\theta$ to fit more conveniently with the notation C_u^θ above. For $\varepsilon > 0$ let

$$\begin{aligned}
V_\varepsilon = \{\theta \in [0, \pi) :\exists\, k,\ u,\ v \text{ such that } u, v \in I^k,\ u \neq v \\
\text{and } C_u^\theta \text{ and } C_v^\theta \text{ are } \varepsilon\text{-relatively close}\}.
\end{aligned}$$

It follows from Lemma 2.7 that it suffices to show that for every $\varepsilon > 0$,

$$\mathcal{L}^1([0, \pi)\setminus V_\varepsilon) = 0.$$

Then also $\mathcal{L}^1([0, \pi)\setminus \bigcap_{\varepsilon>0} V_\varepsilon) = \mathcal{L}^1([0, \pi)\setminus \bigcap_{j=1}^{\infty} V_{\frac{1}{j}}) = 0$. So let $\varepsilon > 0$ and $\theta \in [0, \pi)$. By Lemma 2.5(2), $C_i^\theta \cap C_j^\theta \neq \varnothing$ for some $i \neq j$. This means that there are $x \in C_i$ and $y \in C_j$ such that $p_\theta x = p_\theta y$. Let $k > 1$ be an integer. Then $x \in C_u$ and $y \in C_v$ for some $u, v \in I^k$ with $u \neq v$. Let $\theta_0 \in [0, \pi)$ be such that $p_{\theta_0}(C_u) = p_{\theta_0}(C_v)$ (that is, p_{θ_0} maps the squares of side-length 4^{-k} which contain C_u and C_v onto the same interval). Then $|\theta - \theta_0| < c4^{-k}$ with some constant $c > 1$. Moreover, $C_u^{\theta_0}$ and $C_v^{\theta_0}$ are '0-relatively close', and a simple geometric inspection shows that C_u^φ and C_v^φ are ε-relatively close when $|\varphi - \theta_0| < b\varepsilon 4^{-k}$, where $b < 1$ is a constant. Hence $[\theta - 2c4^{-k}, \theta + 2c4^{-k}] \cap V_\varepsilon$ contains an interval of length $b\varepsilon 4^{-k}$. Since this is true for every k, it follows that $\mathcal{L}^1([0, \pi)\setminus V_\varepsilon) = 0$ as required. □

Since $\mathcal{L}^1(p_\theta(C(1/4))) = 0$ for almost all $\theta \in (0, \pi)$, the integrals, the average length of projections,

$$I_k := \int_0^\pi \mathcal{L}^1(p_\theta(U_k^{1/4}))\, d\theta$$

tend to 0 when k tends to ∞; recall the definition of $U^{\frac{1}{4}}$ from (2.1). But how fast do they converge? To get a lower bound, we first prove a general result:

Theorem 2.8. Let $A \subset \mathbb{R}^2$ be Lebesgue measurable and let $\mu \in \mathcal{M}(A)$ with $\mu(A) = 1$ and $I_1(\mu) < \infty$. Then

$$\int_0^\pi \mathcal{L}^1(\pi_\theta(A))\, d\theta \geq \frac{\pi^2}{I_1(\mu)}.$$

Proof. The measurability of the function $\theta \mapsto \mathcal{L}^1(P_\theta(A))$ is easily checked for compact sets A and from that it follows for measurable sets by approximation. The argument in the proof of (2.4) shows that for almost all $\theta \in [0, \pi)$ the measure μ_θ is absolutely continuous and the derivative $D(\mu_\theta, \cdot)$ is in L^2. This means that we can consider μ_θ as an L^2 function, identifying it with $D(\mu_\theta, \cdot)$. Moreover, that proof gives

$$\int_0^\pi \int \mu_\theta(u)^2\, du\, d\theta \leq I_1(\mu).$$

By Schwartz's inequality,

$$1 = \mu_\theta(\mathbb{R})^2 = \left(\int_{\pi_\theta(A)} \mu_\theta(u)\, du\right)^2 \leq \mathcal{L}^1(\pi_\theta(A)) \int \mu_\theta(u)^2\, du.$$

A combination of these two inequalities gives

$$\int_0^\pi \mathcal{L}^1(\pi_\theta(A))^{-1}\, d\theta \leq \int_0^\pi \int \mu_\theta(u)^2\, du\, d\theta \leq I_1(\mu).$$

Thus by Schwartz's inequality,

$$\int_0^\pi \mathcal{L}^1(\pi_\theta(A))\, d\theta \geq \left(\int_0^\pi \mathcal{L}^1(\pi_\theta(A))^{-1}\, d\theta\right)^{-1} \pi^2 \geq \pi^2 I_1(\mu)^{-1}.$$

$\square$

Theorem 2.8 gives easily the lower bound

$$\int_0^\pi \mathcal{L}^1(p_\theta(U_k^{1/4}))\, d\theta \gtrsim k^{-1}. \tag{2.5}$$

To prove this it is enough to check that $I_1(\mu_k) \lesssim k$ when μ_k is the normalized Lebesgue measure on $U_k^{1/4}$ and then apply Theorem 2.8. M. Bateman and A. Volberg have improved this to

$$\int_0^\pi \mathcal{L}^1(p_\theta(U_k^{1/4}))\, d\theta \gtrsim (\log k) k^{-1}. \tag{2.6}$$

Getting good upper bounds has turned out to be a very difficult problem. F. Nazarov, Y. Peres and A. Volberg proved in 2010 with delicate Fourier analytic and combinatorial arguments that for every $\delta > 0$,

$$\int_0^\pi \mathcal{L}^1(p_\theta(U_k^{1/4}))\, d\theta \lesssim_\delta k^{\delta - 1/6}. \tag{2.7}$$

3. BESICOVITCH SETS

We say that a Borel set in $\mathbb{R}^n, n \geq 2$, is a *Besicovitch set*, or a Kakeya set, if it has zero Lebesgue measure and it contains a line segment of unit length in every direction. This means that for every $e \in \mathcal{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ there is $b \in \mathbb{R}^n$ such that $\{te + b : 0 < t < 1\} \subset B$. It is not clear that Besicovitch sets exist but they do in every $\mathbb{R}^n, n \geq 2$, as we shall now prove. We shall also show that in the plane their Hausdorff dimension is 2.

We show that Besicovitch sets exist using duality between points and lines.

Theorem 3.1. For any $n \geq 2$ there exists a Borel set $B \subset \mathbb{R}^n$ such that $\mathcal{L}^n(B) = 0$ and B contains a whole line in every direction. Moreover, there exist compact Besicovitch sets in $\mathbb{R}^n$.

Proof. It is enough to find B in the plane since then we can take $B \times \mathbb{R}^{n-2}$ in higher dimensions.

Let $C \subset \mathbb{R}^2$ be a compact set such that $\pi(C) = [0,1]$, where $\pi(x,y) = x$ for $(x,y) \in \mathbb{R}^2$, and $\mathcal{L}^1(p_\theta(C)) = 0$ for $\mathcal{L}^1$ almost all $\theta \in [0,\pi)$. Here p_θ is again the projection onto the line through the origin forming angle θ with the x-axis. We can take as C a suitably rotated and dilated copy of $C(1/4)$ or we can modify the construction of $C(1/4)$ by placing the first four disjoint closed squares of side-length $\frac{1}{4}$ inside $[0,1] \times [0,1]$ so that their projections cover $[0,1]$. Consider the lines

$$\ell(a,b) = \{(x,y) : y = ax + b\}, \quad (a,b) \in C,$$

and define B as their union:

$$B = \bigcup_{(a,b) \in C} \ell(a,b) = \{(x, ax+b) : x \in \mathbb{R},\ (a,b) \in C\}.$$

From the latter representation it is easy to see that B is σ-compact and hence a Borel set. If we restrict x to $[0,1]$, B will be compact, which will give us compact Besicovitch sets. Since $\pi(C) = [0,1]$, B contains a line $\ell(a,b)$ for some b for all $0 \leq a \leq 1$. Taking a union of four rotated copies of B we get a Borel set that contains a line in every direction. It remains to show that $\mathcal{L}^2(B) = 0$.

