Fractal sets in analysis, Exercise 1, 28.1.2015

1. Prove that $\mathcal{H}^s(A) = 0$ if and only if $\mathcal{H}^s_\infty(A) = 0$.

2. Prove that

$$\mathcal{H}^s_\delta(A) = \inf\{\sum_j d(U_j)^s : A \subset \bigcup_j U_j, d(U_j) < \delta, U_j \text{ open}\}.$$

So in the definition of Hausdorff measures we can restrict to open covering sets.

3. Prove that for any $A \subset \mathbb{R}^n$ there is a Borel set $B \subset \mathbb{R}^n$ such that $A \subset B$ and $\mathcal{H}^s(B) = \mathcal{H}^s(A)$.

4. Prove that if $0 \leq s < t$ and $\mathcal{H}^s(A) < \infty$, then $\mathcal{H}^t(A) = 0$. Conclude that in the definition of Hausdorff dimension

$$\dim A = \inf\{s : \mathcal{H}^s(A) = 0\} = \sup\{s : \mathcal{H}^s(A) = \infty\}$$

the second equality holds.

5. Show that Hausdorff dimension has the countable stability property: for any $A_i \subset \mathbb{R}^n, i = 1, 2, \ldots,$

$$\dim \bigcup_i A_i = \sup_i \dim A_i.$$

6. Show that closed set F is AD-regular if and only there are a Borel measure μ and positive numbers s and C such that $\mu(\mathbb{R}^n \setminus F) = 0$ and

$$r^s/C \leq \mu(B(x,r)) \leq Cr^s \quad \text{for } x \in F, 0 < r < d(F).$$