ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

TUOMAS HYTÖNEN

or:
Classical regularity theory of
second-order divergence-form elliptic
partial differential equations with
bounded measurable coefficients
(but this is probably too long to fit your transcript)

Background reading

The course is primarily based on the book [GT77], especially its Chapter 8, with some necessary prerequisites from earlier chapters. Time permitting, there may be some topics from [Ken94] as well.

1. Set-up

Consider L (an operator) acting on u (a function), given by the formula

$$Lu = \sum_{i,j=1}^{n} \partial_i a_{ij} \partial_j u = \nabla \cdot A \nabla u,$$

where

$$\partial_i = \frac{\partial}{\partial x_i}, \qquad \nabla = (\partial_i)_{i=1}^n$$

and

$$A = (a_{ij})_{i,j=1}^n = (a_{ij}(x))_{i,j=1}^n$$

is a matrix whose entries are bounded measurable functions, or in other words, a bounded measurable matrix-valued function. We are interested in properties of solutions u to $Lu = 0$. The first topic is what we mean by "solutions".

Remark 1.1. Much of what we do could be extended to the more general operators of the form

$$Lu = \nabla \cdot (A\nabla u + \vec{b}u) + \vec{c} \cdot \nabla u + du,$$

with lower order terms as well; nevertheless, we concentrate on the pure second order case. This case is enough for applications to quasilinear equations in Part II of [GT77] (as pointed out in [GT77], bottom of p. 167).

1.A. **Notion of weak solutions.** A common paradigm in the modern theory of PDE is to separate the questions of *existence* and *properties* of solutions:

(1) Establish the existence of solutions in some weak sense, so that this becomes relatively easy.
(2) Show that these weak solutions actually satisfy stronger properties than initially required by the notion of solutions.

Version: May 5, 2015.

To motivate the definition of solutions, consider a formal calculation: Suppose that $Lu = 0$ in Ω (a domain of $\mathbb{R}^n$). Let

$$\phi \in C_c^1(\Omega) := \text{continuously differentiable compactly supported functions in } \Omega.$$

[The book [GT77] denotes this space by $C_0^1(\Omega)$ instead.] Then, integrating by parts and observing that the boundary terms disappear because of the compact support,

$$0 = \int (Lu)\phi = \int \nabla \cdot (A\nabla u)\phi = -\int A\nabla u \cdot \nabla\phi \qquad \forall \phi \in C_c^1(\Omega).$$

Now, we forget about the formal intermediate steps (justifying the integration by parts etc.) and take the result of this computation as a definition of what it means that "$Lu = 0$": We say that

$$u \in C^1(\Omega) := \text{continuously differentiable functions in } \Omega \text{ (not necessarily compact support)}$$

satisfies $Lu = 0$ if $\int A\nabla u \cdot \nabla\phi = 0$ for all $\phi \in C_c^1(\Omega)$. This last integral is well-defined for all such functions. In this way, we can make sense of the action of a second-order differential operator on u, although u is only required to have the first derivative.

This notion of solutions can be generalized further by relaxing the notion of the derivative. This leads us to:

Definition 1.2. Consider two functions

$$\begin{aligned} u, v \in L^1_{\text{loc}}(\Omega) &:= \text{locally integrable functions on } \Omega \\ &:= \Big\{ u : \Omega \to \mathbb{R} \text{ measurable} \Big| \int_K |u| < \infty \ \forall K \subset \Omega \text{ compact} \Big\}. \end{aligned}$$

Then v is called a weak ith partial derivative of u provided that

$$\int v\phi = -\int u\partial_i\phi \qquad \forall \phi \in C_c^1(\Omega). \tag{1.3}$$

Remark 1.4 (Mollification)**.** We recall the following construction from Real Analysis. Let $\phi \in C_c^\infty(B(0,1))$ with $\int \phi = 1$ and $u \in L^1_{\text{loc}}(\Omega)$. Then the *convolutions*

$$u * \phi_\varepsilon(x) := \int u(y)\phi_\varepsilon(x-y)\,\mathrm{d}y := \int u(x-y)\phi_\varepsilon(y)\,\mathrm{d}y := \int u(x-y)\frac{1}{\varepsilon^n}\phi\Big(\frac{y}{\varepsilon}\Big)\,\mathrm{d}y$$

are well defined for $\mathrm{dist}(x, \Omega^c) > \varepsilon$, and they are C^∞ functions there. Moreover, we have the convergence

$$u * \phi_\varepsilon(x) \to u(x) \qquad \text{at a.e. (almost every) } x \in \Omega \text{ as } \varepsilon \to 0.$$

This last statement is a version of the Lebesgue differentiation theorem.

From the mollification technique it follows that the weak partial derivatives are unique (in the a.e. sense). Namely,

$$v(x) = \lim_{\varepsilon\to 0} \int v(y)\phi_\varepsilon(x-y)\,\mathrm{d}y = \lim_{\varepsilon\to 0} -\int u(y)\frac{\partial}{\partial y_i}[\phi_\varepsilon(x-y)]\,\mathrm{d}y,$$

by the defining formula (1.3) applied to the function $y \mapsto \phi_\varepsilon(x-y)$ in place of ϕ, and this formula determines $v(x)$ uniquely at almost every $x \in \Omega$.

We shall the denote the weak partial derivative of u, whenever it exists, by $\partial_i u$, i.e., by the same notation as for the classical partial derivative. Let us also define

$$\begin{aligned} W^1(\Omega) &:= \text{weakly differentiable functions on } \Omega \\ &:= \{u \in L^1_{\text{loc}}(\Omega) : \partial_i u \in L^1_{\text{loc}}(\Omega) \text{ exists for all } i = 1, \ldots, n\}. \end{aligned}$$

With this definition, we can revise the notion of weak solutions as follows:

Definition 1.5. $u \in W^1(\Omega)$ is a weak solution of "$Lu = g$", where $g \in L^1_{\text{loc}}(\Omega)$, if

$$-\int A\nabla u \cdot \nabla\phi = \int g\phi \qquad \forall \phi \in C_c^1(\Omega).$$

1.B. **Standing assumptions on the operator L, or its matrix A.** We will only consider *real-valued* functions. There are also results known for complex-coefficient matrices $A(x) = (a_{ij}(x))_{i,j=1}^n$, but in general these require different techniques, and the choice of the present course is to concentrate on the methods available in the real-valued case.

The other key assumptions are:

(1) The matrix A is (or, if you prefer: its coefficients a_{ij} are) bounded and measurable, and quantitatively

$$\|A(x)\|_{op} \leq \Lambda \quad \Leftrightarrow \quad \Big| \sum_{i,j=1}^n a_{ij}(x)\eta_i\xi_j \Big| \leq \Lambda|\eta||\xi| \qquad \text{a.e. } x \in \Omega, \quad \forall \xi, \eta \in \mathbb{R}^n.$$

(2) The matrix A is strictly elliptic, which means that:

$$\xi \cdot A\xi = \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \qquad \text{a.e. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^n.$$

In these conditions, we assume that $0 < \lambda \leq \Lambda < \infty$.

Exercise 1.6. Show that, under the standing assumptions, the matrix function $A(x)$ is a perturbation of the identity matrix I in the following sense: There exists $m \in \mathbb{R}$ (independent of x) and a matrix function $K(x)$ such that $A(x) = m(I + K(x))$, where $\|K(x)\|_{op} \leq k < 1$ for a.e. $x \in \Omega$. Hint: You need to show that $|\eta \cdot (I - A(x)/m)\xi| \leq k < 1$ for all $\xi, \eta \in \mathbb{R}^n$ of norm one. Write ξ as a sum of two vectors $\xi_{\parallel}$ and $\xi_{\perp}$, parallel with η and orthogonal to η, and use ellipticity to see that the part $\xi_{\parallel}$ will 'help' you to decrease the norm. At some point you should encounter a simple optimization problem to find a good value for m.

1.C. **Sobolev spaces and boundary values.** In the sequel, we will be concerned with solution u of $Lu = 0$ living in certain subspaces of weakly differentiable functions. We define the (first order) Sobolev space

$$W^{1,p}(\Omega) := \{u \in W^1(\Omega) : u, \partial_i u \in L^p(\Omega) \ \forall i = 1, \ldots, n\}$$

equipped with the norm

$$\|u\|_{W^{1,p}} := \Big(\int_\Omega |u|^p + \sum_{i=1}^n \int_\Omega |\partial_i u|^p \Big)^{1/p}.$$

It is easy to see that all $u \in C_c^1(\Omega)$ belong to this space, i.e., $C_c^1(\Omega) \subset W^{1,p}(\Omega)$ is a subspace. We define

$$\begin{aligned} W_0^{1,p}(\Omega) &:= \text{closure of } C_c^1(\Omega) \text{ in } W^{1,p}(\Omega) \\ &:= \{u \in W^{1,p}(\Omega) | \ \exists \ u_k \in C_c^1(\Omega) : \|u_k - u\|_{W^{1,p}} \to 0\} \end{aligned}$$

Intuitively, one should think of $W_0^{1,p}(\Omega)$ as the subspace of $u \in W^{1,p}(\Omega)$ with zero boundary values, "$u|_{\partial\Omega} = 0$". Note that it is not meaningful to define these boundary values in a pointwise way, since the Sobolev functions are only almost-everywhere equivalence classes, and the boundary $\partial\Omega$ usually has measure zero.

So we adopt the convention that for $u \in W^{1,p}(\Omega)$,

$$u = 0 \text{ on } \partial\Omega \quad \overset{\text{def}}{\Leftrightarrow} \quad u \in W_0^{1,p}(\Omega).$$

This allows to define several related notions: for instance, two functions $u, v \in W^{1,p}(\Omega)$ have same boundary values, "$u = v$ on $\partial\Omega$", if and only if $u - v = 0$ on $\partial\Omega$, if and only if $u - v \in W_0^{1,p}(\Omega)$.

We can even define inequality on the boundary. As before, other cases can be deduced once we define "$u \leq 0$ on $\partial\Omega$". For a real number $t \in \mathbb{R}$, observe that $t \leq 0$ if and only if $t^+ := \max(t, 0) = 0$. Accordingly, we define

$$u \leq 0 \text{ on } \partial\Omega \quad \overset{\text{def}}{\Leftrightarrow} \quad u^+ := \max(0, u) \in W_0^{1,p}(\Omega).$$

The maximum $u^+ := \max(0, u)$ is well-defined in the pointwise a.e. sense, and one can ask if it belongs to the space $W_0^{1,p}(\Omega)$ or not. We will soon check that it always belongs to $W^{1,p}(\Omega)$ (if $u \in W^{1,p}(\Omega)$), but the above definition can be made even without knowing this.

Now of course $u \leq v$ on $\partial\Omega$ means that $u - v \leq 0$ on $\partial\Omega$. Taking v to be a constant function $v \equiv k \in \mathbb{R}$, we can define the boundary supremum as

$$\sup_{\partial\Omega} u := \inf\{k \in \mathbb{R} : u \leq k \text{ on } \partial\Omega\} = \inf\{k \in \mathbb{R} : (u-k)^+ \in W_0^{1,p}(\Omega)\},$$

with the understanding that $\inf \varnothing = \infty$. (Note that this notion of supremum depends on the Sobolev space $W^{1,p}(\Omega)$ that we are considering — in particular, on the exponent p —, so a more complete notation could be $W^{1,p}$- sup; however, the space $W^{1,p}(\Omega)$ will be always understood from the context, and we will not apply such heavy notation.)

Exercise 1.7. Let $u \in W^{1,p}(\Omega)$ with $\sup_{\partial\Omega} u < \infty$. Prove that in this case the infimum in the definition of $\sup_{\partial\Omega} u$ is actually reached as a minimum. (Hint: It is enough to show (why?) that $(u-\varepsilon)^+ \to u^+$ in $W^{1,p}(\Omega)$ as $\varepsilon \to 0$. Estimate the L^p-norm of the function $u^+ - (u-\varepsilon)^+$ directly, and the L^p norm of its weak derivative with the help of the results from the next section.)

The first result on the elliptic equations that we want to prove is the *maximum principle*: If $u \in W^{1,2}(\Omega)$ (note that we have chosen $p = 2$ here) satisfies $Lu \geq 0$ in a bounded domain Ω, then

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u.$$

Here the inequality "$Lu \geq 0$" is again interpreted in the integrated form:

$$Lu \geq 0 \quad \overset{\text{def}}{\Leftrightarrow} \quad -\int A\nabla u \cdot \nabla\phi \geq 0 \quad \forall \phi \in C_c^1(\Omega) \text{ such that } \phi \geq 0.$$

Before going to the proof of the maximum principle, it is convenient to develop some more theory of the weak derivatives.

2. More on weak derivatives

2.A. **Mollification and relation to classical derivatives.** Recall the mollification of $u \in L^1_{\text{loc}}(\Omega)$ by $\phi \in C_c^\infty(B(0,1))$, given by the formula

$$u * \phi_\varepsilon(x) = \int u(x-y)\phi_\varepsilon(y)\,\mathrm{d}y = \int u(y)\phi_\varepsilon(x-y)\,\mathrm{d}y, \qquad \operatorname{dist}(x, \partial\Omega) > \varepsilon,$$

from Remark 1.4. In addition to the pointwise approximation $u*\phi_\varepsilon(x) \to u(x)$ almost everywhere, this also has the following norm approximation property:

Lemma 2.1. *If $K \subset \Omega$ is a compact subset and $u \in L^1_{\text{loc}}(\Omega)$, then $1_K(u * \phi_\varepsilon) \to 1_K u$ in L^1 as $\varepsilon \to 0$.*

Proof. Note that the expression $1_K(u * \phi_\varepsilon)$ is well defined as soon as $\varepsilon < \delta := \operatorname{dist}(K, \partial\Omega)$. Recall from Real Analysis that $v * \phi_\varepsilon \to v$ in $L^1(\mathbb{R}^n)$ for all $v \in L^1(\mathbb{R}^n)$ (globally, instead of locally, integrable).