We do this by showing that almost every vertical line meets B in a set of length zero and then using Fubini's theorem. For any $t \in \mathbb{R}$,

$$\begin{aligned} B \cap \{(x,y) : x = t\} &= \{(t, at+b) : (a,b) \in C\} \\ &= \{t\} \times \pi_t(C), \end{aligned} \tag{3.1}$$

where $\pi_t(x,y) = tx + y$. The map π_t is essentially a projection p_θ for some θ, and hence we have $\mathcal{L}^1(\pi_t(C)) = 0$ for $\mathcal{L}^1$ almost all $t \in \mathbb{R}$. Thus $\mathcal{L}^2(B) = 0$. ☐

Reversing the above argument, we now use Marstrand's projection theorem to prove that Besicovitch sets must have Hausdorff dimension 2 in the plane.

Theorem 3.2. For every Besicovitch set $B \subset \mathbb{R}^2$, $\dim B = 2$.

Proof. If B is a Besicovitch set, then B is contained in a G_δ (countable intersection of open sets) set B' which contains a unit line segment in every direction and for which $\dim B' = \dim B$. Thus we can assume that B is a G_δ Besicovitch set in the plane. For $a \in (0,1), b \in \mathbb{R}$ and $q \in \mathbb{Q}$ denote by $I(a,b,q)$ the line segment $\{(q+t, at+b) : 0 \leq t \leq 1/2\}$ of length less than 1. Let C_q be the set of (a,b) such that $I(a,b,q) \subset B$ and $C = \cup_{q\in\mathbb{Q}} I(a,b,q)$. Then each C_q is a G_δ-set, because for any open set G, the set of (a,b) such that $I(a,b,q) \subset G$ is open. Since for every $a \in (0,1)$, some $I(a,b,q) \subset B$, we have $\pi(\cup_{q\in\mathbb{Q}} C_q) = (0,1)$, with $\pi(x,y) = x$, and so there is $q \in \mathbb{Q}$ for which $\mathcal{H}^1(C_q) > 0$. Then by Theorem 2.3, for almost all $t \in \mathbb{R}$, $\dim \pi_t(C_q) = 1$, where again $\pi_t(x,y) = tx + y$. We have now for $0 \leq t \leq 1/2$,

$$\{q+t\} \times \pi_t(C_q) = \{(q+t, at+b) : (a,b) \in C_q\} \subset B \cap \{(x,y) : x = q+t\}.$$

Hence for a positive measure set of t, vertical t-sections of B have dimension 1. By a relatively easy property of Hausdorff measures this implies $\dim B = 2$. $\square$

It is conjectured that in $\mathbb{R}^n, n \geq 2$, the Besicovitch sets have Hausdorff dimension. This conjecture, called Kakeya conjecture, is open when $n \geq 3$. It is related to many central questions of modern Fourier analysis.

4. Self-similar sets

This topic is discussed in [BP] and [F1].

Let $0 < d < 1/2$ and consider the functions

$$f_1 : \mathbb{R} \to \mathbb{R}, f_1(x) = dx, \quad f_2 : \mathbb{R} \to \mathbb{R}, f_2(x) = dx + 1 - d.$$

Then for the Cantor set C_d of Chapter 1,

$$C_d = f_1(C_d) \cup f_2(C_d).$$

In Chapter 2 we observed that the Cantor set $C(d)$ in the plane and its projections satisfy similar equations with four mappings. These are examples of self-similar sets which we now define and study more generally.

Let (X, d) be a metric space. A mapping $f : X \to X$ is called a *contraction* if there is $0 < r < 1$ such that

$$d(f(x), f(y)) \leq rd(x, y) \quad \text{for all } x, y \in X.$$

Theorem 4.1. [Banach's fixed point theorem] If (X, d) is complete and $f : X \to X$ is a contraction, then f has a unique fixed point x_0:

$$f(x_0) = x_0.$$

The proof of this standard result can be found in [BP] and many other books.

Definition 4.2. Let $f_j : X \to X, j = 1, \ldots, N, N \geq 2$, be contractions. A non-empty compact subset K of X is called an *attractor* of $f_j, j = 1, \ldots, N$, if

$$K = \bigcup_{j=1}^{N} f_j(K).$$

The Cantor sets mentioned above are examples of such attractors. Other examples can be found for instance in [BP] and [F1]. For example, the Sierpinski gasket, Sierpinski carpet (Figure 1.3.4 in [BP]) and the top third of the von Koch snowflake (Figure 1.2.2 in [BP]) can be represented in this form.

In a complete metric space any finite sequence of contractions leads to a unique attractor due to a theorem of Hutchinson from 1981:

Theorem 4.3. Let (X, d) be a complete metric space and let $f_j : X \to X, j = 1, \ldots, N, N \geq 2$, be contractions.

(1) There exists a unique non-empty compact subset K of X such that

$$K = \bigcup_{j=1}^{N} f_j(K).$$

(2) If $p_j \in [0, 1]$ with $\sum_{j=1}^{N} p_j = 1$, then there exists a unique Borel probability measure $\mu \in \mathcal{M}(K)$ such that

$$\mu = \sum_{j=1}^{N} p_j f_{j\#}\mu.$$

If $p_j > 0$ for all j, $\operatorname{spt}\mu = K$.
μ is called the *invariant measure* of the system $f_1, \dots, f_N, p_1, \dots, p_N$.

Both statements can be proven with the help of Banach's fixed point theorem, but to do that we need metrics on the spaces of compact subsets of X and probability measures on X. For the first this is the following:

Definition 4.4. Let (X, d) be a complete metric space and let $\mathcal{K}(X)$ be the set of all non-empty compact subsets of X. Set

$$d_H(K_1, K_2) = \inf\{\varepsilon : K_1 \subset K_2(\varepsilon) \text{ and } K_2 \subset K_1(\varepsilon)\},$$

where $K_j(\varepsilon) = \{x \in X : d(x, K_j) < \varepsilon\}$. Then d_H is called the *Hausdorff metric* on $\mathcal{K}(X)$.

The completenes part of the following is called Blaschke's selection theorem.

Theorem 4.5. $(\mathcal{K}(X), d_H)$ is a complete metric space.

For the second part of Theorem 4.3 we consider only measures on the compact set K and we define the metric in this case:

Definition 4.6. Let (K, d) be a compact metric space and let $\mathcal{P}(K)$ be the set of all Borel probability measures on K. Set

$$d_L(\mu, \nu) = \sup\{|\int g\,d\mu - \int g\,d\nu| : Lip(g) \leq 1\}, \mu, \nu \in \mathcal{P}(K),$$

where $Lip(g)$ is the Lipschitz constant of $g : K \to \mathbb{R}$, that is, the smallest L such that $d(g(x), g(y)) \leq Ld(x, y)$ for all $x, y \in X$.. Then d_L is called the *dual Lipschitz metric* on $\mathcal{P}(K)$.

Theorem 4.7. $(\mathcal{P}(K), d_L)$ is a compact metric space.

The proofs of Theorems 4.5 and 4.7 can be found, for example, in [BP].

Theorem 4.3 is now proved combining Banach's fixed point theorem with these two theorems. For that we need to use suitable contractions on $(\mathcal{K}(X), d_H)$ and $(\mathcal{P}(K), d_L)$. But these are obvious:

$$F : \mathcal{K}(X) \to \mathcal{K}(X), F(K) = \bigcup_{j=1}^{N} f_j(K),$$

$$M : \mathcal{P}(K) \to \mathcal{P}(K), M(\mu) = \sum_{j=1}^{N} p_j f_{j\#}\mu.$$

Then one shows that these mappings really are contractions, which completes the proof of Theorem 4.3.

Now we go to $\mathbb{R}^n$. A contraction $f : \mathbb{R}^n \to \mathbb{R}^n$ is a *similitude* (or a contractive similarity) if there is $0 < r < 1$ such that

$$|f(x) - f(y)| = r|x - y| \quad \text{for all } x, y \in \mathbb{R}^n.$$

It can be shown that this means that f is of the form

$$f(x) = rg(x) + a, x \in \mathbb{R}^n, \quad \text{for some } g \in O(n), a \in \mathbb{R}^n.$$

In particular, f is an affine mapping (linear + translation).

The number r is called the *contraction ratio* of f, we denote it by $r(f)$.

