Fractal sets in analysis, Exercise 6, 11.3.2015

1. Prove that for any continuous function $f : [a, b] \to \mathbb{R}$, the graph G_f is closed and $\mathcal{L}^2(G_f) = 0$.

2. Prove that if $f : [a, b] \to \mathbb{R}$ is Lipschitz, then $\mathcal{H}^1(G_f) < \infty$.
Hint: Show that the projection $p : G_f \to \mathbb{R}, p(x, y) = x$, is injective with Lipschitz inverse.

3. Let f be the Cantor singular function;

$$f(x) = \mu_{1/3}([0, x]), \quad x \in [0, 1],$$

where $\mu_{1/3}$ is the Cantor measure on the $1/3$ Cantor set $C_{1/3}$. What are the Hausdorff and Minkowski dimensions of the graph of f?

4. Prove that the Weierstrass function $f_{\alpha,b}$ is differentiable, if $\alpha > 1$.

5. Construct a continuous function $f : [0, 1] \to \mathbb{R}$ whose graph has upper Minkowski dimension 2, and, if you want, even Hausdorff dimension 2.