Let $K_\eta := \{x \in \mathbb{R}^n : \operatorname{dist}(x, K) \leq \eta\}$ for some $\eta < \delta$, so that K_η is a compact subset of Ω. For $\varepsilon < \eta$, we have $1_K(u * \phi_\varepsilon) = 1_K((1_{K_\eta}u) * \phi_\varepsilon)$, and thus

$$\|1_K(u * \phi_\varepsilon - u)\|_{L^1} = \|1_K((1_{K_\eta}u) * \phi_\varepsilon - 1_{K_\eta}u)\|_{L^1} \leq \|(1_{K_\eta}u) * \phi_\varepsilon - 1_{K_\eta}u\|_{L^1} \to 0$$

by the mentioned global L^1 convergence for the function $1_{K_\eta}u \in L^1$. □

As a C^∞ function, $u * \phi_\varepsilon$ in particular possesses all first order derivatives $\partial_i(u * \phi_\varepsilon)$ in the classical sense. The following lemma records a connection between these classical derivatives and the weak derivatives of u:

Lemma 2.2. *Let $u \in L^1_{\text{loc}}(\Omega)$ have a weak derivative $\partial_i u \in L^1_{\text{loc}}(\Omega)$. Then*

$$\partial_i(u * \phi_\varepsilon)(x) = (\partial_i u) * \phi_\varepsilon(x) \qquad \textit{for all } x \in \Omega \textit{ such that } \operatorname{dist}(x, \partial\Omega) > \varepsilon.$$

Proof. With the help of the dominated convergence theorem, one can justify the differentiation under the integral to compute

$$\begin{aligned}\partial_i(u*\phi_\varepsilon)(x) &= \frac{\partial}{\partial x_i}\int u(y)\phi_\varepsilon(x-y)\,\mathrm{d}y \\ &= \int u(y)\frac{\partial}{\partial x_i}\phi_\varepsilon(x-y)\,\mathrm{d}y = \int u(y)\Big(-\frac{\partial}{\partial y_i}\phi_\varepsilon(x-y)\Big)\,\mathrm{d}y,\end{aligned}$$

the last step being immediate from the chain rule. Now, we can apply the defining formula of the weak derivative, with the function $\phi_{\varepsilon,x}\in C_c^1(\Omega)$ given by $\phi_{\varepsilon,x}(y)=\phi_\varepsilon(x-y)$, to deduce that

$$-\int u(y)\frac{\partial}{\partial y_i}\phi_\varepsilon(x-y)\,\mathrm{d}y = \int \partial_i u(y)\phi_\varepsilon(x-y)\,\mathrm{d}y = (\partial_i u)*\phi_\varepsilon(x),$$

as we wanted to prove. □

We are now ready for a useful characterization that provides alternative points of view to the notion of weak derivatives:

Proposition 2.3. *Let $u,v\in L^1_{\mathrm{loc}}(\Omega)$. Then the following statements are equivalent:*

1. *v is the weak derivative of u, namely, $v=\partial_i u$.*
2. *There is a sequence of functions $u_k\in C^\infty(\Omega)$ such that $u_k\to u$ and $\partial_i u_k\to v$ in $L^1_{\mathrm{loc}}(\Omega)$.*
3. *There is a sequence of functions $u_k\in C^1(\Omega)$ such that the same convergences hold.*

If all partial derivatives $\partial_i u$, $i=1,\dots,n$, exist, then it is possible to choose one approximating sequence u_k for which the convergence $\partial_i u_k\to\partial_i u$ holds for all i.

By "$u_k\to u$ in $L^1_{\mathrm{loc}}(\Omega)$" we mean: For every compact $K\subset\Omega$, we have $\|1_K(u_k-u)\|_{L^1}\to 0$.

Proof. (2)⇒(3) is trivial, since the sequence in C^∞ also qualifies for the sequence in C^1.

(3)⇒(1): This is a direct verification of the definition. For $\phi\in C_c^1(\Omega)$, we have

$$\begin{aligned}\int u\partial_i\phi &= \lim_{k\to\infty}\int u_k\partial_i\phi &&\text{by convergence in } L^1_{\mathrm{loc}}(\Omega) \text{ and compact support of } \phi\\ &= -\lim_{k\to\infty}\int(\partial_i u_k)\phi &&\text{by integration by parts for classical derivatives}\\ &= -\int v\phi &&\text{by convergence in } L^1_{\mathrm{loc}}(\Omega) \text{ and compact support of } \phi\text{, again.}\end{aligned}$$

This proves that $v=\partial_i u$ in the weak sense, by definition.

(1)⇒(2): This is the main step that requires a construction of the approximating sequence. Morally, we would like to take $u_k=(u*\phi_{\varepsilon_k})$ with $\varepsilon_k\to 0$, but the problem is that $u*\phi_\varepsilon(x)$ is only well-defined for $\mathrm{dist}(x,\partial\Omega)<\varepsilon$, i.e., not for all $x\in\Omega$. To fix this problem, we consider cut-off functions $\chi^\varepsilon\in C_c^\infty(\Omega)$ with the following properties:

- $\chi^\varepsilon(x)=1$ if $\mathrm{dist}(x,\partial\Omega)>3\varepsilon$.
- $\chi^\varepsilon(x)=0$ if $\mathrm{dist}(x,\partial\Omega)<2\varepsilon$.
- $\chi^\varepsilon(x)$ is between 0 and 1, and smooth in between.

We take for granted the existence of such functions. Then, our choice of the approximating sequence is $u_k:=(u*\phi_{\varepsilon_k})\chi^{\varepsilon_k}$ (where we interpret "not-well-defined $\times\ 0 = 0$").

Concerning the convergence $u_k\to u$ in $L^1_{\mathrm{loc}}(\Omega)$, we argue as follows. Given a compact $K\subset\Omega$, as soon as $\varepsilon_k<\mathrm{dist}(K,\partial\Omega)$, we have $\chi^{\varepsilon_k}=1$ throughout K, so that $1_K u_k = 1_K(u*\phi_\varepsilon)\to 1_K u$ by Lemma 2.1.

As for the derivatives, as both factors of u_k are C^∞ (where defined), we can use the classical product rule and then the previous lemma to see that

$$\partial_i u_k = \partial_i(u*\phi_{\varepsilon_k})\chi^{\varepsilon_k} + (u*\phi_{\varepsilon_k})\partial\chi^{\varepsilon_k} = (v*\phi_{\varepsilon_k})\chi^{\varepsilon_k} + (u*\phi_{\varepsilon_k})\partial_i\chi^{\varepsilon_k}.$$

Like before, as soon as $\varepsilon_k<\mathrm{dist}(K,\partial\Omega)$, we have $\chi^{\varepsilon_k}\equiv 1$ and thus $\partial_i\chi^{\varepsilon_k}\equiv 0$ throughout K. Hence $1_K\partial_i u_k = 1_K(v*\phi_{\varepsilon_k})\to 1_K v$ by Lemma 2.1, exactly as before.

The last statement of the proposition is immediate from the construction just given. □

2.B. **The chain rule for weak derivatives.** Our goal is to see how to find weak derivatives of $f \circ u$, where $u \in W^1(\Omega)$ and $f : \mathbb{R} \to \mathbb{R}$ is (at least piecewise) differentiable in the classical sense. We begin with:

Lemma 2.4. *Let $u \in W^1(\Omega)$ and $f \in C^1(\mathbb{R})$ with $f' \in L^\infty(\mathbb{R})$. Then $f \circ u \in W^1(\Omega)$ and*

$$\nabla(f \circ u) = (f' \circ u)\nabla u.$$

Proof. We make use of the characterization provided by Proposition 2.3. By assumption, there is a sequence $u_k \in C^1(\Omega)$ such that $u_k \to u$ in $L^1_{\text{loc}}(\Omega)$ and $\nabla u_k \to \nabla u$ in $L^1_{\text{loc}}(\Omega)$. Recall from Real Analysis that some subsequence (which we still denote by u_k) also converges pointwise almost everywhere.

By the classical chain rule for C^1 functions, it follows that $f \circ u_k \in C^1(\Omega)$ and $\nabla(f \circ u_k) = (f' \circ u_k)\nabla u_k$. It suffices to prove that $f \circ u_k \to f \circ u$ and $\nabla(f \circ u_k) \to (f' \circ u)\nabla u$, since this will show (again by Proposition 2.3) that $f \circ u \in W^1(\Omega)$ with $\nabla(f \circ u) = (f' \circ u)\nabla u$, as claimed.

Let K be a compact set. Then

$$\int_K |f(u_k) - f(u)| \,\mathrm{d}x \leq \int_K \|f'\|_\infty |u_k - u| \,\mathrm{d}x \to 0,$$

since $u_k \to u$ in $L^1_{\text{loc}}(\Omega)$, and this shows that $f \circ u_k \to f \circ u$ in $L^1_{\text{loc}}(\Omega)$.

We turn to the convergence of the derivatives:

$$\begin{aligned}\int_K |f'(u_k)\nabla u_k - f'(u)\nabla u| \,\mathrm{d}x &\leq \int_K |f'(u_k)(\nabla u_k - \nabla u)| \,\mathrm{d}x + \int_K |(f'(u_k) - f'(u))\nabla u| \,\mathrm{d}x \\ &\leq \|f'\|_\infty \int_K |\nabla u_k - \nabla u| \,\mathrm{d}x + \int_K |f'(u_k) - f'(u)||\nabla u| \,\mathrm{d}x.\end{aligned}$$

For the first term, we immediately have convergence to zero, since $\nabla u_k \to \nabla u$ is $L^1_{\text{loc}}(\Omega)$. For the second term, we argue as follows: Recall that we picked a sequence u_k such that $u_k(x) \to u(x)$ at almost every $x \in \Omega$. Since f' is a continuous function, we also have that $f'(u_k(x)) \to f'(u(x))$ at all these same x. Thus the integrand converges to zero pointwise almost everywhere. On the other hand, the integrand is also bounded by

$$|f'(u_k) - f'(u)||\nabla u| \leq 2\|f'\|_\infty |\nabla u| \in L^1(K),$$

and thus the whole integral converges to zero by the dominated convergence theorem. □

Exercise 2.5. Let $f \in C^1(\mathbb{R})$ with $f' \in L^\infty(\mathbb{R})$. Let $u, u_k \in W^{1,p}(\Omega)$ with $u_k \to u$ in $W^{1,p}(\Omega)$. Show that we also have $f \circ u_k \to f \circ u$ in $W^{1,p}(\Omega)$, at least for a subsequence. Hint: Repeat considerations similar to the proof of the previous lemma, but using $L^p(\Omega)$ norms instead of $L^1(K)$ norms.

The following variant is particularly relevant in the context of the maximum principle:

Lemma 2.6. *Let $u \in W^1(\Omega)$. Then $u^+ := \max(u, 0)$ also belongs to $W^1(\Omega)$, and*

$$\nabla u^+(x) = \begin{cases} \nabla u(x), & \textit{if } u(x) > 0, \\ 0, & \textit{else.} \end{cases}$$

Proof. Since the function $f(t) = \max(t, 0)$ is not differentiable at zero, for every $\varepsilon > 0$, we consider an approximation

$$f_\varepsilon(t) := \begin{cases} (t^2 + \varepsilon^2)^{1/2} - \varepsilon, & \text{if } t > 0, \\ 0, & \text{else.} \end{cases}$$

It is straightforward to check that $f_\varepsilon \in C^1(\mathbb{R})$, and

$$f'_\varepsilon(t) := \begin{cases} (t^2 + \varepsilon^2)^{-1/2} t, & \text{if } t > 0, \\ 0, & \text{else,} \end{cases}$$

which is more compactly written as $f_\varepsilon'(t) = t^+(t^2+\varepsilon^2)^{-1/2}$ belongs to $L^\infty(\mathbb{R})$. Hence the previous lemma applies to show that $f_\varepsilon \circ u \in W^1(\Omega)$ and

$$\nabla(f_\varepsilon \circ u) = (f_\varepsilon' \circ u)\nabla u = \frac{u^+}{(u^2+\varepsilon^2)^{1/2}}\nabla u.$$

Integrating against $\phi \in C_c^1(\Omega)$ and using the definition of the weak derivative, we arrive at

$$\int \phi \frac{u^+}{(u^2+\varepsilon^2)^{1/2}}\nabla u = -\int (\nabla\phi)(f_\varepsilon \circ u). \tag{2.7}$$

Note that the factor $u^+(u^2+\varepsilon^2)^{-1/2}$ is bounded by one and tends poitwise to

$$1_{\{u>0\}} = \begin{cases} 1, & \text{if } u > 0, \\ 0, & \text{else} \end{cases}$$

as $\varepsilon \to 0$. Since $\phi\nabla u$ is integrable (ϕ being compactly supported and ∇u locally integrable), we conclude that

$$\int \phi \frac{u^+}{(u^2+\varepsilon^2)^{1/2}}\nabla u \to \int \phi 1_{\{u>0\}}\nabla u$$

by dominated convergence. On the other hand, $0 \le f_\varepsilon(u) \le u$ and $f_\varepsilon(u) \to u^+$ pointwise as $\varepsilon \to 0$, so we also deduce from dominated convergence that

$$-\int (\nabla\phi)(f_\varepsilon \circ u) \to -\int (\nabla\phi)u_+.$$

Thus, taking the limits of both sides of (2.7), we arrive at

$$\int \phi\, 1_{\{u>0\}}\nabla u = -\int (\nabla\phi)\, u^+,$$

which shows that ∇u^+ exists and equals $1_{\{u>0\}}\nabla u$, directly from the definition of the weak derivative. This concludes the proof. □

The following theorem contains and generalizes both previous lemmas about the chain rule for weak derivatives. We say that a function is piecewise C^1 if

- it is C^1 outside a finite number of exceptional points,
- at these points, it has one sided derivatives, which agree with the one-sided limits of the proper derivatives.