Definition 4.8. A non-empty compact subset K of $\mathbb{R}^n$ is called *self-similar*, if it is the (unique) attractor of some system $f_1, \dots, f_N, N \geq 2$, of similitudes on $\mathbb{R}^n$: $K = \bigcup_{j=1}^N f_j(K)$.

Notice that K does not determine uniquely the similitudes $f_1, \dots, f_N$.

What can we say about the Hausdorff dimension and measures of self-similar sets? What helped us for the earlier Cantor sets was that the different parts $f_j(K)$ were disjoint. Suppose $K = \bigcup_{j=1}^N f_j(K)$ is a self-similar set with $f_j(K)$ pairwise disjoint and $0 < \mathcal{H}^s(K) < \infty$. Then

$$0 < \mathcal{H}^s(K) = \sum_{j=1}^N \mathcal{H}^s(f_j(K)) = \sum_{j=1}^N r(f_j)^s \mathcal{H}^s(K) < \infty.$$

Hence $\sum_{j=1}^N r(f_j)^s = 1$. Conversely, given any similitudes $f_1, \dots, f_N, N \geq 2$,, there is a unique number $s, 0 < s < 1$, such that $\sum_{j=1}^N r(f_j)^s = 1$. This number is called the *similarity dimension* of the system $f_1, \dots, f_N$. Now one could hope that this always would give the Hausdorff dimension of the self-similar attractor of this system, but this is not true. However, it is true if the different parts $f_j(K)$ are disjoint, and more generally under the following open set condition. This condition is very useful. For example, it applies to the von Koch curve and the Sierpinski gasket, although the $f_j(K)$ are not disjoint.

Definition 4.9. We say that the similitudes $f_1, \dots, f_N, N \geq 2$, on $\mathbb{R}^n$ satisfy the *open set condition*, or OSC, if there is a non-empty bounded open set $O \subset \mathbb{R}^n$ such that

$$f_j(O) \subset O \text{ for all } j = 1, \dots, N, \text{ and } f_j(O) \cap f_k(O) = \varnothing \text{ for all } j \neq k.$$

The following theorem was proved by Moran in the 1940s, and in this formulation by Hutchinson in 1981:

Theorem 4.10. Suppose that the similitudes $f_1, \dots, f_N, N \geq 2$, satisfy the *open set condition*, let s be the similarity dimension of this system and let K be the corresponding self-similar set. Then $0 < \mathcal{H}^s(K) < \infty$.

The proof of this can be found in [BP] and [F1].

5. Dimension of some sets of real numbers

These lectures follow parts of Sections 1.3-5 of [BP].

The standard Cantor set $C_{1/3}$ can be written as

$$C_{1/3} = \{\sum_{j=1}^{\infty} x_j 3^{-j} : x_j = 0 \text{ or } x_j = 2\}.$$

We begin by studying sets of this type. Let $b \geq 2$ be an integer. For any $x \in [0, 1]$ we have the b-adic expansion

$$x = \sum_{j=1}^{\infty} x_j b^{-j},$$

where $x_j \in \{0, 1, \dots, b-1\}$. If we don't allow x_j to have all those possible values, we can have some interesting fractal sets. For example, choosing a subset S of $\{0, 1, \dots, b-1\}$ of $m < b$ elements, the set of all x as above with $x_j \in S$ form a self-similar Cantor set of dimension $\log m / \log b$.

Here is a bit more complicated example: let $S \subset \mathbb{N} = \{1, 2, \dots\}$ and define

$$A_S = \{\sum_{j=1}^{\infty} x_j 2^{-j} : x_j = 0 \text{ or } x_j = 1, \text{ if } j \in S, \text{ and } x_j = 0, \text{ if } j \notin S\}. \tag{5.1}$$

We shall soon find Minkowski and Hausdorff dimensions of A_S. Minkowski is rather easy. For any $S \subset \mathbb{N}$, define the *upper density* of S as

$$\overline{d}(S) = \limsup_{N \to \infty} \frac{\#(S \cap \{1, \dots, N\})}{N},$$

and the lower density

$$\underline{d}(S) = \limsup_{N \to \infty} \frac{\#(S \cap \{1, \dots, N\})}{N}.$$

Then

$$\overline{\dim}_M A_S = \overline{d}(S),$$

and

$$\underline{\dim}_M A_S = \underline{d}(S).$$

We shall show that $\underline{d}(S)$ also equals the Hausdorff dimension of A_S. For this and other related sets b-adic Hausdorff measures $\widetilde{\mathcal{H}}^s$ are useful. They are defined just as the ordinary Hausdorff measures but instead of covering with arbitrary sets we cover with b-adic intervals. By a b-adic interval we mean any half-open interval of the form

$$[\frac{k-1}{b^m}, \frac{k}{b^m}), \quad k \in \mathbb{Z}, m \in \mathbb{N}.$$

Thus

$$\widetilde{\mathcal{H}}^s(A) = \lim_{\delta \to 0} \widetilde{\mathcal{H}}^s_\delta(A),$$

where, for $0 < \delta \leq \infty$,

$$\widetilde{\mathcal{H}}^s_\delta(A) = \inf\{\sum_j d(I_j)^s : A \subset \bigcup_j I_j, d(I_j) < \delta, I_j \text{ a b-adic interval}\}.$$

Then

$$\mathcal{H}^s(A) \leq \widetilde{\mathcal{H}}^s(A) \leq 2b\mathcal{H}^s(A).$$

The first inequality is obvious, the second follows since every interval of length $d \leq 1/b$ can be covered with $2b$ b-adic intervals of length at most d.

We also need the following *Billingsley's* lemma. In it $I_k(x)$ stands for the unique b-adic interval of length b^{-k} containing x.

Lemma 5.1. Let $A \subset [0,1]$ be a Borel set and let $\mu \in \mathcal{M}([0,1])$ with $\mu(A) > 0$. If

$$\alpha \leq \liminf_{k\to\infty} \frac{\log \mu(I_k(x))}{\log d(I_k(x))} \leq \beta \quad \text{for all } x \in A,$$

then $\alpha \leq \dim A \leq \beta$.

With these tools it is rather easy to show for the set A_S of (5.1) that

$$\dim A_S = \underline{d}(S).$$

Next we study sets defined by their digit frequencies. For $0 < p < 1$, set

$$A_p = \{\sum_{j=1}^{\infty} x_j 2^{-j} : x_j = 0 \text{ or } x_j = 1 \text{ and } \lim_{k\to\infty} \frac{1}{k}\sum_{j=1}^{k} x_j = p\}.$$

Then

$$\dim A_p = h_2(p) := (-p\log p - (1-p)\log(1-p))/\log 2. \tag{5.2}$$

In addition to Billigsley's lemma we need for this a probabilistic result:

Theorem 5.2 (Strong law of large numbers)**.** Let (X, ν) be a probability space (a measure space with $\nu(X) = 1$). Let (f_j) be an orthogonal sequence in $L^2(X, \nu)$:

$$\int f_i f_j \, d\nu = 0 \quad \text{for all } i \neq j.$$

If $\int |f_j|^2 \, d\nu \leq 1$ for all $j \in \mathbb{N}$, then

$$\lim_{k\to\infty} \frac{1}{k}\sum_{j=1}^{k} f_j(x) = 0 \quad \text{for } \nu \text{ almost all } x \in X.$$

In order to use these tools to verify (5.2) we still need a measure. It can be defined like the Cantor measure μ_d. Let μ_1 give the measure $1-p$ for $[0,1/2)$ and p for $[1/2,1)$. Next let μ_2 give the measure $(1-p)^2$ for $[0,1/4)$, $p(1-p)$ for $[1/4,1/2)$, $(1-p)p$ for $[1/2,3/4)$ and p^2 for $[3/4,1)$, and so on. The weak limit μ_p of this sequence is the desired measure. It satisfies for $k = 1,\dots,2^m, m \in \mathbb{N}$,

$$\mu_p\left([\frac{k-1}{2^m}, \frac{k}{2^m})\right) = p^{n(k)}(1-p)^{m-n(k)},$$

where $n(k)$ is the number of 1s in the binary expression of k.

6. Graphs of continuous functions

This chapter is based on [BP], Chapter 5, see also [F1], Section 8.2, and [F2], Chapter 11.

