**The lecture on Thursday, February 5, will be 15.00 - 16.00.**

**Fractal sets in analysis, Exercise 3, 11.2.2015**

1. Prove that for closed sets $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$,

$$\dim A + \dim B \leq \dim A \times B.$$

Hint: You may use Frostman's lemma.

2. Prove that for bounded sets $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$,

$$\overline{\dim}_M A \times B \leq \overline{\dim}_M A + \overline{\dim}_M B.$$

3. Show that if $A \subset \mathbb{R}^n$ is a closed set, then

$$\dim A = \sup\{s : \exists \mu \in \mathcal{M}(A) \text{ such that } \int |x-y|^{-s} \, d\mu y \leq 1 \forall x \in \mathbb{R}^n\}.$$

Define the Riesz $s$-capacity, $s > 0$, of a non-empty compact set $A \subset \mathbb{R}^n$ by

$$C_s(A) = \sup\{I_s(\mu)^{-1} : \mu \in \mathcal{M}(A), \mu(A) = 1\}.$$

4. Prove that $C_s(A) = 0$, if $C_t(A) = 0$ and $0 < t < s$.

5. Prove that $C_s(A) = 0$, if $\mathcal{H}^s(A) < \infty$.

6. Prove that $\dim A \leq s$ if $C_s(A) = 0$. Conclude that

$$\dim A = \inf\{s : C_s(A) = 0\}.$$