Theorem 2.8. *Let $f : \mathbb{R} \to \mathbb{R}$ be piecewise C^1 with $f' \in L^\infty(\mathbb{R})$, and $u \in W^1(\Omega)$. Then $f \circ u \in W^1(\Omega)$, and*

$$\nabla(f \circ u)(x) = \begin{cases} f'(u(x))\nabla u(x), & \text{if } u(x) \notin E, \\ 0, & \text{if } u(x) \in E, \end{cases}$$

where E is the finite set of exceptional points of f.

Exercise 2.9. Prove the previous theorem. Hint: Make an induction on the number n of exceptional points. The case $n = 0$ is already handled, so it remains to make the induction step. Let t_0 be one of the exceptional points. By replacing $f(t)$ by $f(t+t_0)$ and u by $u-t_0$ (check the details!), it may be assumed that $t_0 = 0$. The restriction of the function f to $\mathbb{R}_-$ and $\mathbb{R}_+$ may be extended to piecewise C^1 functions f_- and f_+, each of which has only the exceptional points that f has on $\mathbb{R}_-$ and $\mathbb{R}_+$. Find a formula for $f(u)$ in terms of $f_-(u^-)$ and $f_+(u^+)$ (where $u^- := \min(u, 0)$) and apply the induction hypothesis, checking that each of f_- and f_+ has fewer exceptional points than f itself.

3. The maximum principle

We now return to considerations around the elliptic differential operator L.

Theorem 3.1. *Let Ω be a bounded domain and $u \in W^{1,2}(\Omega)$ satisfy $Lu \geq 0$ in the weak sense, i.e.,*

$$\int A\nabla u \cdot \nabla\phi \leq 0 \qquad \forall \phi \in C_c^1(\Omega) \text{ such that } \phi \geq 0. \tag{3.2}$$

Then

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u.$$

Remark 3.3. It is understood that $\sup_\Omega u$ is actually the essential supremum, i.e., the supremum outside a set of measure zero. Recall that the boundary supremum is defined as

$$\sup_{\partial\Omega} u := \inf\{k \in \mathbb{R} : (u-k)^+ \in W_0^{1,2}(\Omega)\}.$$

The idea of the proof is to use a "clever" choice of the test function ϕ in the condition (3.2). To allow more possibilities for this choice, we would like to be able to replace the test function space $C_c^1(\Omega)$ by the somewhat larger $W_0^{1,2}(\Omega)$.

Lemma 3.4. *The expression*

$$\mathscr{L}(u,\phi) := \int_\Omega A\nabla u \cdot \nabla\phi$$

defines a continuous bilinear form on $W^{1,2}(\Omega)$.

Proof. By the boundedness of A and the Cauchy–Schwarz inequality, we have

$$|\mathscr{L}(u,\phi)| \leq \int_\Omega \Lambda|\nabla u||\nabla\phi| \leq \Lambda\|\nabla u\|_{L^2}\|\nabla\phi\|_{L^2} \leq \Lambda\|u\|_{W^{1,2}}\|\phi\|_{W^{1,2}},$$

so indeed $\mathscr{L}(u,\phi)$ is well-defined for $u,\phi \in W^{1,2}(\Omega)$. The linearity of $\mathscr{L}$ with respect to each argument is obvious, and continuity follows from the above bound and $\mathscr{L}(u,\phi_n) - \mathscr{L}(u,\phi) = \mathscr{L}(u,\phi_n-\phi)$, and a similar identity in the first argument. □

Lemma 3.5. *Under the assumptions of Theorem 3.1, the inequality* (3.2) *remains valid with ϕ replaced by any $v \in W_0^{1,2}(\Omega)$ such that $v \geq 0$.*

On a quick thought, this could seem obvious from the previous lemma and the definition of $W_0^{1,2}(\Omega)$ as the closure of $C_c^1(\Omega)$. Namely, for each $v \in W_0^{1,2}(\Omega)$, we can choose a sequence $v_n \in C_c^1(\Omega)$ such that $\|v_n - v\|_{W^{1,2}} \to 0$, and therefore $\mathscr{L}(u,v_n) \to \mathscr{L}(u,v)$ by the continuity of $\mathscr{L}$ on $W^{1,2}(\Omega)$. Now, if $\mathscr{L}(u,v_n) \leq 0$, then also $\mathscr{L}(u,v) \leq 0$. But, to deduce that $\mathscr{L}(u,v_n) \leq 0$ from the assumption (3.2), we need that $v_n \geq 0$, i.e., we need the following:

Lemma 3.6. *If $v \in W_0^{1,2}(\Omega)$ satisfies $v \geq 0$, then an approximating sequence $v_n \in C_c^1(\Omega)$ with $\|v_n - v\|_{W^{1,2}} \to 0$ may be chosen so that $v_n \geq 0$ as well.*

Exercise 3.7. Prove this lemma. Hint: Let v_n be any approximating sequence of v, and let f_ε be the auxiliary function from the proof of Lemma 2.6. Check that $f_\varepsilon \circ v \to v^+$ (which is v for $v \geq 0$) as $\varepsilon \to 0$ and $f_\varepsilon \circ v_n \to f_\varepsilon \circ v$ as $n \to \infty$ for any fixed ε. Which functions appearing in this hint are nonnegative and $C_c^1(\Omega)$?

As discussed above, Lemmas 3.4 and 3.6 imply Lemma 3.5, and we are ready for:

Proof of Theorem 3.1. By Lemma 3.5, we have

$$\int A\nabla u \cdot \nabla v \leq 0 \qquad \forall v \in W_0^{1,2}(\Omega) \text{ such that } v \geq 0.$$

On the other hand, by the definition of $\sup_{\partial\Omega} u$, there is a sequence of numbers $k_j \searrow \sup_{\partial\Omega} u$ (i.e., approach from above) such that $(u-k_j)^+ \in W^{1,2}(\Omega)$. Let us consider one such k_j and simply denote it by k. Our choice for v is then $v = (u-k)^+$, which lies in $W_0^{1,2}(\Omega)$ by the choice of k, and clearly satisfies $v \geq 0$, being the positive part of a function. Thus, we have

$$\begin{aligned} 0 \geq \int_\Omega A\nabla u \cdot \nabla v &= \int_\Omega A\nabla u \cdot \nabla (u-k)^+ = \int_\Omega A\nabla u \cdot 1_{\{u>k\}} \nabla u \\ &= \int_\Omega A(1_{\{u>k\}}\nabla u) \cdot 1_{\{u>k\}}\nabla u = \int_\Omega A\nabla v \cdot \nabla v \geq \int_\Omega \lambda |\nabla v|^2. \end{aligned}$$

Since $\lambda > 0$, this implies that $\int_\Omega |\nabla v|^2 = 0$, which in turn implies that $\nabla v = 0$ almost everywhere.

Now, we would like to argue that v itself satisfies $v = 0$ almost everywhere. Assuming this for the moment, we could conclude the proof as follows: Recalling that $v = (u-k)^+$, we have $(u-k)^+ = 0$, thus $u \leq k$ almost everywhere on Ω, i.e., $\sup_\Omega u \leq k$. Since this is true for all $k = k_j \searrow \sup_{\partial\Omega} u$, we have $\sup_\Omega u \leq \sup_{\partial\Omega} u$, as we claimed.

To check that $v = 0$, we make use of *Sobolev's inequality*, which we prove in the next section. Here we need the following case:

$$\|w\|_{L^{n/(n-1)}(\Omega)} \leq \|\nabla w\|_{W^{1,1}(\Omega)} \qquad \forall w \in W_0^{1,1}(\Omega). \tag{3.8}$$

Let us observe that our v is such a function. By assumption, it belongs to $W^{1,2}(\Omega)$, which means that $\|v - \phi_n\|_{W^{1,2}} \to 0$ for some functions $\phi_n \in C_c^1(\Omega)$. But recalling the definition of the norm of $W^{1,p}$ in terms of the L^p norms of the function and its partial derivatives, and using $\|f\|_{L^1(\Omega)} \leq \|f\|_{L^2(\Omega)}|\Omega|^{1/2}$, we immediately check that $\|v-\phi_n\|_{W^{1,1}} \leq c_\Omega \|v - \phi_n\|_{W^{1,2}}$, so that v also belongs to $W_0^{1,1}(\Omega)$. Thus we may apply (3.8) with $w = v$, for which the right side, and thus also the left side, is zero. And this clearly implies that $v = 0$ almost everywhere. □

4. Sobolev's inequality

The proof of the maximum principle already used a particular case of the following fundamental estimate, which has a wide range of applications in Analysis:

Theorem 4.1 (Sobolev's inequality)**.** *The following inequality*

$$\|u\|_{L^{np/(n-p)}(\mathbb{R}^n)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^n)} \qquad \forall u \in W_0^{1,p}(\mathbb{R}^n)$$

is valid for all $1 \leq p < n$, with some $C = C(n,p)$.

Remark 4.2. (a) The Sobolev exponent $pn/(n-p)$ will sometimes be abbreviated as

$$p^* := \frac{np}{n-p},$$

when the dimension n is understood from the context. Note that

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n},$$

so that $1/p^* < 1/p$ and hence $p^* > p$.

(b) As $p \nearrow n$, the exponent $np/(n-p)$ tends to ∞. However, this formal limit of Sobolev's inequality is in general not valid, except for $n = p = 1$, when it is easy. For $u \in C_c^1(\mathbb{R})$, we can write

$$u(x) = \int_{-\infty}^x u'(y)\,\mathrm{d}y \qquad \Rightarrow \qquad |u(x)| \leq \int_{-\infty}^x |u'(y)|\,\mathrm{d}y \leq \|u'\|_{L^1(\mathbb{R})}, \tag{4.3}$$

and obtain the general case of $u \in W^{1,1}(\mathbb{R})$ by approximation. (The details of this approximation argument will be indicated in Exercise 4.8 below.)

The simple argument above serves as a model for the main case of Theorem 4.1 stated as follows:

Proposition 4.4.

$$\|u\|_{L^{n/(n-1)}(\mathbb{R}^n)} \leq C\|\nabla u\|_{L^1(\mathbb{R}^n)} \qquad \forall u \in C_c^1(\mathbb{R}^n).$$

Proof. Now we can write an analogue of (4.3) in each coordinate direction:

$$u(x)=u(x_1,\ldots,x_n)=\int_{-\infty}^{x_i}\partial_i u(x_1,\ldots,x_{i-1},y_i,x_{i+1},\ldots,x_n)\,\mathrm{d}y_i=\int_{-\infty}^{x_i}\partial_i u(\hat{x}_i,y_i)\,\mathrm{d}y_i,$$

where

$$\hat{x}_i=(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)\in\mathbb{R}^{n-1}$$

is the vector x with the ith component removed, and $(\hat{x}_i,y_i)$ is the vector x with the ith component replaced by y_i. Thus

$$|u(x)|\leq\int_{\mathbb{R}}|\partial u_i(\hat{x}_i,y_i)|\,\mathrm{d}y_i=:u_i(\hat{x}_i),$$

where in the last step we just gave a name to this new function of $n-1$ variables. Multiplying the analogue of this bound for each $i=1,\ldots,n$ and taking the $(n-1)$th root, we arrive at

$$|u(x)|^{n/(n-1)}\leq\prod_{i=1}^n u_i(\hat{x}_i)^{1/(n-1)}=:\prod_{i=1}^n v_i(\hat{x}_i).$$

Integrating over $x\in\mathbb{R}^n$, we can use a variant of Hölder's inequality recorded in the following lemma:

$$\begin{aligned}\int_{\mathbb{R}^b}|u(x)|^{n/(n-1)}\,\mathrm{d}x&\leq\int_{\mathbb{R}^n}\prod_{i=1}^n v_i(\hat{x}_i)\,\mathrm{d}x\leq\prod_{i=1}^n\Big(\int_{\mathbb{R}^{n-1}}v_i(\hat{x}_i)^{n-1}\,\mathrm{d}\hat{x}_i\Big)^{1/(n-1)}\\&=\prod_{i=1}^n\Big(\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}}|\partial_i u(\hat{x}_i,y_i)|\,\mathrm{d}y_i\,\mathrm{d}\hat{x}_i\Big)^{1/(n-1)}\\&=\prod_{i=1}^n\Big(\int_{\mathbb{R}^n}|\partial_i u(x)|\,\mathrm{d}x\Big)^{1/(n-1)}\\&\leq\prod_{i=1}^n\Big(\int_{\mathbb{R}^n}|\nabla u(x)|\,\mathrm{d}x\Big)^{1/(n-1)}=\Big(\int_{\mathbb{R}^n}|\nabla u(x)|\,\mathrm{d}x\Big)^{n/(n-1)},\end{aligned}$$

and taking the $n/(n-1)$th root of both sides gives the claimed inequality. □

Above, we needed the following variant of Hölder's inequality dealing with functions that depend on a restricted number of variables only:

Lemma 4.5. *Let* $v_i:\mathbb{R}^{n-1}\to[0,\infty)$ *be measurable functions, and* $\hat{x}_i:=(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$. *Then*

$$\int_{\mathbb{R}^n}\prod_{i=1}^n v_i(\hat{x}_i)\,\mathrm{d}x\leq\prod_{i=1}^n\Big(\int_{\mathbb{R}^{n-1}}v_i(\hat{x}_i)^{n-1}\,\mathrm{d}\hat{x}_i\Big)^{1/(n-1)}.\tag{4.6}$$

Remark 4.7. (a) Recall that the usual Hölder inequality states that $\int uv\,\mathrm{d}x\leq\|u\|_p\|u\|_{p'}$, where $1/p+1/p'=1$, and by induction on n it is easy to derive from this that

$$\int\prod_{i=1}^n u_i(x)\,\mathrm{d}x\leq\prod_{i=1}^n\Big(\int u_i(x)^{p_i}\,\mathrm{d}x\Big)^{1/p_i}$$

for any numbers $p_i\geq 1$ satisfying $\sum_{i=1}^n 1/p_i=1$. One possible choice of such exponents would be $p_1=\ldots=p_n=n$. Note that this is a different inequality than that asserted by the lemma, where the exponent is $n-1$, not n. In the lemma, we have the additional structural assumption that each function depends only on $n-1$ variables, not all n variables.