The *graph* of a function $f : X \to Y$ is

$$G_f = \{(x, y) : x \in X, y = f(x)\}.$$

If $f : [a, b] \to \mathbb{R}$ is smooth, or just Lipschitz, then $\mathcal{H}^1(G_f) < \infty$ and $\dim G_f = 1$. What about more irregular functions? We study mainly functions on intervals $I \subset \mathbb{R}$, although many results could easily be generalized to higher dimensions.

Definition 6.1. A mapping $f : A \to \mathbb{R}^m, A \subset \mathbb{R}^n$, is *Hölder* of order $\alpha, 0 < \alpha \leq 1$, if there is $C < \infty$ such that

$$|f(x) - f(y)| \leq C|x - y|^\alpha \quad \text{for all } x, y \in A.$$

Lemma 6.2. If $f : A \to \mathbb{R}^m, A \subset \mathbb{R}^n$, is Hölder of order α with C as in Definition 6.1, then for all $s > 0$,

$$\mathcal{H}^{s/\alpha}(f(A)) \leq C^{s/\alpha}\mathcal{H}^s(A) \quad \text{and } \dim f(A) \leq \dim A/\alpha.$$

Lemma 6.3. If $f : [a, b] \to \mathbb{R}$ a is Hölder of order α, then

$$\overline{\dim}_M G_f \leq 2 - \alpha.$$

Definition 6.4. A function $f : I \to \mathbb{R}, I \subset \mathbb{R}$ an interval, satisfies the *reverse Hölder condition* of order $\alpha, 0 \leq \alpha \leq 1$, if there is $c > 0$ such that for any subinterval $J \subset I$ there exist $x, y \in J$, such that

$$|f(x) - f(y)| \geq cd(J)^\alpha.$$

Lemma 6.5. If $f : [a, b] \to \mathbb{R}$ satisfies the reverse Hölder condition of order α, then

$$\underline{\dim}_M G_f \geq 2 - \alpha.$$

Our main object of study will be the Weierstrass function:

Definition 6.6. The *Weierstrass function* $f_{\alpha,b}$ with parameters $b \in \mathbb{N}, b \geq 2$, and $\alpha > 0$ is defined by

$$f_{\alpha,b}(x) = \sum_{n=1}^{\infty} b^{-n\alpha} \cos(b^n x), \quad x \in [-\pi, \pi].$$

We shall discuss the proof of the following theorem:

Theorem 6.7. (i) If $0 < \alpha < 1$, $f_{\alpha,b}$ is *Hölder* of order α.
(ii) If $0 < \alpha \leq 1$, $f_{\alpha,b}$ is nowhere differentiable.
(iii) If $0 < \alpha < 1$, $f_{\alpha,b}$ satisfies the reverse Hölder condition of order α.
(iv) If $0 < \alpha < 1$, $\underline{\dim}_M G_{f_{\alpha,b}} = \overline{\dim}_M G_{f_{\alpha,b}} = 2 - \alpha$.
(v) If $0 < \alpha < 1$, $\dim G_{f_{\alpha,b}} > 1$.

It is conjectured, but not known, that $\dim G_{f_{\alpha,b}} = 2 - \alpha$.

7. The uniqueness problem for trigonometric series

We don't need much of the standard theory of the Fourier series, but for those who havn't yet learned it, this is probably the time to learn. Here is a brief sketch, the details can be found in many books on Fourier analysis. Very nice and quick presentations are given by

J. Duoandikoetxea, *Fourier Analysis*, Graduate Studies in Mathematics, Volume 29, 2001, American Mathematical Society,

Y. Katznelson, *An Introduction to Harmonic Analysis*, Dover Publications, 1968, 1976.

By a *trigonometric series* we mean any formal series $\sum_{n=-\infty}^{\infty} c_n e^{inx}, c_n \in \mathbb{C}, x \in [0, 2\pi)$. Such a trigonometric series is a *Fourier series* of a function $f \in L^1([0, 2\pi)$ if the coefficients c_n are obtained from f by integration:

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx}\, dx := \widehat{f}(n), \quad \text{the Fourier coefficient of } f.$$

Not every converging trigonometric series is a Fourier series, but it is if $\sum_{n=-\infty}^{\infty} |c_n| < \infty$, and much more generally.

The convergence of Fourier series is a very difficult question even for continuous 2π-periodic functions, that is, functions f which are continuous on $[0, 2\pi]$ with $f(0) = f(2\pi)$. But the difficulty depends on the kind of convergence we use: the pointwise, even almost everywhere, is much more difficult than the convergence in L^p-norms.

In this chapter when we speak about continuous functions on $[0, 2\pi]$, they will always be 2π-periodic, and we denote their space by $C([0, 2\pi])$ equipped with the norm $\|f\| = \max\{|f(x)| : x \in [0, 2\pi]\}$. Let us write

$$S_N f(x) = \sum_{n=-N}^{N} \widehat{f}(n) e^{inx}, \quad x \in [0, 2\pi), N \in \mathbb{N},$$

for the Nth partial sum of the Fourier series of $f \in L^1([0, 2\pi)$. Then using the formula of the geometric sums for $\sum_{n=-N}^{N} e^{iny}$ we obtain

$$S_N f(x) = \int_0^{2\pi} f(x-y) D_N(y)\, dy = D_N * f(x),$$

where D_N is the *Dirichlet kernel*,

$$D_N f(x) = \sum_{n=-N}^{N} e^{inx} = \frac{\sin((n+1/2)x)}{\sin(x/2)}.$$

Here and later all functions on $[0, 2\pi)$ will be extended to $\mathbb{R}$ as 2π-periodic functions.

The *Fejér kernel* F_N is much easier to deal with than the Dirichlet kernel:

$$F_N(x) = \frac{1}{N+1} \sum_{n=0}^{N} D_n f(x) = \frac{1}{N+1} \left(\frac{\sin((n+1)x/2)}{\sin(x/2)} \right)^2.$$

The first reason is that a sequence (a_n) converges much more easily in the Cesáro sense; $(a_1 + \cdots + a_N)/N \to a$, than in the usual sense. Secondly, Fejér's kernel behaves much better than Dirichlet's kernel. It is almost like the functions φ_ε used in standard convolution approximation: for large n it becomes quickly very small outside $[-1/N, 1/N]$ and it has integral 1. Thus one can show rather easily the following theorem:

Theorem 7.1.

$$\|F_N * f - f\|_p \to 0 \text{ as } N \to \infty, \text{ if } f \in L^p([0,2\pi)) \text{ and } 1 \leq p < \infty,$$

and

$$F_N * f \to f \text{ uniformly as } N \to \infty, \text{ if } f \text{ is continuous and } 2\pi \text{ periodic on } [0,2\pi].$$

The L^p norms will always be over the interval $[0, 2\pi]$.

This has the following corollary. By a trigonometric polynomial we mean any finite sum $\sum_{n=-N}^{N} c_n e^{inx}$.

Corollary 7.2. (1) Trigonometric polynomials are dense in $L^p([0,2\pi)), 1 \leq p < \infty$, and in $C([0,2\pi])$.

(2) If $f \in L^1([0,2\pi)$ and $\widehat{f}(n) = 0$ for all n, the $f = 0$.

The basic exponentials $e_n, e_n(x) = e^{inx}$, are obviously orthogonal: $\int_0^{2\pi} e^{imx}e^{-inx}\,dx = 0$, if $m \neq n$, while it is 2π, when $m = n$. Combining this with part (2) of the previous corollary means that the system $e_n, n \in \mathbb{N}$, is a complete orthogonal system in the Hilbert space $L^2([0,2\pi))$. Then the following theorem follows immediately from the general, and rather easy, Hilbert space theory.

Theorem 7.3. Let $f, g \in L^2([0,2\pi)$. Then

$$\|S_N * f - f\|_2 \to 0 \text{ as } N \to \infty.$$

Moreover, we have the Parseval and Plancherel formulas:

$$\int_0^{2\pi} f(x)g(x)\,dx = \sum_{n=-\infty}^{\infty} \widehat{f}(n)\overline{\widehat{g}(n)},$$

$$\|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2.$$

Now we begin the true contents of these lectures: uniqueness of trigonometric series: when does $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ determine the coefficients uniquely?

