Fractal sets in analysis, Exercise 8, 25.3.2015

1. Prove the formula for the Dirichlet kernel:

$$D_N(x) := \sum_{n=-N}^{N} e^{inx} = \frac{\sin((n+1/2)x)}{\sin(x/2)}.$$

2. Let $f : \mathbb{R} \to \mathbb{C}$ be a continuously differentiable 2π-periodic function. Find the Fourier coefficients of the derivative f' in terms of the Fourier coefficients of f. Show that

$$\limsup_{n\to\infty} |n\widehat{f}(n)| < \infty.$$

3. Let $\mu, \nu \in \mathcal{M}(\mathbb{R})$ with compact support:

$$spt\mu = \{x : \mu(B(x,r)) > 0 \; \forall r > 0\}.$$

Their convolution $\mu * \nu$ is the measure for which

$$\int f \, d\mu * \nu = \iint f(x+y) \, d\mu x \, d\nu y$$

for all continuous functions f on $\mathbb{R}$. Show that

$$spt(\mu * \nu) \subset spt\mu + spt\nu := \{x + y : x \in \operatorname{spt}\mu, y \in \operatorname{spt}\nu\}.$$

4. Prove that $\widehat{\mu * \nu} = \widehat{\mu}\widehat{\nu}$ for μ and ν as in the previous exercise.

5. Let $A \subset \mathbb{R}$. Suppose that there exists $\mu \in \mathcal{M}(\mathbb{R})$ such that $\operatorname{spt}\mu$ is a compact subset of A and which for some $\alpha > 0$ satisfies

$$|\widehat{\mu}(x)| \leq |x|^{-\alpha} \quad \text{for all } x \in \mathbb{R}.$$

Prove that there exists a positive integer k such that the k-fold sum set

$$A^k := \{a_1 + \cdots + a_k : a_1, \ldots, a_k \in A\}$$

has non-empty interior.

You may use the fact, which is a consequence of the Fourier inversion theorem: if $\nu \in \mathcal{M}(\mathbb{R})$ is such that $\int_{\mathbb{R}} |\widehat{\nu}(x)| \, dx < \infty$, then ν is an absolutely continuous measure such that for all Borel sets $B \subset \mathbb{R}$,

$$\nu(B) = \int_B f \quad \text{where} \quad f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\nu}(y) e^{ixy} \, dy.$$