(b) If you forget the inequality (4.6), but remember that some estimate of the type

$$\int_{\mathbb{R}^n}\prod_{i=1}^n v_i(\hat{x}_i)\,\mathrm{d}x\leq\prod_{i=1}^n\Big(\int_{\mathbb{R}^{n-1}}v_i(\hat{x}_i)^q\,\mathrm{d}\hat{x}_i\Big)^{1/q}$$

is valid, there is an easy way to check what the value of q must be. (The same trick applies as a useful "reality check" many other inequalities as well.) Namely, replace each function $v_i(\hat{x}_i)$ by $v_i(t\hat{x}_i)$ for some $t>0$, and make the change of variables $tx=y$ on both sides. Then t disappears from inside the functions v_i, and $\mathrm{d}x$ becomes $t^{-n}\,\mathrm{d}y$ and $\mathrm{d}\hat{x}_i$ becomes $t^{-(n-1)}\,\mathrm{d}\hat{y}_i$ (since this is only $(n-1)$-dimensional). Taking out the powers of t, the left side scales like t^{-n} and the right side like $t^{-n(n-1)/q}$. This leads to a contradiction by considering the limits $t\to 0$ and $t\to\infty$, unless the exponents match, i.e., unless $n = n(n-1)/q$, and thus $q = n-1$. So the estimate above can only be valid for this exponent.

Proof of Lemma 4.5. We proceed by induction on n. The case $n = 1$ is not meaningful, so the base of induction is $n = 2$. In this case, $\hat{x}_1 = x_2$, $\hat{x}_2 = x_1$, and the claimed estimate becomes an *identity*:

$$\int_{\mathbb{R}^2} v_1(x_2)v_2(x_1)\,\mathrm{d}x = \int_{\mathbb{R}} v_1(x_2)\,\mathrm{d}x_2 \int_{\mathbb{R}} v_2(x_1)\,\mathrm{d}x_1$$

by Fubini's theorem.

Let us then assume that the claim is true for some $n \geq 2$, and prove it for $n+1$. We keep denoting by x a point of $\mathbb{R}^n$, so that a generic point of $\mathbb{R}^{n+1}$ becomes (x, x_{n+1}). Thus we now want to estimate

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}}\prod_{i=1}^n v_i(\hat{x}_i, x_{n+1})v_{n+1}(x)\,\mathrm{d}x_{n+1}\,\mathrm{d}x = \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}}\prod_{i=1}^n v_i(\hat{x}_i, x_{n+1})\,\mathrm{d}x_{n+1}\Big)v_{n+1}(x)\,\mathrm{d}x.$$

For each fixed x, we apply Hölder's inequality, with exponents $p_1 = \ldots = p_n = n$, to the n functions $v_i(\hat{x}_i, x_{n+1})$ of the variable $x_{n+1}\in\mathbb{R}$, arriving at

$$\leq \int_{\mathbb{R}^n}\prod_{i=1}^n\Big(\int_{\mathbb{R}} v_i(\hat{x}_i, x_{n+1})^n\,\mathrm{d}x_{n+1}\Big)^{1/n} v_{n+1}(x)\,\mathrm{d}x.$$

We also use Hölder's inequality with exponents n and $n' = n/(n-1)$ to the two functions $v_{n+1}(x)$ and the other (more complicated) factor in the integrand above, arriving at

$$\leq \Big(\int_{\mathbb{R}^n}\prod_{i=1}^n\Big(\int_{\mathbb{R}} v_i(\hat{x}_i, x_{n+1})^n\,\mathrm{d}x_{n+1}\Big)^{1/(n-1)}\mathrm{d}x\Big)^{(n-1)/n}\Big(\int_{\mathbb{R}^n} v_{n+1}(x)^n\,\mathrm{d}x\Big)^{1/n}.$$

Finally, we apply the induction assumption to the n functions $\big(\int_{\mathbb{R}} v_i(\hat{x}_i, x_{n+1})^n\,\mathrm{d}x_{n+1}\big)^{1/(n-1)}$ of $\hat{x}_i$, which gives

$$\leq \Big(\prod_{i=1}^n\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}} v_i(\hat{x}_i, x_{n+1})^n\,\mathrm{d}x_{n+1}\,\mathrm{d}x\Big)^{1/n}\Big(\int_{\mathbb{R}^n} v_{n+1}(x)^n\,\mathrm{d}x\Big)^{1/n},$$

and this is exactly the right side of the lemma with $n+1$ in place of n, since each function v_i is raised to the power $n = (n+1)-1$, and integrated over its "own" variables. This completes the induction, and thereby the proof. ☐

Exercise 4.8. Complete the proof of Sobolev's inequality as stated in Theorem 4.1. Hint: (a) Case $1 < p < n$ for $u \in C_c^1(\mathbb{R}^n)$: Check that $|u|^\gamma \in C_c^1(\mathbb{R}^n)$ for $\gamma > 1$, apply Proposition 4.4 to this function, use Hölder's inequality to extract the norms of u and ∇u, and make an appropriate choice of the parameters to reach the claim. (b) The general case: Deduce from the previous cases that an approximating sequence $u_k \in C_c^1(\mathbb{R}^n)$ of $u \in W_0^{1,p}(\mathbb{R}^n)$ is Cauchy with respect to the norm of $L^{pn/(n-p)}$, and check that the $L^{pn/(n-p)}$-limit must agree with the $W^{1,p}$-limit u.

The following corollary to Sobolev's inequality will be needed in the next section:

Corollary 4.9. *Let Ω be a bounded domain. On the space $W_0^{1,p}(\Omega)$, we have equivalent norms*

$$\|u\|_{W^{1,p}} \eqsim \|\nabla u\|_{L^p} \qquad \forall u \in W_0^{1,p}(\Omega).$$

The notation "$\eqsim$" indicated that the two quantities are equivalent "up to constants", i.e.,

$$c\|u\|_{W^{1,p}} \leq \|\nabla u\|_{L^p} \leq C\|u\|_{W^{1,p}}$$

for some $0 < c \leq C < \infty$, where these constants are independent of the particular function u. They may, however, depend on p and Ω.

Proof. Recall that $\|u\|_{W^{1,p}} \eqsim \|u\|_{L^p} + \|\nabla u\|_{L^p}$, so that it is enough to show that $\|u\|_{L^p} \lesssim \|\nabla u\|_{L^p}$ for all $u \in W_0^{1,p}(\Omega)$. We will use the fact that, on bounded domain, the L^p norms form an increasing scale in the sense that $\|u\|_{L^p} \leq \|u\|_{L^q} |\Omega|^{1/p-1/q} \lesssim \|u\|_{L^q}$ whenever $p \leq q$, by Hölder's inequality. The proof uses Sobolev's inequality and splits into two cases:

Case $1 \leq p < n$: Since $p < p^*$, we have from Hölder's and Sobolev's inequalities that $\|u\|_p \lesssim \|u\|_{p^*} \lesssim \|u\|_p$.

Case $p \geq n$: We choose an auxiliary $q < n \leq p$. Since $q^* \to \infty$ as $q \to n$, we may choose q^* as large as we like, in particular, so that $q^* \geq p$. Then $\|u\|_p \lesssim \|u\|_{q^*} \lesssim \|\nabla u\|_q \leq \|\nabla u\|_p$. Actually, with a little algebra it is easy to check that one can choose q so that $q^* = p$ and the first "$\lesssim$" becomes "=". Either way, the proof is complete in both cases. □

5. Solvability of the Dirichlet problem

We have developed enough tools to actually 'solve' a partial differential equation now. The quotation marks may be necessary, since we only prove the existence of solutions, without providing any actual formula to express them.

Theorem 5.1. *Let Ω be a bounded domain and $L = \nabla \cdot A\nabla$ satisfy the standing assumptions. Let $\varphi \in W^{1,2}(\Omega)$, $g \in L^2(\Omega)$ and $\vec{f} = (f_1, \ldots, f_n) \in L^2(\Omega; \mathbb{R}^n)$. Then the Dirichlet problem*

$$\begin{cases} Lu = g + \nabla \cdot \vec{f} & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{5.2}$$

which is an abbreviation for

$$\int A\nabla u \cdot \nabla \phi = \int [\vec{f} \cdot \nabla \phi - g\phi] \quad \forall \phi \in C_c^1(\Omega), \qquad u - \varphi \in W_0^{1,2}(\Omega), \tag{5.3}$$

has a unique solution $u \in W^{1,2}(\Omega)$.

The uniqueness can be seen immediately from the maximum principle: If u_1, u_2 are two solutions, then $u := u_1 - u_2$ satisfies $Lu = 0$ in Ω and $u = 0$ on $\partial\Omega$. Hence the maximum principle implies that $\sup_\Omega u \leq \sup_{\partial\Omega} u = 0$. Since $-u$ satisfies the same conditions, the maximum principle also shows that $-\inf_\Omega u = \sup_\Omega(-u) \leq \sup_{\partial\Omega}(-u) = 0$, thus $\inf_\Omega u \geq 0$, and altogether $u = 0$ a.e. on Ω.

The existence of the solution will be derived from the following abstract principle, which incidentally provides another proof of uniqueness as well:

Lemma 5.4 (Lax–Milgram)**.** *Let $B : H \times H \to \mathbb{R}$ be a bounded (i.e., $|B(u,v)| \leq C\|u\|\|v\|$) bilinear (i.e., linear with respect to each argument) form on a real Hilbert space H, which is also* coercive:

$$B(u,u) \geq c\|u\|^2 \qquad \forall u \in H,$$

where $c > 0$ is independent of $u \in H$. Let $F : H \to \mathbb{R}$ be a bounded linear functional. Then the equation

$$B(u,v) = F(v) \qquad \forall v \in H \tag{5.5}$$

has a unique solution $u \in H$.

Proof. The working engine of the proof is a fundamental result of Functional Analysis called the *Riesz Representation Theorem*: it says that every bounded linear functional $F : H \to \mathbb{R}$ is of the form $F(v) = (f, v)$ for some $f \in H$ that depends only on F (not on v), where (f, v) denotes the inner product of f and v. Thus the right side of (5.5) can be written as (f, v) for some $f \in H$ as a direct application of the mentioned theorem.

We also rewrite the left side in a similar way: For each fixed $u \in H$, we observe that $v \mapsto B(u, v)$ is a bounded linear functional from H to $\mathbb{R}$, and thus of the form $(T(u), v)$ for some $T(u) \in H$ depending only on u (and the fixed bilinear form B). Since $B(u, v)$ is linear in u, it easily follows that $u \mapsto T(u)$ is linear as well, and the boundedness of the form B implies the boundedness of operator $T : H \to H$. So $B(u, v) = (Tu, v)$ for some bounded linear operator T, and (5.5) is equivalent to $(Tu, v) = (f, v)$ for all $v \in H$, or more simply to $Tu = f$. It remains to check that $T : H \to H$ is a bijection, so that the unique solution is given by $u = T^{-1}f$.

To this end, we record the following consequence of coercivity:

$$c\|u\|^2 \leq B(u, u) = (Tu, u) \leq \|Tu\|\,\|u\|,$$

and hence

$$c\|u\| \leq \|Tu\| \leq C\|u\| \qquad \forall u \in H, \tag{5.6}$$

using also the boundedness of T. From this it is immediate that T is an *injection*: if $Tu_1 = Tu_2$, then $u = u_1 - u_2$ satisfies $Tu = 0$ and hence $u = 0$. Thus $T : H \to H'$ is a bijection from H to its range H', which is easily seen to be a subspace of H by the linearity of T.

This subspace is also closed: If Tu_k converges to some $h \in H$, then (5.6) shows that $c\|u_k - u_j\| \leq \|Tu_k - Tu_j\| \to 0$; thus u_k is a Cauchy sequence and hence convergent to a limit $u \in H$. But then (5.6) shows that $Tu_k \to Tu$, so that $h = Tu \in H'$, proving the closedness of H'.

We finally check that $H = H'$ by contraposition. If this is not the case, then there is some nonzero vector $y \in H$ that is orthogonal to the closed subspace H', i.e., we have $(Tu, y) = 0$ for all $u \in H$. But, choosing $u = y$, we deduce that $c\|y\|^2 \leq (Ty, y) = 0$, contradicting the choice of $y \neq 0$, and thus H' must be all of H.

Altogether, we have established that $T : H \to H$ is a bijection, and therefore $Tu = f$ has the unique solution $u = T^{-1}f$. □

Proof of Theorem 5.1. We first make the following reduction: Writing $w := u - \varphi$, the original problem is equivalent to finding a $w \in W^{1,2}(\Omega)$ that satisfies

$$\begin{cases} Lw = Lu - L\varphi = g + \nabla \cdot (\vec{f} - A\nabla\varphi) =: g + \nabla \cdot \vec{h} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

Since $\varphi \in W^{1,2}(\Omega)$, we have $\nabla\varphi \in L^2(\Omega; \mathbb{R}^n)$, and since A is uniformly bounded, also $A\nabla\varphi \in L^2(\Omega; \mathbb{R}^n)$. Thus $\vec{h}$ satisfies the same assumptions as the original $\vec{f}$.