Classical references for this are

A. Zygmund, *Trigonometric Series, volumes I and II*, Cambridge University Press, 1959, (the first edition 1935 in Warsaw),

J.–P. Kahane and R. Salem, *Ensembles parfaits et séries trigonométriques*, 1963, Hermann.

A very nice presentation is given in

Alexander S. Kechris, SET THEORY AND UNIQUENESS FOR TRIGONOMETRIC SERIES, http://www.math.caltech.edu/ kechris/papers/uniqueness.pdf

The first problem is: if a trigonometric series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ converges to 0 for all $x \in \mathbb{R}$, do all the coefficients have to be 0? The answer is 'yes' and even more

Theorem 7.4. If $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ converges to 0 for all $x \in \mathbb{R} \setminus F$ for some finite set F, then $c_n = 0$ for all n.

This is due to Cantor from 1871. Lebesgue extended this to all countable closed sets in 1903 and Young to all countable sets in 1909.

The second problem is: for which subsets E of $[0, 2\pi)$ (or of $\mathbb{R}$ by periodicity) is it true that if a trigonometric series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ converges to 0 for all $x \in [0, 2\pi) \setminus E$, then $c_n = 0$ for all n?

Sets with this property are called *sets of uniqueness*, or U-sets. A set which is not a set of uniqueness is called a *set of multipilicity*, or M-set.

We now know that finite, and even closed countable, sets are U-sets. The characterizion of sets, even closed sets, of uniqueness is a very difficult problem which leads to interesting questions of descriptive set theory. Kechris's article above is much about this aspect.

To the other direction we have the rather easy result:

Theorem 7.5. If a Lebesgue measurable set $E \subset [0, 2\pi)$ is a U-set, then $\mathcal{L}^1(E) = 0$.

Are the Cantor sets $C_d, 0 < d < 1/2$, which we introduced in Chapter 1, U- or M-sets? This turns out to be a very interesting question because some of them are U-sets and some M-sets, but the crucial property determining this is not size (of d or of the dimension of C_d), but the number theoretic nature of d.

Theorem 7.6. If $d = 1/N$ where $N \geq 3$ is an integer, then C_d is a U-set.

More generally,

Theorem 7.7. If $0 < d < 1/2$, then C_d is a U-set if an only $1/d$ is a Pisot number.

A real number $\theta > 1$ is a *Pisot number* if it is an algebraic integer whose conjugates have modulus less than 1. Algebraic integers are special type of algebraic numbers; they are solutions of polynomial equations with integer coefficients and with leading coefficient 1. That is, θ is an algebraic integer if there are integers $m_0, \ldots, m_{k-1}$ such that $P(\theta) = 0$ where $P(x) = x^k + m_{k-1}x^{k-1} + \cdots + m_0$. The conjugates of θ are the other complex solutions of $P(z) = 0$.

There is a characterization of Pisot numbers which brings us closer to trigonometric series: A real number $\theta > 1$ is a Pisot number if and only if there exists a real number $\lambda \neq 0$ such that

$$\sum_{k=0}^{\infty} \sin^2(\lambda\theta^k) < \infty. \tag{7.1}$$

Obviously all integers greater than 1 are Pisot numbers. The smallest non-integral Pisot number is $1.3247\dots$. It is a solution of $x^3-x-1=0$. Some quadratic equations giving Pisot numbers are $x^2-x-1=0$, which gives the golden ratio $\frac{1+\sqrt{5}}{2} = 1.618034...$, and $x^2-2x-1=0$, which gives $1+\sqrt{2} = 2.414214....$

The uniqueness and multiplicity is closely related to the behaviour of Fourier transforms of measures on the set under consideration. The *Fourier transform* of a finite Borel measure $\mu \in \mathcal{M}(\mathbb{R})$ is by definition

$$\widehat{\mu}(x) = \int_{\mathbb{R}} e^{-ixy}\, d\mu y.$$

The Fourier coefficients of μ are $\widehat{\mu}(n), n \in \mathbb{Z}$.

Theorem 7.8. Let $E \subset [0,2\pi), E \neq [0,2\pi)$, be a closed set and let $\mu \in \mathcal{M}(E)$. Then the following are equivalent:

(1)
$$\lim_{n\to\infty} \widehat{\mu}(n) = 0.$$

(2)
$$\sum_{n=-\infty}^{\infty} \widehat{\mu}(n)e^{inx} = 0 \quad \text{for all } x \in [0,2\pi) \setminus E.$$

We have for the measures μ_d on the Cantor sets C_d:

Theorem 7.9. Let $\mu_d, 0 < d < 1/2$, be the Cantor measure as in Chapter 1. Then

$$\lim_{x\to\infty} \widehat{\mu_d}(x) = 0$$

if and only if $1/d$ is not a Pisot number.

More generally, one can prove that if $1/d$ is a Pisot number, then there is no measure in $\mathcal{M}(C_d)$ whose Fourier transform would tend to zero at infinity.

The proof for the Cantor sets $C_{1/N}, N \in \mathbb{N}, N \geq 2,$ that they are U-sets and do not carry measures with Fourier coefficients vanishing at infinity can be based on the following property they possess: A set $E \subset [0,\pi]$ is an *H-set* if there is a non-empty open interval $I \subset [0,2\pi]$ and positive integers $n_1 < n_2 < \dots$ such that $(n_k E \mod 2\pi) \cap I = \varnothing$ for all k.

Theorem 7.10. Every H-set $E \subset [0,2\pi)$ is a U-set. Moreover, for every measure $\mu \in \mathcal{M}(E)$, the Fourier coefficients satisfy $\limsup_{n\to\infty} |\widehat{\mu}(n)| > 0$.

We shall discuss the symmetric Cantor sets and measures on them. Let us first recall the construction. For $0 < d < 1/2$ we have the Cantor set C_d with dissection ratio d:

$$C_d = \bigcap_{k=1}^{\infty} \bigcup_{i=1}^{2^k} I_{k,i}.$$

Here the closed intervals $I_{k,i}, i = 1, \dots, 2^k$, have length d^k. As before, let μ_d be the 'natural' probability measure on C_d. This is the unique Borel measure $\mu_d \in \mathcal{M}(C_d)$ which is uniformly distributed in the sense that

$$\mu_d(I_{k,i}) = 2^{-k} \quad \text{for } i = 1, \dots, 2^k, k = 1, 2 \dots. \tag{7.2}$$

Recall that

$$\mathcal{H}^{s_d} \llcorner C_d = \mu_d \quad \text{and} \quad \mathcal{H}^{s_d}(C_d) = 1 \quad \text{with} \quad s_d = \log 2 / \log(1/d).$$

In order to compute the Fourier transform of μ_d it is helpful to express μ_d as a weak limit of finite linear combinations of Dirac measures indexed by binary sequences. To do this we observe that

$$C_d = \{\sum_{j=1}^{\infty} \varepsilon_j (1-d) d^{j-1} : \varepsilon_j = 0 \text{ or } \varepsilon_j = 1\}. \tag{7.3}$$

Let

$$\mathcal{E}_k = \{(\varepsilon_1, \dots, \varepsilon_k) : \varepsilon_j = 0 \text{ or } \varepsilon_j = 1\},$$

$$a(\varepsilon) = \sum_{j=1}^{k} \varepsilon_j (1-d) d^{j-1} \text{ for } \varepsilon = (\varepsilon_j) \in \mathcal{E}_k,$$

and define

$$\nu_k = 2^{-k} \sum_{\varepsilon \in \mathcal{E}_k} \delta_{a(\varepsilon)}.$$

Then

$$\nu_k \to \mu_d \text{ weakly as } k \to \infty.$$

By the definition of the Fourier transform,

$$\widehat{\delta_a}(u) = e^{-iau} \quad \text{for } a, u \in \mathbb{R},$$

so

$$\widehat{\nu_k}(u) = 2^{-k} \sum_{\varepsilon \in \mathcal{E}_k} \widehat{\delta_{a(\varepsilon)}}(u) = 2^{-k} \sum_{\varepsilon \in \mathcal{E}_k} e^{-ia(\varepsilon)u} = 2^{-k} \sum_{\varepsilon \in \mathcal{E}_k} e^{i \sum_{j=1}^{k} \varepsilon_j u_j}$$

where $u_j = -(1-d) d^{j-1} u$. Here

$$\sum_{\varepsilon \in \mathcal{E}_k} e^{i \sum_{j=1}^{k} \varepsilon_j u_j} = \Pi_{j=1}^{k} (1 + e^{iu_j})$$

