Fractal sets in analysis, Exercise 9, 22.4.2015

1. Let μ be the measure on the space $\mathcal{L}$ of lines in $\mathbb{R}^2$ as defined by Bishop and Peres in Chapter 10: for $G \subset \mathcal{L}$, $\mu(G)$ is the Lebesgue measure of the set of the points $(r, \theta), 0 \le r < \infty, \theta \in (0, 2\pi)$, for which $re^{i\theta}$ is the closest point to the origin on some line of G. Show that

$$\mu(G) = \int_0^\pi \mathcal{L}^1(\{t \in \mathbb{R} : L_{\theta,t} \in G\})\, d\theta,$$

if $\{(\theta, t) : L_{\theta,t} \in G\})$ is Lebesgue measurable. Here $L_{\theta,t} = \{u(-\sin\theta, \cos\theta) + t(\cos\theta, \sin\theta) : u \in \mathbb{R}\}$.

2. Let $K \subset \mathbb{R}^2$ be convex. Show that $\mu(L(K))$ is the length of the boundary curve Γ of K. You may use the Crofton formula for Γ.

3. Let R be a rectangle with sidelengths 1 and β. Compute the μ measure of the set of lines meeting both sides of R of length β.

4. For $E \subset \mathbb{R}^2, x \in \mathbb{R}^2$ and $r > 0$, let

$$\beta_E(x, r) = r^{-1} \inf_{L \in L(B(x,r))} \sup_{z \in E \cap B(x,r)} dist(z, L).$$

Prove that

$$\sum_{Q \in \mathcal{D}} \beta_E(3Q)^2 |Q| \approx \int_0^\infty \int_{\mathbb{R}^2} \frac{\beta_E(x, r)^2}{r^2}\, dx\, dr.$$

5. Let $E \subset \mathbb{R}^2$ be a compact 1-dimensional AD-regular (recall Chapter 1) set. Show that

$$\sum_{Q \in \mathcal{D}} \beta_E(3Q)^2 |Q| \approx \int_0^\infty \int_E \frac{\beta_E(x, r)^2}{r}\, d\mathcal{H}^1 x\, dr.$$

6. Let $K \subset \mathbb{R}^2$ be a compact self-similar set (recall Definition 4.8). Show that either K is a subset of some line or there is $c > 0$ such that $\beta_K(x, r) > c$ for all $x \in K, 0 < r < 1$.