Thus, we may assume without loss of generality that the boundary function $\varphi = 0$, so that we are looking for a solution $u \in W_0^{1,2}(\Omega)$. The strategy is to put this case of the Dirichlet problem (5.3) into the framework of the Lax–Milgram lemma. Indeed, on the left we have

$$\mathscr{L}(u, \phi) := \int A\nabla u \cdot \nabla\phi,$$

which was already checked to be a bounded bilinear form on $W^{1,2}(\Omega)$ (and hence also on the subspace $W_0^{1,2}(\Omega)$) in Lemma 3.4. Moreover, we have

$$\mathscr{L}(u, u) = \int A\nabla u \cdot \nabla u \geq \int \lambda|\nabla u|^2 = \lambda\|\nabla u\|_{L^2}^2 \gtrsim \|u\|_{W^{1,2}} \qquad \forall u \in W_0^{1,2}(\Omega)$$

by Corollary 4.9 in the last step. Hence $\mathscr{L} : W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \to \mathbb{R}$ is a bounded, coercive bilinear form.

We turn to the right side of (5.3), which we suggestively denote by

$$F(\phi) := \int [g\phi - \vec{f} \cdot \nabla\phi],$$

which is linear in ϕ and satisfies

$$|F(\phi)| \leq \int [|g||\phi| + |\vec{f}||\nabla\phi|] \leq \|g\|_{L^2}\|\phi\|_{L^2} + \|\vec{f}\|_{L^2}\|\nabla\phi\|_{L^2} \leq (\|g\|_{L^2} + \|\vec{f}\|_{L^2})\|\phi\|_{W^{1,2}},$$

so that this is bounded on $W^{1,2}(\Omega)$ and in particular on its subspace $W_0^{1,2}(\Omega)$.

Altogether, the problem (5.3) with $\varphi = 0$ is seen to be: Find $u \in W_0^{1,2}(\Omega)$ such that

$$\mathscr{L}(u, \phi) = F(\phi) \quad \forall \phi \in C_c^1(\Omega),$$

where $\mathscr{L}$ and F are as in the Lax–Milgram lemma on the Hilbert space $H = W_0^{1,2}(\Omega)$. The only deviation from Lax–Milgram is that ϕ ranges over the subspace $C_c^1(\Omega)$ instead of all $W_0^{1,2}(\Omega)$, but the continuity of both sides in ϕ implies that requiring the identity on the dense subspace $C_c^1(\Omega)$

is equivalent to requiring it for all $\phi \in W_0^{1,2}(\Omega)$. Thus Theorem 5.1 follows from the Lax–Milgram lemma. □

Exercise 5.7. Show that the condition that $g \in L^2(\Omega)$ in Theorem 5.1 can be relaxed (preserving the validity of the conclusions of the theorem) to $g \in L^q(\Omega)$, where

$$\begin{cases} q = 2n/(n+2), & \text{if } n > 2, \\ q > 1, & \text{if } n = 2. \end{cases}$$

Hint: Note that the assumptions on g were used in checking that the functional F is bounded on $W_0^{1,2}(\Omega)$, and use the Sobolev embedding and Hölder's inequality to see that the new assumptions are enough here.

6. Boundedness of solutions

Now that we know that the functions we are studying (solutions of elliptic equations) actually exist, we are on a firmed ground to continue the investigation of their properties. In this section, we prove the following:

Theorem 6.1. *Let $\vec{f} \in L^q(\Omega;\mathbb{R}^n)$ and $g \in L^{q/2}(\Omega)$ for some $q > n$, and let $u \in W^{1,2}(\Omega)$ satisfy*

$$Lu \geq \nabla \cdot \vec{f} + g \text{ in } \Omega, \qquad u \leq 0 \text{ on } \partial\Omega$$

in the weak sense, i.e.,

$$\int_\Omega A\nabla u \cdot \nabla \phi \leq \int_\Omega (\vec{f} \cdot \nabla \phi - g\phi) \quad \forall 0 \leq \phi \in C_c^1(\Omega), \qquad u^+ \in W_0^{1,2}(\Omega). \tag{6.2}$$

Then

$$\sup_\Omega u \leq C(\|u^+\|_{L^2} + k), \qquad k = \lambda^{-1}(\|\vec{f}\|_q + \|g\|_{q/2}),$$

where $C = C(n, q, |\Omega|)$.

In particular, if u is a solution of the equation (not just inequality)

$$Lu = \nabla \cdot \vec{f} + g \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega,$$

then u satisfies the assumptions of the theorem as stated, and $-u$ with $\vec{f}$ and g replaced by $-\vec{f}$ and $-g$. Thus we have

$$\|u\|_{L^\infty} = \sup_\Omega |u| = \max\Big(\sup_\Omega u, \sup_\Omega(-u)\Big) \leq C\big(\max(\|u^+\|_{L^2}, \|u^-\|_{L^2}) + k\big) \leq C(\|u\|_{L^2} + k).$$

Thus the solution u, which was only assumed to be in $W_0^{1,2}(\Omega)$ to begin with, also belongs to $L^\infty(\Omega)$. This is much better than the integrability conditions for $W_0^{1,2}(\Omega)$ functions coming from Sobolev's inequality; the solutions of elliptic equations are better than just arbitrary Sobolev functions!

Exercise 6.3. Use the same reduction to zero boundary values as in the proof of Theorem 5.1 to see what you can say, with the help of Theorem 6.1 about the boundedness of the solutions to

$$\begin{cases} Lu = \nabla \cdot \vec{f} + g, & \text{in } \Omega, \\ u = \varphi, & \text{on } \partial\Omega. \end{cases}$$

What assumptions on φ do you need to estimate $\|u\|_\infty$, and what is the bound that you get?

The claim of Theorem 6.1 is trivial if $\sup_\Omega u < 0$, and otherwise this supremum is the same as $\|u^+\|_\infty$. Our approach to this L^∞ norm is via the L^p norms with finite $p \to \infty$, namely:

Lemma 6.4. *For any measurable function w, we have $\|w\|_\infty \leq \liminf_{q\to\infty} \|w\|_q$.*

Proof. If $\|w\|_\infty > \lambda$, then $|w(x)| > \lambda$ for all x in some set E with positive measure $|E| > 0$. Hence

$$\|w\|_q \geq \|\lambda 1_E\|_q = \lambda|E|^{1/q} \to \lambda \quad \text{as } q \to \infty.$$

Thus $\liminf_{q\to\infty} \|w\|_q \geq \lambda$. Being valid for all $\lambda < \|w\|_\infty$, this shows that $\liminf_{q\to\infty} \|w\|_q \geq \|w\|_\infty$. □

This leads to the following criterion for estimating L^∞ norms:

Lemma 6.5 (Moser's iteration)**.** *With $p \in [1,\infty)$, $\kappa > 1$ and $C > 0$ fixed, suppose that a function $w \in L^p(\Omega)$ satisfies*
$$\|w\|_{\kappa\beta p} \leq (C\beta)^{1/\beta}\|w\|_{\beta p} \qquad \forall \beta \in [1,\infty).$$
Then $w \in L^\infty(\Omega)$, and $\|w\|_\infty \leq C'\|w\|_p$, where $C' = C'(C,\kappa)$.

Proof. We apply the assumption with $\beta = \kappa^j$ for all $j = 0,1,2,\ldots$:
$$\|w\|_{\kappa^{j+1}p} \leq (C\kappa^j)^{\kappa^{-j}}\|w\|_{\kappa^j p}.$$
Observing that the second factor on the right has the same form as the left side, but with j in place of $j+1$, we can iterate this estimate, to the results that
$$\|w\|_{\kappa^N p} \leq \Big(\prod_{j=0}^{N-1}(C\kappa^j)^{\kappa^{-j}}\Big)\|w\|_p.$$
The product can be written as
$$\prod_{j=0}^{N-1}(C\kappa^j)^{\kappa^{-j}} = C^{\sum_{j=0}^{N-1}\kappa^{-j}}\kappa^{\sum_{j=0}^{\infty}j\kappa^{-j}} \leq C^\sigma\kappa^\tau,$$
where
$$\sigma := \sum_{j=0}^{\infty}\kappa^{-j} < \infty, \qquad \tau := \sum_{j=0}^{\infty}j\kappa^{-j} < \infty.$$
This shows that $\|w\|_{\kappa^N p} \leq C'\|w\|_p$ with $C' = C^\sigma\kappa^\tau = C'(C,\kappa)$, uniformly in N. Since $\kappa^N p \to \infty$ as $N \to \infty$, the claim follows from Lemma 6.4. □

We also record a number of further lemmas on Sobolev functions:

Lemma 6.6. *Let $f \in C^1(\mathbb{R})$, $f' \in L^\infty(\mathbb{R})$ and $f(0) = 0$. If $u \in W_0^{1,p}(\Omega)$, then $f(u) \in W_0^{1,p}(\Omega)$.*

Proof. By definition, there is a sequence $u_k \in C_c^1(\Omega)$ such that $u_k \to u$ in $W^{1,p}(\Omega)$. By Exercise 2.5, we also have that $f \circ u_k \to f \circ u$ in $W^{1,p}(\Omega)$, at least for a subsequence. It suffices to check that $f \circ u_k \in C_c^1(\Omega)$, for this shows that $f \circ u$ belongs to the $W^{1,p}$-closure of $C_c^1(\Omega)$. That $f \circ u_k$ belongs to $C^1(\Omega)$ follows from the chain rule, since both f and u_k are C^1 functions. Also, since $u_k = 0$ outside a compact set K_k, and $f(0) = 0$, we also have that $f \circ u_k = 0$ outside the same compact set K_k. Thus $f \circ u_k$ is also compactly supported. □

Lemma 6.7. *If $u \in W^{1,p}(\Omega)$ and $f_\varepsilon(t)$ is the C^1 approximation of t^+ from the proof of Lemma 2.6, then $f_\varepsilon \circ u \to u^+$ in $W^{1,p}(\Omega)$.*

Proof. Recall that
$$f_\varepsilon(t) = \begin{cases}(t^2+\varepsilon^2)^{1/2} - \varepsilon, & \text{if } t > 0,\\ 0, & \text{if } t \leq 0,\end{cases} \qquad f'_\varepsilon(t) = \frac{t^+}{(t^2+\varepsilon^2)^{1/2}}.$$
Then $0 \leq f_\varepsilon(t) \leq t^+$ and $f_\varepsilon(t) \to t^+$ as $\varepsilon \to 0$, and $0 \leq f'_\varepsilon(t) \leq 1_{(0,\infty)}(t)$ and $f'_\varepsilon(t) \to 1_{(0,\infty)}(t)$ as $\varepsilon \to 0$. Thus
$$\int |f_\varepsilon(u) - u^+|^p \to 0, \qquad \int |\nabla(f_\varepsilon \circ u) - \nabla u^+|^p = \int |(f'_\varepsilon(u) - 1_{(0,\infty)}(u))\nabla u|^p \to 0,$$
by the pointwise convergence of the integrands and the dominated convergence theorem. □

Lemma 6.8. *If $u \in W_0^{1,p}(\Omega)$, then also $u_N := \min(u,N)$ belongs to $W_0^{1,p}(\Omega)$ for every $N > 0$.*

Proof. We have $u_N = \min(u,N) = u - \max(u-N,0) = u - (u-N)_+$. Let $f_\varepsilon(t)$ be the C^1 approximation of t^+ form the previous Lemma. Then $f_\varepsilon(u-N) \to (u-N)_+$ as $\varepsilon \to 0$, and thus $u_N = \lim_{\varepsilon\to 0}(u - f_\varepsilon(u-N)) = \lim_{\varepsilon\to 0} g_{\varepsilon,N} \circ u$, where $g_{\varepsilon,N}(t) = t - f_\varepsilon(t-N) \in C^1(\mathbb{R})$, $g'_{\varepsilon,N}(t) = 1 - f'_\varepsilon(t-N) \in L^\infty(\mathbb{R})$, and $g_{\varepsilon,N}(0) = 0 - f_\varepsilon(-N) = 0$. By Lemma 6.6, $g_{\varepsilon,N} \circ u \in W_0^{1,p}(\Omega)$, and hence $u_N \in W_0^{1,p}(\Omega)$ as the limit of these functions. □

Now we are ready for:

Proof of Theorem 6.1. With the idea of applying Moser's iteration in mind, we would like to estimate the integral of $(u^+)^\beta$ for large value of β. We first do this with u^+ replaced by $u_N^+ = \min(u^+, N)$, which takes its values between 0 and N.

There is slight technical problem in that the function $f(t) = t^\beta$, for $\beta > 1$, does not satisfy the condition $f' \in L^\infty$ required by many of our results dealing with $f \circ u$. To fix this problem, for auxiliary numbers $\beta \geq 1$ and $N > 0$, we consider the modified power function

$$H(t) := \begin{cases} \beta k^{\beta-1} t, & t < 0, \\ (k+t)^\beta - k^\beta, & t \in [0, N], \\ (k+N)^\beta - k^\beta + \beta(k+N)^{\beta-1}(t-N), & t > N. \end{cases}$$

The main part is that for $t \in [0, N]$, and in fact $H(u_N^+) = (k + u^+)^\beta - k^\beta$, since u_N^+ takes all its values on $[0, N]$. Outside this interval, we have chosen the unique affine extensions that make H globally C^1, i.e., we have matched the one-sided limits of both H and H' at $t \in \{0, N\}$. Due to these affine extensions off the finite interval, the derivative H' is constant on $(-\infty, 0)$ and (N, ∞), and we see that $H' \in L^\infty$. We also have that $H(0) = 0$. Since $u^+ \in W_0^{1,2}(\Omega)$ by assumption, we have $u_N^+ \in W_0^{1,2}(\Omega)$ by Lemma 6.8, and then also that $H(u_N^+) \in W_0^{1,2}(\Omega)$ by Lemma 6.6.

We also define another auxiliary function

$$G(t) := \int_0^t H'(s)^2 \,\mathrm{d}s.$$

As the primitive of a continuous function, G is also C^1, and moreover $G' = (H')^2 \in L^\infty$ and $G(0) = 0$, so that $G(u_N^+) \in W_0^{1,2}(\Omega)$ as well. Since H' and hence G' is increasing on $\mathbb{R}_+$, we also have $G(t) \leq tG'(t)$ for $t > 0$.