as one can see by expanding the right hand side as a sum and checking that it agrees with the left hand side. Thus

$$\widehat{\nu_k}(u) = \Pi_{j=1}^k \frac{(1+e^{iu_j})}{2} = \Pi_{j=1}^k e^{iu_j/2} \Pi_{j=1}^k \cos(u_j/2) = e^{\sum_{j=1}^k iu_j/2} \Pi_{j=1}^k \cos(u_j/2),$$

where we have used the formula

$$\frac{1+e^{ix}}{2} = e^{ix/2}\cos(x/2).$$

Recalling the definition of u_j we see that

$$\sum_{j=1}^k iu_j/2 = \sum_{j=1}^k -i(1-d)d^{j-1}u/2 = -i(1-d^k)u/2.$$

Therefore we obtain

$$\widehat{\nu_k}(u) = e^{i(1-d^k)u/2} \Pi_{j=1}^k \cos((1-d)d^{j-1}u/2).$$

Letting $k \to \infty$ we finally obtain

$$\widehat{\mu_d}(u) = e^{-iu/2} \Pi_{j=1}^\infty \cos((1-d)d^{j-1}u/2). \tag{7.4}$$

When $d = 1/3$ we have for the classical ternary Cantor set

$$\widehat{\mu_{1/3}}(u) = e^{-iu/2} \Pi_{j=1}^\infty \cos(3^{-j}u).$$

It follows that $\widehat{\mu_{1/3}}(u)$ does not tend to 0 as u tends to ∞; look at $u = 3^k\pi, k = 1, 2, \dots$.

We shall now show that if $1/d \geq 3$ is an integer, then there is no measure in $\mathcal{M}(C_d)$ whose Fourier transform would tend to zero at infinity. The proof relies on the fact that then C_d is an H-set, and Theorem 7.11 and its proof are valid for all H-sets. That is, it also gives Theorem 7.10.

More precisely, letting $I = (d, 1-d)$ and $N = 1/d$,

$$[N^k x] \notin I \quad \text{for all } x \in C_d, k = 1, 2, \dots, \tag{7.5}$$

where for $y \geq 0$, $[y]$ stands for the fractional part of y, that is, $[y] \in [0,1)$ and $y - [y] \in \mathbb{N}$. To see this recall that by (7.3) C_d consists of points

$$x = \sum_{j=1}^\infty \varepsilon_j (1-d)d^{j-1} = (N-1)\sum_{j=1}^\infty \varepsilon_j N^{-j}$$

where $\varepsilon_j = 0$ or $\varepsilon_j = 1$. Then

$$N^k x = (N-1)\sum_{j=1}^\infty \varepsilon_j N^{k-j} = (N-1)(\sum_{j=0}^{k-1} \varepsilon_{k-j} N^j + \sum_{j=1}^\infty \varepsilon_{k+j} N^{-j}).$$

Thus

$$[N^k x] = (N-1)\sum_{j=1}^\infty \varepsilon_{k+j} N^{-j} \in C_d \subset [0,1] \setminus I.$$

Theorem 7.11. If $1/d \geq 3$ is an integer, then for any $\mu \in \mathcal{M}(C_d)$, $\limsup_{n\to\infty} |\widehat{\mu}(2\pi n)| > 0$.

Proof. Suppose there would exist $\mu \in \mathcal{M}(2\pi C_d)$ such that $\widehat{\mu}(k) \to 0$ as $k \in \mathbb{Z}, |k| \to \infty$. Choose a function $\varphi \in \mathcal{S}(\mathbb{R})$ such that $\operatorname{spt}\varphi \subset (2\pi d, 2\pi(1-d))$ and $\int \varphi = 1$. Let again $N = 1/d$ and define for $j = 1, 2, \dots,$

$$\varphi_j(x) = \varphi(N^j x \mod 2\pi) \quad \text{for } x \in [0,1].$$

Then by (7.5) $\operatorname{spt}\varphi_j \cap C_d = \varnothing$, and by the Fourier inversion formula

$$\varphi_j(x) = \sum_{k\in\mathbb{Z}} \widehat{\varphi}(k) e^{ixN^j k}, \quad x \in [0,1],$$

so $\widehat{\varphi_j}(N^j k) = \widehat{\varphi}(k)$ and the other Fourier coefficients of φ_j vanish. Therefore by the Parseval formula for any j and any $m > 1$,

$$\begin{aligned} 0 = \int \varphi_j \, d\mu &= \sum_{k\in\mathbb{Z}} \overline{\widehat{\varphi_j}(k)} \widehat{\mu}(k) \\ &= \sum_{k\in\mathbb{Z}} \overline{\widehat{\varphi_j}(N^j k)} \widehat{\mu}(N^j k) = \sum_{k\in\mathbb{Z}} \overline{\widehat{\varphi}(k)} \widehat{\mu}(N^j k) \\ &= \overline{\widehat{\varphi}(0)} \widehat{\mu}(0) + \sum_{1\leq |k| \leq m} \overline{\widehat{\varphi}(k)} \widehat{\mu}(N^j k) + \sum_{|k|>m} \overline{\widehat{\varphi}(k)} \widehat{\mu}(N^j k). \end{aligned}$$

The first term is $\mu(C_d) > 0$. For the last term we have

$$|\sum_{|k|>m} \overline{\widehat{\varphi}(k)} \widehat{\mu}(N^j k)| \leq \mu(C_d) \sum_{|k|>m} |\widehat{\varphi}(k)|,$$

which we can make arbitrarily small choosing m large, since $\varphi \in \mathcal{S}(\mathbb{R})$. For any m we have for the middle term

$$|\sum_{1\leq|k|\leq m} \overline{\widehat{\varphi}(k)} \widehat{\mu}(N^j k)| \leq 2m \sup_{|l|\geq N^j, l\in\mathbb{Z}} |\widehat{\mu(l)}|$$

which goes to zero as $j \to \infty$. It follows that $\mu(C_d) = 0$, which is a contradiction. □

Proof of Theorem 7.9. Let $\theta = 1/d$. Suppose that $\widehat{\mu_d}(u)$ does not tend to 0 at infinity. Then there exist $\delta > 0$ and an increasing sequence (u_k) such that $u_k \to \infty$ and

$$|\widehat{\mu_d}(u_k)| > \delta$$

for all k. We can write

$$(1-d)u_k/2 = \lambda_k \theta^{m_k}$$

where $1 \leq \lambda_k < \theta$ and (m_k) is an increasing sequence of positive integers. Replacing the sequence (λ_k) by a subsequence if needed we can assume that $\lambda_k \to \lambda, 1 \leq \lambda \leq \theta$. By (7.4),

$$\begin{aligned} \delta < |\widehat{\mu_d}(u_k)| &= |\Pi_{j=1}^{\infty} \cos((1-d)d^{j-1}u_k/2)| \\ &= |\Pi_{j=1}^{\infty} \cos(\lambda_k \theta^{m_k - j+1})| \leq |\Pi_{j=0}^{m_k} \cos(\lambda_k \theta^j)|, \end{aligned}$$

which gives

$$\Pi_{j=0}^{m_k}(1-\sin(\lambda_k\theta^j)^2) \geq \delta^2.$$

Using the elementary inequality $x \leq -\log(1-x)$ for $0 < x < 1$ this yields

$$\sum_{j=0}^{m_k} \sin^2(\lambda_k\theta^j) \leq \log(1/\delta^2).$$

Hence for $l > k$,

$$\sum_{j=0}^{m_k} \sin^2(\lambda_l\theta^j) \leq \sum_{j=0}^{m_l} \sin^2(\lambda_l\theta^j) \leq \log(1/\delta^2).$$

Keeping k fixed and letting $l \to \infty$ we get

$$\sum_{j=0}^{m_k} \sin^2(\lambda\theta^j) \leq \log(1/\delta^2),$$

and letting $k \to \infty$,

$$\sum_{j=0}^{\infty} \sin^2(\lambda\theta^j) \leq \log(1/\delta^2).$$

Hence $\theta = 1/d$ is a Pisot number.

