Fractal sets in analysis, Exercise 10, 6.5.2015

1. Let μ be a Borel measure on $\mathbb{C}$ with compact support and $0 < \mu(\mathbb{C}) < \infty$ such that the potential $x \mapsto \int |x-y|^{-1}\, d\mu y, x \in \mathbb{C}$, is bounded. Prove that the Cauchy transform C_μ,

$$C_\mu(z) = \int \frac{1}{w-z}\, d\mu w, \quad z \in \mathbb{C},$$

is a bounded non-constant complex analytic function in the complement of $\operatorname{spt}\mu$.

2. Let μ as in the previous exercise satisfy $\mu(B(z,r)) \leq r^s$ for all $z \in \mathbb{C}$ and $r > 0$ where $1 < s < 2$. Prove that C_μ is Hölder continuous with exponent $s-1$.

3. Prove that if K_1 and K_2 are compact disjoint removable subsets of $\mathbb{C}$, then $K_1 \cup K_2$ is removable.
This is true without disjointness but the proof using disjointness and the Cauchy integral formula should be easier.

4. Let $C(1/4)$ be the four-corner Cantor set of Chapter 2. Show that C_μ is not bounded in $\mathbb{C} \setminus C(1/4)$ where μ is the restriction of the one-dimensional Hausdorff measure to $C(1/4)$.

5. Let μ be a Borel measure on $\mathbb{C}$ with compact support and $0 < \mu(\mathbb{C}) < \infty$ such that the potential $x \mapsto \int |x-y|^{-1}\, d\mu y, x \in \mathbb{C}$, is bounded. Prove that

$$\int \left| \int \frac{1}{w-z}\, d\mu w \right|^2 d\mu z = \frac{1}{6} \iiint c(z_1,z_2,z_3)^2\, d\mu z_1\, d\mu z_2\, d\mu z_3.$$

Here $c(z_1,z_2,z_3)$ is the Menger curvature as defined in the lecture notes. You may use Melnikov's identity.
Hint: Write the left hand integral as a triple integral, permute the variables and use Fubini's theorem.