By density and continuity, we may take any nonnegative $v \in W_0^{1,2}(\Omega)$ in place of $\phi \in C_c^1(\Omega)$ in (6.2). We choose $v = G \circ u_N^+$. Then (6.2) reads as

$$\begin{aligned} \int A\nabla u \cdot G'(u_N^+)\nabla u_N^+ &\leq \int [\vec{f} \cdot G'(u_N^+)\nabla u_N^+ - gG(u_N^+)] \\ &\leq \int [|\vec{f}|G'(u_N^+)|\nabla u_N^+| + |g|G'(u_N^+)u_N^+]. \end{aligned} \tag{6.9}$$

Since $\nabla u_N^+ = 1_{(0,N)}(u)\nabla u$ by Theorem 2.8, we have $\nabla u = \nabla u_N^+$ on the support of the latter, and the left side above is

$$\int A\nabla u \cdot G'(u_N^+)\nabla u_N^+ = \int G'(u_N^+) A\nabla u_N^+ \cdot \nabla u_N^+ \geq \lambda \int G'(u_N^+)|\nabla u_N^+|^2$$

On the other hand,

$$|\vec{f}|G'(u_N^+)|\nabla u_N^+| = \lambda^{-1/2}|\vec{f}|G'(u_N^+)\lambda^{1/2}|\nabla u_N^+| \leq \frac{1}{2}G'(u_N^+)\Big(\lambda|\nabla u_N^+|^2 + \frac{1}{\lambda}|\vec{f}|^2\Big).$$

Absorbing the integral of the first term to the left of (6.9), we get

$$\begin{aligned} \int G'(u_N^+)|\nabla u_N^+|^2 &\leq \int G'(u_N^+)\Big(\frac{1}{\lambda^2}|\vec{f}|^2 + \frac{2}{\lambda}|g|u^+\Big) \\ &\leq \int G'(u_N^+)(u_N^+ + k)^2\Big(\frac{|\vec{f}|^2}{(\lambda k)^2} + 2\frac{|g|}{k}\Big), \end{aligned}$$

using $1 \leq (u_N^+ + k)/k$ and $u_N^+ \leq u_N^+ + k$ in the last step.

Recalling that $G' = (H')^2$, we find that

$$\int G'(u_N^+)|\nabla u_N^+|^2 = \int |H'(u_N^+)\nabla u_N^+|^2 = \int |\nabla H(u_N^+)|^2 \geq \frac{1}{C^2}\|H(u_N^+)\|_{2n/(n-2)}^2,$$

provided that dimension $n > 2$, where the Sobolev embedding is valid, since $H(u_N^+) \in W_0^{1,2}(\Omega)$, as already observed. On the other hand, by Hölder's inequality,

$$\int G'(u_N^+)(u_N^+ + k)^2\Big(\frac{|\vec{f}|^2}{(\lambda k)^2} + 2\frac{|g|}{\lambda k}\Big) \leq \|H'(u_N^+)(u_N^+ + k)\|_{2q/(q-2)}^2 \Big\|\frac{|\vec{f}|^2}{(\lambda k)^2} + 2\frac{|g|}{\lambda k}\Big\|_{q/2},$$

and the last factor is dominated by

$$\frac{1}{k^2}\Big\|\frac{|\vec{f}|}{\lambda}\Big\|_q^2 + \frac{2}{k}\Big\|\frac{|g|}{\lambda}\Big\|_{q/2} \leq 1 + 2 = 3$$

by the choice of k. So altogether,

$$\|H(u_N^+)\|_{2n/(n-2)} \leq C\|H'(u_N^+)(u_N^+ + k)\|_{2q/(q-2)}.$$

Since u_N^+ takes its values on the interval $[0, N]$, where H is given by $H(t) = (t + k)^\beta - k^\beta$, we can rewrite the above line as

$$\|(u_N^+ + k)^\beta - k^\beta\|_{2n/(n-2)} \leq C\|\beta(u_N^+ + k)^{\beta-1}(u_N^+ + k)\|_{2q/(q-2)},$$

or in other words

$$\|w^\beta\|_{2n/(n-2)} := \|(u_N^+ + k)^\beta\|_{2n/(n-2)} \leq C\beta\|w^\beta\|_{2q/(q-2)} + \|k^\beta\|_{2n/(n-2)} \leq C'\beta\|w^\beta\|_{2q/(q-2)},$$

since all L^p norms of the constant k^β are comparable (with constant depending only on $|\Omega|$), and $k \leq w$. Using the simple identity $\|w^\beta\|_{L^p} = \|w\|_{L^{p\beta}}^\beta$, the above estimate can be further rewritten as

$$\|w\|_{\beta\kappa p} \leq (C'\beta)^{1/\beta}\|w\|_{\beta p},$$

where

$$p := \frac{2q}{q-2}, \qquad \kappa := \frac{2n}{n-2}\frac{q-2}{2q} = \frac{\frac{1}{2} - \frac{1}{q}}{\frac{1}{2} - \frac{1}{n}} > 1,$$

since $q > n$. But this is precisely the assumption of Moser's iteration Lemma 6.5, and the conclusion of that Lemma says that

$$\|w\|_\infty \leq C\|w\|_p.$$

This can be further improved by observing that

$$\|w\|_p = \Big(\int w^p\Big)^{1/p} \leq \Big(\int \|w\|_\infty^{p-2} w^2\Big)^{1/p} = \|w\|_\infty^{1-2/p}\|w\|_2^{2/p};$$

hence $\|w\|_\infty \leq C\|w\|_\infty^{1-2/p}\|w\|_2^{2/p}$. Diving both sides by $\|w\|_\infty^{1-2/p}$ (which is finite, since $w = u_N^+$ is pointwise bounded by N) and simplifying, we arrive at

$$\|w\|_\infty \leq C\|w\|_2,$$

and hence

$$\|u_N^+\|_\infty \leq \|w\|_\infty \leq C\|w\|_2 \leq C(\|u_N^+\|_2 + k).$$

Letting $N \to \infty$, we deduce the statement of the Theorem by monotone convergence.

This completes the proof for $n > 2$, and the case $n = 2$ is left as an exercise. □

Exercise 6.10. Complete the proof of Theorem 6.1 by presenting the necessary modifications for the case that the dimension is $n = 2$. Hint: Check that the Sobolev inequality $\|v\|_{2n/(n-2)} \leq C\|\nabla v\|_2$ can now be replaced by $\|v\|_t \leq C\|\nabla v\|_2$ for any $t < \infty$, where C depends on t and $|\Omega|$.

7. Harnack's inequality

We now move from the study of global properties of solutions to *local properties*. By global, we understand estimates dealing with the entire domain Ω, such as the statement of Theorem 6.1, where we estimate the norm of u^+ in $L^\infty(\Omega)$. In contrast to this, local properties are about the behaviour of the solutions in, say, some balls B contained in Ω. Of course, if a solution is globally bounded on Ω, it is trivially bounded in every ball $B \subset \Omega$ as well, but the point of the local theory is that we may expect some more precise information on the local scale.

To simplify the technicalities, from this point on we only deal with solutions to $Lu = 0$, instead of the subsolutions $Lu \geq \nabla \cdot \vec{f} + g$ (so we have both replaced "$\geq$" by "$=$", and the functions on the right by zero. In the local theory, the boundary values on $\partial\Omega$ are less important, since we are only dealing with $B \subset \Omega$, which may be far away from the boundary.

The main goal of this section is the following theorem:

Theorem 7.1 (Harnack's inequality)**.** *Let $u \in W^{1,2}(\Omega)$ be a solution of $Lu = 0$ in Ω. Suppose moreover that $u \geq 0$ on some ball $4B = B(y, 4R) \subset \Omega$. Then*

$$\sup_B u \leq C \inf_B u,$$

where $C = C(n, \Lambda/\lambda)$.

In fact, we establish this via several intermediate subgoals, namely: under the same assumptions, we have

$$\sup_B u \leq C_p \Big(\fint_{2B} u^p \Big)^{1/p} \qquad \forall\, p > 0 \tag{7.2}$$

$$\Big(\fint_{2B} u^p \Big)^{1/p} \leq C_p \inf_B u \qquad \forall\, p < 0 \tag{7.3}$$

$$\Big(\fint_{2B} u^p \Big)^{1/p} \leq C_p \Big(\fint_{2B} u^{-p} \Big)^{-1/p} \qquad \text{for some } p > 0. \tag{7.4}$$

It is clear that a combination of these estimates will prove Harnack's inequality. (Above, we have denoted by $\fint_B \cdots := \frac{1}{|B|} \int_B \cdots$ the "average integral" over B)

As always, the starting point is the choice of a clever test function in the defining formula

$$\int A\nabla u \cdot \nabla v = 0 \qquad \forall\, v \in W_0^{1,2}(\Omega).$$

The relevant choice now is

$$v = \eta^2 \bar{u}^\beta, \qquad \bar{u} := u + k,$$

where $\eta \in C_c^1(4B)$ and $k > 0$ is an auxiliary parameter that is eventually taken to the limit $k \to 0$. In principle, we would like to consider all $\beta \in \mathbb{R} \setminus \{0\}$, but there are technical obstacles similar to those in the proof of Theorem 6.1 about the behaviour of the function $(t + k)^\beta$. Let us take

$$f(t) := \begin{cases} k^\beta + \beta k^{\beta-1} t, & \text{if } t < 0, \\ (t + k)^\beta, & \text{if } t \geq 0, \end{cases}$$

so that this is a C^1 extension of $(t+k)^\beta$ from $[0, \infty)$ to all $\mathbb{R}$. Now it is immediate that $f' \in L^\infty(\mathbb{R})$ if and only if $\beta \leq 1$, and in this case $\|f'\|_\infty = |\beta| k^{\beta-1}$. We concentrate for the moment on this case only.

Since $u \in W^{1,2}(\Omega)$ and $f \in C^1(\mathbb{R})$ with $f' \in L^\infty(\mathbb{R})$, we have $\bar{u}^\beta = f \circ u \in W^{1,2}(\Omega)$ as well. Since η, and hence η^2, belongs to $C_c^1(\Omega)$, we have $\eta^2 \bar{u}^\beta \in W_0^{1,2}(\Omega)$; indeed, $f \circ u = \lim_{k\to\infty} v_k$ for some $v_k \in C^1(\Omega)$, and thus $\eta^2 (f \circ u) = \lim_{k\to\infty} \eta^2 v_k$, where $\eta^2 v_k \in C_c^1(\Omega)$. So indeed $v = \eta^2 \bar{u}^k$ is a legal test function. Observe the first appearance of the local flavour through the multiplying function η^2; in the global theory, it was enough to deal with test functions of the form $f \circ u$ for suitably chosen f.

Now, by the product rule

$$\nabla v = \nabla(\eta^2 \bar{u}^\beta) = 2\eta \bar{u}^\beta \nabla\eta + \beta \eta^2 \bar{u}^{\beta-1} \nabla u,$$

hence

$$0=\int A\nabla u\cdot\nabla v=2\int \eta\bar u^{\beta}A\nabla u\cdot\nabla\eta+\beta\int\eta^2\bar u^{\beta-1}A\nabla u\cdot\nabla u,$$

thus the absolute values of the two terms are equal, and therefore

$$\lambda|\beta|\int\eta^2\bar u^{\beta-1}|\nabla u|^2\le|\beta|\int\eta^2\bar u^{\beta-1}A\nabla u\cdot\nabla u\le 2\int\eta\bar u^{\beta}|A\nabla u\cdot\nabla\eta|\le 2\Lambda\int\eta\bar u^{\beta}|\nabla u||\nabla\eta|.$$

We split the right side as follows:

$$2\eta\bar u^{\beta}|\nabla u||\nabla\eta|=2\varepsilon^{1/2}\eta\bar u^{(\beta-1)/2}|\nabla u|\times\varepsilon^{-1/2}\bar u^{(\beta+1)/2}|\nabla\eta|\le\varepsilon\eta^2\bar u^{\beta-1}|\nabla u|^2+\varepsilon^{-1}\bar u^{\beta+1}|\nabla\eta|^2 \quad (7.5)$$

Choosing ε so that $\Lambda\varepsilon=\frac12\lambda|\beta|$, we can absorb the integral of the first term to the left, arriving at

$$\frac12\lambda|\beta|\int\eta^2\bar u^{\beta-1}|\nabla u|^2\le\frac{2\Lambda^2}{\lambda|\beta|}\int\bar u^{\beta+1}|\nabla\eta|^2. \quad (7.6)$$

To proceed, we would like to identify $\bar u^{(\beta-1)/2}\nabla u$ as a gradient of something. Note that the function

$$w:=\begin{cases}\bar u^{(\beta+1)/2}, & \beta\ne-1,\\ \log\bar u, & \beta=-1,\end{cases}$$

satisfies

$$\nabla w=\begin{cases}\frac12(\beta+1)\bar u^{(\beta-1)/2}\nabla u, & \beta\ne-1,\\ \bar u^{-1}\nabla u=\bar u^{(\beta-1)/2}\nabla y, & \beta=-1.\end{cases}$$

Thus (7.6) can be rewritten as

$$\int\eta^2|\nabla w|^2\le c_\beta\int w^2|\nabla\eta|^2,\quad c_\beta:=\Big(\frac{\Lambda(\beta+1)}{\lambda\beta}\Big)^2,\qquad \beta\ne-1, \quad (7.7)$$

and

$$\int\eta^2|\nabla w|^2\le c_{-1}\int|\nabla\eta|^2,\quad c_{-1}:=\Big(\frac{2\Lambda}{\lambda}\Big)^2,\qquad\beta=-1. \quad (7.8)$$

We make some observations: The constant c_β is uniformly bounded for all $|\beta|\ge\beta_0>0$. Also, there is a qualitative difference between (7.7) and (7.8): the function w appear on the right of the former, but not of the latter. We shall use (7.7) to prove the bounds (7.2) and (7.3), while (7.8) will be crucial for (7.4).