To prove the converse, suppose that $\theta = 1/d$ is a Pisot number. Then there exists a real number $\lambda \neq 0$ such that

$$\sum_{j=0}^{\infty} \sin^2(\lambda\theta^j) < \infty.$$

Reversing the above argument this implies that

$$p = \Pi_{j=0}^{\infty}|\cos(\lambda\theta^j)| > 0.$$

Using the formula (7.4) we get for $u_k = \lambda\theta^k/(\pi(1-d))$,

$$\begin{aligned}|\widehat{\mu_d}(u_k)| &= |\Pi_{j=1}^{\infty}\cos(\lambda d^{j-1}\theta^k)| = |\Pi_{j=1}^{k}\cos(\lambda\theta^j)||\Pi_{j=0}^{\infty}\cos(\lambda\theta^{-j})| \\ &\geq p|\Pi_{j=0}^{\infty}\cos(\lambda\theta^{-j})| = pq,\end{aligned}$$

where $q > 0$ by similar calculus as above; $\sum_{j=0}^{\infty}\sin^2(\lambda\theta^{-j}) < \infty$ since $\theta > 1$. Hence $\widehat{\mu_d}(u)$ does not tend to 0 at infinity which proves the theorem. □

Because of Theorems 7.9 and 7.8, C_d is an M-set if $1/d$ is not a Pisot number. In fact, we have also the converse: C_d is a set of uniqueness if and only if $1/d$ is a Pisot number, see Kechris or the book of Kahane and Salem for a proof. Theorem 7.9 gives an indication that this converse might be true, but it is not enough to prove it. In the book of Kahane and Salem one also finds: there is $\mu \in \mathcal{M}(C_d)$ such that $\lim_{u\to\infty}\widehat{\mu}(u) = 0$ if and only if $1/d$ is not a Pisot number. But even this is not quite enough to prove that C_d is a U-set if $1/d$ is not a Pisot number.

8. Fourier transform and Hausdorff dimension

Here we just quickly see how Fourier transforms of measures are related and can be applied to Hausdorff dimension. For the background on Fourier transform see, for example, J. Duoandikoetxea: Fourier Analysis, Graduate Studies in Mathematics Volume 29, American Mathematical Society, 2001, and L. Grafakos: Classical Fourier Analysis, Springer–Verlag, 2008.

The Fourier transform of $f \in L^1(\mathbb{R}^n)$ is from now on defined by

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = \int f(x)e^{-2\pi i\xi\cdot x}\,dx, \quad \xi \in \mathbb{R}^n. \tag{8.1}$$

The Fourier transform of a finite Borel measure μ on $\mathbb{R}^n$ is defined by

$$\widehat{\mu}(\xi) = \int e^{-2\pi i\xi\cdot x}\,d\mu x, \quad \xi \in \mathbb{R}^n. \tag{8.2}$$

Then $\widehat{f}$ and $\widehat{\mu}$ are a bounded continuous functions.

Recall from Chapter 1 the s-energy of $\mu \in \mathcal{M}(\mathbb{R}^n)$:

$$I_s(\mu) = \iint |x-y|^{-s}\,d\mu x\,d\mu y = \int k_s * \mu\,d\mu,$$

where k_s is the Riesz kernel:

$$k_s(x) = |x|^{-s}, \quad x \in \mathbb{R}^n.$$

We had

Theorem 8.1. For a closed set $A \subset \mathbb{R}^n$,

$$\dim A = \sup\{s : \text{there is } \mu \in \mathcal{M}(A) \text{ such that } I_s(\mu) < \infty\}.$$

We can write the energy with Fourier transform which immediately gives the connection between Fourier transform and Hausdorff dimension.

Theorem 8.2. Let $\mu \in \mathcal{M}(\mathbb{R}^n)$ and $0 < s < n$. Then

$$I_s(\mu) = \gamma(n,s)\int |\widehat{\mu}(x)|^2|x|^{s-n}\,dx. \tag{8.3}$$

Here $\gamma(n,s)$ is a positive constant. It comes from the identity

$$\widehat{k_s} = \gamma(n,s)k_{n-s}.$$

As k_s does belong to any L^p, this must be interpreted in the distributional sense; it means that

$$\int k_s\widehat{\varphi} = \int \gamma(n,s)k_{n-s}\varphi$$

for all smooth functions φ which together with their derivatives tend to zero very quickly at infinity, they are called Schwartz functions.

Formally Theorem 8.3 follows easily by the basic Parseval and convolution formulas:

$$I_s(\mu) = \int k_s * \mu\,d\mu = \int \widehat{k_s * \mu}\widehat{\mu} = \int \widehat{k_s}|\widehat{\mu}|^2 = \gamma(n,s)\int |\widehat{\mu}(x)|^2|x|^{s-n}\,dx,$$

but some further arguments are needed since $\widehat{k_s} = \gamma(n,s)k_{n-s}$ only holds in the distributional sense. They are given in [Ma].

Recall one half one Marstrand's projections theorem 2.3:

Let $A \subset \mathbb{R}^2$ be a Borel set. If $\dim A > 1$, then

$$\mathcal{L}^1(p_\theta(A)) > 0 \quad \text{for almost all } \theta \in [0,\pi).$$

Let us give a simple proof for this with Fourier transform:

Choose by Theorems 8.1 and 8.3 a measure $\mu \in \mathcal{M}(A)$ such that $\int |x|^{-1}|\widehat{\mu}(x)|^2\,dx < \infty$. Letting again $\mu_\theta(B) = \mu(p_\theta^{-1}(B))$, we see directly from the definition of the Fourier transform that $\widehat{\mu_\theta}(t) = \widehat{\mu}(t(\cos\theta,\sin\theta))$ for $t \in \mathbb{R}, \theta \in [0,\pi)$. Integrating in polar coordinates we obtain

$$\int_0^\pi \int_{-\infty}^\infty |\widehat{\mu_\theta}(t)|^2\,dt\,d\theta = 2\int_0^\pi \int_0^\infty |\widehat{\mu}(t(\cos\theta,\sin\theta))|^2\,dt\,d\theta = 2\int |x|^{-1}|\widehat{\mu}(x)|^2\,dx < \infty.$$

Thus for almost all $\theta \in [0,\pi)$, $\widehat{\mu_\theta} \in L^2(\mathbb{R})$ which means that μ_θ is absolutely continuous with L^2 density and hence $\mathcal{L}^1(P_e(A)) > 0$. Here we used the fact Fourier transform can be extended to L^2 as an isometry.

Another application of Fourier transform is to distance sets. The *distance set* of $A \subset \mathbb{R}^n$ is

$$D(A) = \{|x-y| : x,y \in A\} \subset [0,\infty).$$

The following Falconer's conjecture seems plausible:

Conjecture 8.3. If $n \geq 2$ and $A \subset \mathbb{R}^n$ is a Borel set with $\dim A > n/2$, then $\mathcal{L}^1(D(A)) > 0$.

This is open in all dimensions $n \geq 2$. In $\mathbb{R}$ it is false; it is easy to construct examples of compact sets $A \subset \mathbb{R}$ with $\dim A = 1$ and $\mathcal{L}^1(D(A)) = 0$.

The following partial result was proved by Falconer in 1985:

Theorem 8.4. Let $A \subset \mathbb{R}^n, n \geq 2$, be a Borel set. If $\dim A > (n+1)/2$, then $\mathcal{L}^1(D(A)) > 0$.

The proof is a bit more involved than the one above, but it uses a similar technique as with the projections; we map a measure $\mu \in \mathcal{M}(A)$ to its distance measure $\delta(\mu) \in \mathcal{M}(D(A))$ defined for Borel sets $B \subset \mathbb{R}$ by

$$\delta(\mu)(B) = \int \mu(\{y : |x-y| \in B\})\,d\mu x.$$

In other words, $\delta(\mu)$ is the image of $\mu \times \mu$ under the distance map $(x,y) \to |x-y|$, or equivalently, for any continuous function φ on $\mathbb{R}$,

$$\int \varphi d\delta(\mu) = \iint \varphi(|x-y|)\,d\mu x\,d\mu y.$$

Obviously,

$$\operatorname{spt}\delta(\mu) \subset D(\operatorname{spt}\mu). \tag{8.4}$$

Then one can show that $\delta(\mu)$ is absolutely continuous if $I_{(n+1)/2}(\mu) < \infty$, from which Theorem 8.4 follows.

