Fractal sets in analysis, Exercise 7, 18.3.2015

1. Prove the Riemann-Lebesgue lemma: if $f : [0, 2\pi] \to \mathbb{C}$ is Lebesgue integrable, then $\lim_{|n|\to\infty} \widehat{f}(n) = 0$.

2. Prove that if $f : [0, 2\pi] \to \mathbb{C}$ is continuous with $f(0) = f(2\pi)$ and $\sum_{n=-\infty}^{\infty} |\widehat{f}(n)| < \infty$, then the partial sums $\sum_{n=-N}^{N} \widehat{f}(n)e^{inx}$ converge uniformly to f, as $N \to \infty$.

3. Prove that if $f : [0, 2\pi] \to \mathbb{C}$ is continuous with $f(0) = f(2\pi)$ and $\sum_{n=-\infty}^{\infty} |\widehat{f}(n)| < \infty$ and $\mu \in \mathcal{M}([0, 2\pi])$, then $\int_{[0,2\pi]} f \, d\mu = \sum_{n=-\infty}^{\infty} \widehat{f}(-n)\widehat{\mu}(n)$.

4. Prove that if $\mu \in \mathcal{M}([0, 2\pi])$, then $\widehat{\mu}$ is a Lipschitz function.

5. Let $f, g : \mathbb{R} \to \mathbb{C}$ be 2π-periodic functions which belong to $L^1([0, 2\pi])$. Compute the Fourier coefficients of the convolution $f * g$,

$$f * g(x) = \int_0^{2\pi} f(x - y)g(y) \, dy,$$

in terms of the Fourier coefficients of f and g. Do the same for the product fg assuming in addition that $f, g \in L^2([0, 2\pi])$, $\sum_{n=-\infty}^{\infty} |\widehat{f}(n)| < \infty$ and $\sum_{n=-\infty}^{\infty} |\widehat{g}(n)| < \infty$.