Let's proceed with (7.7). By Sobolev's inequality (for dimension $n>2$, with a necessary modification for $n=2$), we have

$$\|\eta w\|_{2n/(n-2)}\le C\|\nabla(\eta w)\|_2\le C\|\eta\nabla w\|_2+C\|w\nabla\eta\|_2\le C_\beta\|w\nabla\eta\|_2. \quad (7.9)$$

Here we used the fact that $w=\bar u^{(\beta+1)/2}\in W^{1,2}$ (since $(\beta+1)/2\le1$) so that $\eta w\in W^{1,2}_0$ for $\eta\in C^1_c$.

Next, we make a choice of η (which was so far only specified to be in $C^1_c(4B)$): For concentric balls $B_{r_i}=B(y,r_i)$ with $R\le r_1<r_2\le 2R$, we choose η so that $\eta\equiv1$ in B_{r_1}, $\eta\equiv0$ outside B_{r_2}, and $|\nabla\eta|\le 2/(r_2-r_1)$ in $B_{r_2}\setminus B_{r_1}$. Then (7.9) gives

$$\|w\|_{L^{2n/(n-2)}(B_{r_1})}\le\|\eta w\|_{L^{2n/(n-2)}}\le C_\beta\|w\nabla\eta\|_2\le\frac{C_\beta}{r_2-r_1}\|w\|_{L^2(B_{r_2})}.$$

Substituting $w=\bar u^{(\beta+1)/2}=:\bar u^{\gamma/2}$, where $\gamma:=\beta+1$, and using $\|w^\alpha\|_{L^p}=\|w\|^\alpha_{L^{p\alpha}}$ (with obvious meaning also when $p\alpha<1$, even $p\alpha<0$, although "$\|\ \|_{L^{p\alpha}}$" is no longer a norm in this case), we get

$$\|\bar u\|^{\gamma/2}_{L^{\gamma n/(n-2)}(B_{r_1})}\le\frac{C_\gamma}{r_2-r_1}\|\bar u\|^{\gamma/2}_{L^\gamma(B_{r_2})},\qquad\gamma=\beta+1\notin\{0,1\},$$

where the restrictions correspond to the forbidden values $\beta\notin\{-1,0\}$. Raising to the power $2/\gamma$, and observing the reversal of the inequality when raised to a negative power, we finally have

$$\|\bar u\|_{L^{\gamma n/(n-2)}(B_{r_1})}\le\Big(\frac{C_\gamma}{r_2-r_1}\Big)^{2/\gamma}\|\bar u\|_{L^\gamma(B_{r_2})},\qquad\gamma>0\ (\gamma\ne1), \quad (7.10)$$

which is used to prove (7.2), and

$$\|\bar{u}\|_{L^{\gamma}(B_{r_2})} \leq \Big(\frac{C_\gamma}{r_2-r_1}\Big)^{2/|\gamma|}\|\bar{u}\|_{L^{\gamma n/(n-2)}(B_{r_1})}, \qquad \gamma<0, \tag{7.11}$$

which will give us (7.3). As a matter of fact, we should also remember the restriction that $\beta\leq 1$, thus $\gamma\leq 2$, which means that (7.10) has been thus far only established in this range; however, it turns out that it is actually true in the full range as stated. Concerning the constant C_γ, we recall that this stays bounded when β stays away from 0, thus when γ stays away from 1. This is always true in (7.11), but in (7.10), the constant C_γ actually blows up when $\gamma\to 1$.

Both (7.10) and (7.11) should remind the reader of the assumption of Moser's iteration Lemma 6.5; however, there is the added complication that not only the exponents but also the domains B_{r_i} change on the two sides of the estimate. Thus, we cannot directly apply Lemma 6.5 as stated, but we can still adapt the same method. Altogether, one should not think of Moser's iteration as a certain fixed lemma, but rather as a technique that arises in different forms and different situations.

Exercise 7.12. Derive (7.3) from (7.11). Hint: Let $\kappa:=n/(n-2)$, and consider $\gamma:=\kappa^k p$, $r_2=R(1+2^{-k})$, $r_1=R(1+2^{-(k+1)})$, and recall that $C_\gamma\leq C$ for some fixed C. Follow Moser's iteration argument, and check that $\|u\|_{L^q(B)}\to\inf_B u$ as $q\to-\infty$.

Exercise 7.13. Derive (7.2) from (7.10), assuming that this would be already known for all γ as written. Hint: Adapt the hint of the previous exercise, and observe in addition the following: By Hölder's inequality, it is enough to consider a sequence of exponents $p\to 0$. Taking, for instance, $p_j=\kappa^{-1/2-j}$, it follows that $\kappa^k p_j$ is never too close to 1, so that the estimate (7.10) is uniform (i.e., $C_\gamma\leq C$) over such a choice of the parameter γ.

Exercise 7.14. Let $H\in C^1(\mathbb{R})$ with $H'\in L^\infty(\mathbb{R})$, $H(0)=0$, and both H and H' be positive and increasing on $[0,\infty)$. Under the assumptions of Theorem 7.1 and for $\eta\in C_c^1(4B)$, prove that

$$\|H(u)\eta\|_{2n/(n-2)}\leq C\|H(u)\nabla\eta\|_2. \tag{7.15}$$

Hint: Adapt the computations above with $v:=\eta^2 G(u)$, where $G(t):=\int_0^t H'(s)^2\,ds$. The analogue of the splitting estimate should give $G(u)^2/G'(u)$ in place of $\bar{u}^{\beta+1}$ in the second term. Check that this is dominated by $H(u)^2$.

We now apply the result of Exercise 7.14 to the function

$$H_N(t):=\begin{cases}0, & \text{if } t\leq 0,\\ t^\beta, & \text{if } 0<t\leq N,\\ N^\beta+\beta N^{\beta-1}(t-N), & \text{if } t>N,\end{cases}$$

where $\beta>1$ and $N>0$. One easily observes that this satisfies the assumptions of Exercise 7.14, and therefore the conclusion (7.15). It is also easy to check that, for each fixed $t\in\mathbb{R}$, $N\mapsto H_N(t)$ is an increasing function of N that tends to t^β for each $t>0$. Thus, by monotone convergence, (7.15) with $H=H_N$ and $N\to\infty$ implies that

$$\|u^\beta\eta\|_{2n/(n-2)}\leq C\|u^\beta\nabla\eta\|_2, \qquad \beta>1.$$

Choosing, as before concentric balls $B_{r_i}=B(y,r_i)$ with $R\leq r_1<r_2\leq 2R$ and $\eta\equiv 1$ in B_{r_1}, $\eta\equiv 0$ outside B_{r_2}, and $|\nabla\eta|\leq 2/(r_2-r_1)$, this implies

$$\|u^\beta\|_{L^{2n/(n-2)}(B_{r_1})}\leq\frac{C}{r_2-r_1}\|u^\beta\|_{L^2(B_{r_2})}, \qquad \beta>1.$$

Denoting $\beta=\gamma/2$ with $\gamma>2$ (so that $\beta>1$) and simplifying, this is the same as

$$\|u\|_{L^{\gamma n/(n-2)}(B_{r_1})}\leq\Big(\frac{C}{r_2-r_1}\Big)^{2/\gamma}\|u\|_{L^\gamma(B_{r_2})}, \qquad \gamma>2,$$

which is exactly the missing range of γ's from (7.10), which is now completely proven. (An alert reader may notice that we have just u above, in contrast to $\bar{u}=u+k$ in (7.10); however, the result for u immediately implies the same result for $\bar{u}$, since $\bar{u}$ satisfies exactly the same assumptions: we have $L\bar{u} = Lu = 0$ in Ω [since $Lk = 0$ for any constant k] and $\bar{u} \geq u \geq 0$ in $B(y, 4R) \subset \Omega$.)

According to Exercises 7.12 and 7.13, the bounds (7.2) and (7.3) follows from (7.10) and (7.11), both proven now, as already said. Thus, in order to complete the proof of Harnack's inequality, it remains to prove (7.4), which we will derive from (7.8).

In (7.8), we consider $\eta \equiv 1$ in B_r, $\eta \equiv 0$ outside B_{2r}, and $|\nabla \eta| \leq 2/r$, where the balls $B_r = B(z, r) \subset B_{2r} = B(z, 2r) \subset 4B = B(y, 4R)$ need not be concentric with the reference ball $B = B(y, R)$. Then (7.8) implies that

$$\int_{B_r} |\nabla w| \leq \Big(\int_{B_r} |\nabla w|^2 \Big)^{1/2} |B_r|^{1/2} \leq C \Big(\int_{B_{2r}} \Big(\frac{2}{r}\Big)^2 \Big)^{1/2} r^{n/2} \leq C \big(r^n \cdot r^{-2}\big)^{1/2} r^{n/2} = C r^{n-1}$$

for any such $B_r \subset B_{2r} \subset 4B$. This further implies that:

Lemma 7.16. *For any ball $B_r = B(z, r)$, we have*

$$\int_{B_r \cap 2B} |\nabla w| \leq C r^{n-1}. \tag{7.17}$$

Proof. Let us consider three possibilities for the ball B_r:

(a) $B_{2r} \subset 4B$: In this case we already checked the bound (7.17).
(b) $B_r \cap 2B = \varnothing$: In this case the bound is trivial, since the integration domain is empty.
(c) We are in the complement of both (a) and (b).

It remains to consider case (c), in which case we have both $B_{2r} \cap (4B)^c \neq \varnothing$ and $B_{2r} \cap 2B \supset B_r \cap 2B \neq \varnothing$. Thus B_{2r} meets both $2B = B(y, 2R)$ and $(4B)^c$, so the diameter $4r$ of B_{2r} must be larger than $\operatorname{dist}(2B, (4B)^c) = 2R$, hence $2r > R$. On the other hand, the ball $2B$ clearly satisfies $2 \cdot 2B \subset 4B$ (indeed, with equality), so applying the already known case (a), we have in case (c) that

$$\int_{B_r \cap 2B} |\nabla w| \leq \int_{2B} |\nabla w| \leq C(2R)^{n-1} \leq 4^n C \cdot r^{n-1},$$

so (7.17) holds even in this case, possibly with a larger constant C. □

The condition (7.17) actually has a name; it says that $|\nabla w|$ belongs to the *Morrey space* $M^n(2B)$, which will be studied more thoroughly in the next section. At this point, we take for granted the following consequence (to be proven in detail in the next section) of (7.17), and use it to complete the proof of Harnack's inequality:

Lemma 7.18. *Under the condition* (7.17)*, the function $w \in W^{1,1}(2B)$ also satisfies*

$$\fint_{2B} \exp\Big(c|w - \langle w \rangle_{2B}|\Big) \leq C, \qquad \langle w \rangle_{2B} := \fint_{2B} w. \tag{7.19}$$

Proof. This will be a consequence of Theorem 8.4 below. □

Now, since $|x|$ dominates both x and $-x$, and exp is increasing, (7.19) implies the pair of estimates

$$\fint_{2B} e^{cw} e^{-c\langle w \rangle_{2B}} \leq C, \qquad \fint_{2B} e^{-cw} e^{c\langle w \rangle_{2B}} \leq C,$$

or, multiplying the constant factors $e^{\pm c\langle w \rangle_{2B}}$ to the right side,

$$\fint_{2B} e^{cw} \leq C e^{c\langle w \rangle_{2B}}, \qquad \fint_{2B} e^{-cw} \leq C e^{-c\langle w \rangle_{2B}}.$$

If we multiply these two inequalities together, we arrive at

$$\fint_{2B} e^{cw} \fint_{2B} e^{-cw} \leq C e^{c\langle w \rangle_{2B}} e^{-c\langle w \rangle_{2B}} = C.$$

Recalling that $w = \log \bar{u}$, we have $e^{\pm cw} = (e^w)^{\pm c} = \bar{u}^{\pm c} = (u + k)^{\pm c}$, and therefore we have

$$\fint_{2B} u^c \fint_{2B} (u + k)^{-c} \leq \fint_{2B} (u + k)^c \fint_{2B} (u + k)^{-c} \leq C.$$

Finally, letting $k \to 0$, we find that $(u+k)^{-c}$ increases monotonically to u^{-c}, and we conclude (7.4) with $p = c$ from the above estimate via the monotone convergence theorem. This completes the proof of Harnack's inequality, aside from the verification of Lemma 7.18, which will follow from a development of the theory of Morrey spaces in the next section.

8. Morrey spaces and Riesz potentials

Definition 8.1. A function $f \in L^1_{\mathrm{loc}}(\Omega)$ belongs to the Morrey space $M^p(\Omega)$ if

$$\int_{B_r\cap\Omega} |f| \leq K r^{n/p'} = K r^{n(1-1/p)}$$

for any ball $B_r = B(z,r)$, for some constant K independent of the ball. The smallest such K is denoted by $\|f\|_{M^p(\Omega)}$.

As indicated, the condition (7.17) says that $|\nabla w| \in M^n(2B)$; indeed, $n(1-1/n) = n-1$.

Remark 8.2. (a) We have $L^p(\Omega) \subset M^p(\Omega)$ for all $p \in [1,\infty]$; indeed,

$$\int_{B_r\cap\Omega} |f| = \int_\Omega |f| \cdot 1_{B_r} \leq \|f\|_p \|1_{B_r}\|_{p'} \leq c\|f\|_p \cdot r^{n/p'}.$$

(b) For $p \in \{1,\infty\}$, we have $M^p(\Omega) = L^p(\Omega)$. It only remains to check "$\subset$". For $p = 1$, we have $1/p' = 1/\infty = 0$, and the defining condition of $M^1(\Omega)$ says that $\int_{B_r\cap\Omega} |f| \leq K$. Letting $r \to \infty$ (with a fixed centre for B_r), this gives $\|f\|_{L^1(\Omega)} = \int_\Omega |f| \leq K$. If $p = \infty$, the defining condition of $M^\infty(\Omega)$ says, after dividing by $|B_r|$, that $\fint_{B_r} |f| \leq cK$. Considering $B_r = B(x,r)$ for each $x \in \Omega$, and letting $r \to 0$, this gives $|f(x)| \leq cK$ for a.e. $x \in \Omega$, by Lebesgue's differentiation theorem. Thus indeed $\|f\|_\infty \leq cK$.