9. ANALYST'S TRAVELING SALESMAN THEOREM

The classical traveling salesman problem asks to find the shortest path connecting given N points in the plane: the traveling salesman wants to visit N towns as quickly as possible. This very difficult open problem is relevant for computer science. In these lectures I discuss the continuous version: a finite set is replaced by an arbitrary compact subset of the plane and we ask when it is possible to cover this set with a rectifiable curve and how to estimate the length of the shortest such curve. Peter Jones gave a nice answer to this question in 1990 which has turned out to be very influential in many areas of analysis; complex analysis, potential theory and harmonic analysis.

The lectures will follow Bishop and Peres [BP], Chapter 10.

We now denote by $\mathcal{D}_n, n \in \mathbb{Z}$, the grid of closed dyadic squares Q of side-length $l(Q) = 2^{-n}$, and we set $\mathcal{D} = \cup_{n \in Z} \mathcal{D}_n$. Using the notation of [BP], we denote the diameter of as set A by $|A|$. For $\lambda > 0,\ \lambda Q$ is the square concentric with Q with side-length $\lambda l(Q)$. If $E \subset \mathbb{R}^2$, we set

$$L(E) = \{L : L \text{ is line in the plane such that } L \cap E \neq \varnothing.\}$$

For $E \subset \mathbb{R}^2$ and for any square Q we define Jones's $\beta_E(Q)$ number by

$$\beta_E(Q) = |Q|^{-1} \inf_{L \in L(Q)} \sup_{z \in E \cap Q} dist(z, L).$$

Then the theorem of Jones is

Theorem 9.1. Let $E \subset \mathbb{R}^2$ be compact. Then there is a rectifiable curve (curve of finite length) Γ containing E if and only if

$$\beta(E) := |E| + \sum_{Q \in \mathcal{D}} \beta_E(3Q)^2 |Q| < \infty.$$

Moreover, the length of the shortest such curve is comparable to $\beta(E)$.

Notice that $\beta_E(Q)$ measures how well E can be approximated by lines inside Q; $\beta_E(Q) = 0$ if and only E lies on a line, and $\beta_E(Q) \approx 1$ if E is spread all over Q.

10. Removable sets for bounded analytic

This topic is discussed in [Ma] and in particular in

X. Tolsa: Analytic Capacity, The Cauchy Transform, and Non-homogeneous Calderón–Zygmund Theory, Birkhäuser, 2014.

Here we investigate a problem in classical complex analysis: for which compact sets $K \subset \mathbb{C}$ is it true that every bounded complex analytic function in the complement of K can be analytically extended over K?

Definition 10.1. A compact set $K \subset \mathbb{C}$ is *removable* (for bounded analytic functions) if the following is true: if $K \subset U \subset \mathbb{C}, U$ is open, $f : U \setminus K \to \mathbb{C}$ is bounded and analytic, then there is a bounded analytic function $g : U \to \mathbb{C}$ such that $g|K = f$.

It is easy to see that removable sets cannot have interior points which implies that the extension is unique. Moreover, by the Riemann mapping theorem removable sets are totally disconnected.

The first main tool is the Cauchy integral formula:

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta,$$

where Γ is a smooth closed Jordan curve, f is analytic in some some open domain containing Γ, and z is a point in the open bounded domain G whose boundary is Γ. If $\Gamma' \subset G$ is another smooth closed Jordan curve, f is analytic in some some open domain containing Γ and Γ' (but not necessarily in all of G) and z is a point in the domain bounded by Γ and Γ', then this formula takes the form

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\int_{\Gamma'} \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

Let us see how this can be used to prove Riemann's classical result that isolated singularities are removable, that is, singletons are removable sets: let f be bounded and analytic in $U \setminus \{z_0\}$ with U open containing z_0. Choose $0 < r < R$ such that $B(z_0, R) \subset U$. Then by the Cauchy integral formula for $z \in U(z_0, R) \setminus B(z_0, r)$ (here $U(z_0, R)$ is the open disc)

$$f(z) = \frac{1}{2\pi i}\int_{\partial B(z_0,R)} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\int_{\partial B(z_0,r)} \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

When $r \to 0$, the second integral tends to 0 because of the boundedness of f. Then the first integral gives the analytic extension of f to z_0.

A somewhat similar application of the Cauchy integral formula shows that we could always take $U = \mathbb{C}$ in the definition. Thus we have by Liouville's theorem

Theorem 10.2. A compact set $K \subset \mathbb{C}$ is removable if and only every bounded analytic function $f : \mathbb{C} \setminus K \to \mathbb{C}$ is constant.

What about the Hausdorff dimension of removable and non-removable sets? The first part of the following theorem is due to Painlevé from 1888 and the second part to Frostman from the 1930s:

Theorem 10.3. Let $K \subset \mathbb{C}$ be compact.
(1) If $\mathcal{H}^1(K) = 0$, then K is removable.
(2) If $\dim K) > 1$, then K is not removable.

The proof of (1) is again a simple application of the Cauchy integral formula and the definition of the Hausdorff measure. Or rather the definition of Hausdorff measure zero for which we don't need the measure itself. Notice that Painlevé's result is older than Hausdorff measures. The proof of (2) is by Frostman's lemma. It gives us $\mu \in \mathcal{M}(K)$ such that the potential $\int |y - z|^{-1}\, d\mu y, z \in \mathbb{C}$, is bounded. Then $f, f(z) = \int (\zeta - z)^{-1}\, d\mu\zeta, z \in \mathbb{C} \setminus K$, is a non-constant bounded analytic function in $C \setminus K$.

After this, rather easy, theorem we have problems only with compact sets K with $\mathcal{H}^1(K) > 0$ and $\dim K = 1$. But this turned out to be a very difficult problem. Fortunately it was solved by Tolsa in 2003. I now describe some steps leading to this solution.

First, the sufficient condition $\mathcal{H}^1(K) = 0$ is not necessary. Vitushkin provided the example in the 1959, which was rather complicated. In about 1970 Garnett, and independently Ivanov, showed that $K = C(1/4)$ also is removable although $\mathcal{H}^1(C(1/4)) > 0$. Secondly, Calderón proved in 1977 that if K is a subset of some rectifiable curve, then K is removable if and only if $\mathcal{H}^1(K) = 0$. Thirdly, David proved in 1998 that if $\mathcal{H}^1(K) < \infty$, then K is removable if and only if $\mathcal{H}^1(\Gamma \cap K) = 0$ for every rectifiable curve Γ. Finally, Tolsa's full characterization is the following:

Theorem 10.4. A compact set $K \subset \mathbb{C}$ is NOT removable if and only there is a finite Borel measure $\mu \in \mathcal{M}(K)$ such that $\mu(B(z, r)) \leq r$ for all $z \in \mathbb{C}$ and $r > 0$ and

$$\iiint c(x, y, z)^2\, d\mu x\, d\mu y\, d\mu z < \infty.$$

The non-negative number $c(x, y, z)$ is called the Menger curvature of the triple $(x, y, z) \in \mathbb{C}^3$. It is defined as $c(x, y, z) = 1/R$ where R is the radius of the circle passing through x, y and z. This circle is a line if x, y and z are collinear. Then, and only then, $c(x, y, z) = 0$. So c, like β in the previous chapter, measures approximability by lines.

The reason why Menger curvature is useful is its connection to the Cauchy kernel $1/z$ given by the formula due to Melnikov from 1995: for $z_1, z_2, z_3 \in \mathbb{C}$,

$$c(z_1, z_2, z_3)^2 = \sum_{\sigma} \frac{1}{(z_{\sigma(1)} - z_{\sigma(3)})\overline{(z_{\sigma(2)} - z_{\sigma(3)})}}$$

where σ runs through all six permutations of $\{1, 2, 3\}$.

The proof of this is rather easy plane geometry: one can first prove a more than 2000 years old theorem of the Greeks saying that if A is the area of the triangle with vertices z_1, z_2, z_3, then $c(z_1, z_2, z_3)$ equals $4A/(|z_1 - z_2||z_1 - z_3||z_2 - z_3|)$, and then one can show that the square of this equals the right hand side. For the latter it might help first to normalize to $z_1 = 0$ and $z_2 = x \in \mathbb{R}$.

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