Exercise 8.3. Let $\Omega = B(0,1)$ be the unit ball of $\mathbb{R}^n$, and $p \in (1,\infty)$. Check that the function $f(x) = |x|^{-n/p}$ belongs to the Morrey space $M^p(\Omega)$ but not to $L^p(\Omega)$.

Our main result about Morrey spaces will be the following:

Theorem 8.4. *Let $\Omega \subset \mathbb{R}^n$ be a convex domain, and $v \in W^{1,1}(\Omega)$ with $|\nabla v| \in M^n(\Omega)$. Then*

$$\int_\Omega \exp\Big(\frac{c}{\|\nabla v\|_{M^p}} \frac{|\Omega|}{(\operatorname{diam}\Omega)^n} |v - \langle v\rangle_\Omega|\Big) \leq C(\operatorname{diam}\Omega)^n.$$

Proof of Lemma 7.18 assuming Theorem 8.4. We consider the convex domain $\Omega = 2B$. In this case, $(\operatorname{diam}\Omega)^n = c|\Omega|$ for a dimensional constant $c = c_n$. Moreover, in the setting of Lemma 7.18, the norm $\|w\|_{M^p}$ is dominated by a constant C. Substituting all this information, we see that the conclusion of Theorem 8.4 reduces to the conclusion of Lemma 7.18 in this case. □

As a preparation for the proof of Theorem 8.4, we need to relate the function v to its gradient ∇v. This is similar to the proof of Sobolev's inequality, but we now require a different form of this relation, given by the following:

Lemma 8.5. *Let Ω be a convex domain, and $v \in W^{1,1}(\Omega)$. Then*

$$|v(x) - \langle v\rangle_\Omega| \leq \frac{(\operatorname{diam}\Omega)^n}{n|\Omega|} \int_\Omega \frac{|\nabla v(y)|}{|x-y|^{n-1}}\,\mathrm{d}y, \qquad \textit{for a.e. } x \in \Omega. \tag{8.6}$$

Proof in case $v \in C^1 \cap W^{1,1}$. In this case, we can write for $x, y \in \Omega$ the identity

$$\begin{aligned} v(x) - v(y) &= v(x) - v(x + |x-y|\omega_{yx}), \qquad \omega_{yx} := \frac{y-x}{|y-x|}, \\ &= -\Big|_{t=0}^{|x-y|} v(x + t\omega_{yx}) = -\int_0^{|x-y|} \partial_t v(x + t\omega_{yx})\,\mathrm{d}t. \end{aligned}$$

Note that all points $x + t\omega_{yx}$, $t \in [0, |x-y|]$, belong to Ω by convexity.

Taking the average integral over $y \in \Omega$ (for a fixed x), this implies

$$v(x) - \langle v\rangle_\Omega = -\fint_\Omega \int_0^{|x-y|} \partial_t u(x + t\omega_{yx})\,\mathrm{d}t\,\mathrm{d}y,$$

and hence

$$\begin{aligned}|v(x)-\langle v\rangle_\Omega| &\leq \fint_\Omega \int_0^{|x-y|} |\nabla v(x+t\omega_{yx})|\,\mathrm{d}t\,\mathrm{d}y \\ &= \fint_\Omega \int_0^{|x-y|} V(x+t\omega_{yx})\,\mathrm{d}t\,\mathrm{d}y, \qquad V(x):=|\nabla v(z)|1_\Omega(z), \\ &\leq \frac{1}{|\Omega|}\int_{B(x,d)}\int_0^\infty V(x+t\omega_{yx})\,\mathrm{d}t\,\mathrm{d}y, \qquad d:=\operatorname{diam}\Omega,\end{aligned}$$

where we estimated up by extending the integral from 0 to $|x-y|$ all the way to ∞, and replaced Ω by the larger set $B(x,d)\supset\Omega$.

We continue to express the integral over $y=B(x,d)$ in polar coordinates centred at x, so that $y=x+ru$, $r\in(0,d]$ and $u\in S^{n-1}$, the unit sphere. Note that in this case $\omega_{yx}=u$, and hence

$$\begin{aligned}\int_{B(x,d)}\int_0^\infty V(x+t\omega_{yx})\,\mathrm{d}t\,\mathrm{d}y &= \int_0^d\int_{S^{n-1}}\int_0^\infty V(x+tu)\,\mathrm{d}t\,\mathrm{d}\sigma(u) r^{n-1}\,\mathrm{d}r \\ &= \frac{d^n}{n}\int_{S^{n-1}}\int_0^\infty V(x+tu)\,\mathrm{d}t\,\mathrm{d}\sigma(u),\end{aligned}$$

by a direct computation of the integration in r. Now we can reinterpret the remaining integral as an integral in the polar coordinates (t,u), except that the factor t^{n-1} is missing. We multiply and divide by this to find that

$$\begin{aligned}\int_{S^{n-1}}\int_0^\infty V(x+tu)\,\mathrm{d}t\,\mathrm{d}\sigma(u) &= \int_{S^{n-1}}\int_0^\infty \frac{V(x+tu)}{t^{n-1}}t^{n-1}\,\mathrm{d}t\,\mathrm{d}\sigma(u) \\ &= \int_{\mathbb{R}^n}\frac{V(y)}{|x-y|^{n-1}}\,\mathrm{d}y = \int_\Omega \frac{|\nabla v(y)|}{|x-y|^{n-1}}\,\mathrm{d}y.\end{aligned}$$

A combination of the above estimates and identities yields precisely the claim in the considered case that $v\in W^{1,1}\cap C^1$. □

Before proving the lemma in the full generality of $v\in W^{1,1}$, we make some observations about the operator

$$I_\mu f(x) := \int \frac{f(y)}{|x-y|^{n(1-\mu)}}\,\mathrm{d}y, \qquad \mu\in(0,1), \tag{8.7}$$

which appears in (8.6) with $\mu=1/n$ and $f=|\nabla v|$. The operator I_μ is called the *Riesz potential* or a *fractional integral*, and it has a rich theory. We will only treat it to the extent that is necessary for our immediate needs.

Lemma 8.8. *For a bounded domain Ω, we have $I_\mu : L^1(\Omega)\to L^1(\Omega)$, and more precisely*

$$\|I_\mu f\|_{L^1(\Omega)} \leq \omega_n \frac{(\operatorname{diam}\Omega)^n}{\mu}\|f\|_{L^1(\Omega)} \qquad \forall\mu\in(0,1),$$

where $\omega_n=|B(0,1)|$ is the measure of the unit ball of $\mathbb{R}^n$.

Before the proof, let us record a simple relation between ω_n and the surface measure of the unit sphere $S^{n-1}=\{x\in\mathbb{R}^n : |x|=1\}$. By integrating in the polar coordinates, we have

$$\omega_n = \int_{B(0,1)}\mathrm{d}x = \int_0^1\int_{S^{n-1}}\mathrm{d}\sigma r^{n-1}\,\mathrm{d}r = \frac{1}{n}\int_{S^{n-1}}\mathrm{d}\sigma,$$

and thus $\int_{S^{n-1}}\mathrm{d}\sigma = n\omega_n$.

Proof of Lemma 8.8. Noting that $|I_\mu f(x)|\leq I_\mu(|f|)(x)$, it suffices to consider non-negative functions f. Then

$$\|I_\mu f\|_{L^1(\Omega)} = \int_\Omega I_\mu f(x)\,\mathrm{d}x = \int_\Omega\int_\Omega \frac{f(y)}{|x-y|^{n(1-\mu)}}\,\mathrm{d}y\,\mathrm{d}x = \int_\Omega\int_\Omega \frac{1}{|x-y|^{n(1-\mu)}}\,\mathrm{d}x f(y)\,\mathrm{d}y. \tag{8.9}$$

With $d := \operatorname{diam}\Omega$, we clearly have $\Omega \subset B(y,d)$ for every $y \in \Omega$, and hence

$$\int_\Omega \frac{1}{|x-y|^{n(1-\mu)}}\,\mathrm{d}x \leq \int_{B(0,d)} \frac{1}{|x-y|^{n(1-\mu)}}\,\mathrm{d}x = \int_0^d \int_{S^{n-1}} \mathrm{d}\sigma \frac{1}{r^{n(1-\mu)}} r^{n-1}\,\mathrm{d}r$$

$$\int_0^d r^{-n\mu-1}\,\mathrm{d}r \int_{S^{n-1}} \mathrm{d}\sigma = \frac{d^{n\mu}}{n\mu} n\omega_n = \frac{\omega_n}{\mu} d^{n\mu}.$$

Substituting this into (8.9), we obtain the result of the Lemma. □

Exercise 8.10. Show that the result of Lemma 8.8 remains valid under the weaker assumption that Ω has finite measure (instead of being a bounded domain), and give an explicit upper bound in terms of $|\Omega|$ instead of $\operatorname{diam}\Omega$. Hint: Check that

$$\int_\Omega \frac{1}{|x-y|^{n(1-\mu)}}\,\mathrm{d}x \leq \int_{B(y)} \frac{1}{|x-y|^{n(1-\mu)}}\,\mathrm{d}x,$$

where $B(y)$ is the ball at centre y such that $|B(y)| = |\Omega|$, and compute this integral in terms of $|\Omega|$.

Now we can provide:

Completion of the proof of Lemma 8.5 for general $v \in W^{1,1}$. In the language of the Riesz potentials, we should prove that

$$|v - \langle v\rangle_\Omega| \leq c_\Omega I_{1/n}(|\nabla v|), \tag{8.11}$$

where $c_\Omega = (\operatorname{diam}\Omega)^n/(n|\Omega|)$, and we already know that this is true for $v \in C^1(\Omega) \cap W^{1,1}(\Omega)$. This space is dense in $W^{1,1}(\Omega)$. (We haven't proven this density result in these lectures, but take it here for granted. Recall that $C^1_c(\Omega)$ is, by definition, dense in $W^{1,1}_0(\Omega)$, but the density that we now use is different: there is no subscript 'c' in C^1, and no subscript '0' in $W^{1,1}$.)

By the mentioned density, there exist $v_k \in C^1 \cap W^{1,1}$ such that $v_k \to v$ in the norm of $W^{1,1}$, thus in particular in the norm of L^1, and hence a subsequence converges almost everywhere. We still denote this subsequence simply by v_k. Also, the convergence $v_k \to v$ in $W^{1,1}$ implies that $|\nabla v_k| \to |\nabla v|$ in L^1, hence $I_{1/n}(|\nabla v_k|) \to I_{1/n}(|\nabla v|)$ in L^1 by the continuity of $I_{1/n}$ on L^1, established in Lemma 8.8. Thus a further subsequence, still denoted by v_k also satisfies the property that $I_{1/n}(|\nabla v_k|) \to I_{1/n}(|\nabla v|)$ almost everywhere. On the other hand, we have $v_k \to v$ almost everywhere and $\langle v_k\rangle_\Omega \to \langle v\rangle_\Omega$ by the L^1 convergence $v_k \to v$. Altogether, taking pointwise limits almost everywhere of

$$|v_k - \langle v_k\rangle_\Omega| \leq c_\Omega I_{1/n}(|\nabla v_k|),$$

we deduce (8.11), as we claimed. □

Exercise 8.12. Prove the following variant of Lemma 8.5: Let Ω be a bounded (not necessarily convex) domain and $v \in W^{1,1}_0(\Omega)$. Then

$$|v(x)| \leq \frac{1}{n\omega_n} I_{1/n}(|\nabla v(y)|) \qquad \text{for a.e. } x \in \Omega.$$

Let us now observe a reduction for the proof of Theorem 8.4. On the left, there appears the expression

$$\frac{(\operatorname{diam}\Omega)^n}{|\Omega|}|v - \langle v\rangle_\Omega| \leq n \cdot I_{1/n}(|\nabla v|)$$

by Lemma 8.5. Denoting $f = |\nabla v|$ and absorbing the factor n into the dimensional constant c, we find that Theorem 8.4 will follow if we can prove that

$$\int_\Omega \exp\Big(\frac{c}{\|f\|_{M^n}} I_{1/n} f\Big) \leq C(\operatorname{diam}\Omega)^n. \tag{8.13}$$

This will be our goal in the following, but we first present some further lemmas on the Riesz potentials I_μ and the Morrey space M^p. Note that even if our final application only deals with the case $\mu = 1/n$, its proof will make use of other values of μ as well, and this is the reason for introducing the whole family of operators I_μ, instead of just $I_{1/n}$.

Lemma 8.14. *For a bounded domain Ω and $\mu > 1/p$, we have $I_\mu : M^p(\Omega) \to L^\infty(\Omega)$, and more precisely*

$$\|I_\mu f\|_{L^\infty(\Omega)} \leq \frac{1-\frac{1}{p}}{\mu - \frac{1}{p}} (\operatorname{diam}\Omega)^{n(\mu-1/p)} \|f\|_{M^p(\Omega)}.$$

Here and below, we will need to be reasonably precise about the quantitative estimates, since eventually we will need to be able to sum up a certain infinite series of them, in order to reach the exponential integrability.

Proof. Assuming without loss that $f \geq 0$, for any $x \in \Omega$, we have

$$\begin{aligned} I_\mu f(x) &= \int_\Omega |x-y|^{-n(1-\mu)} f(y)\,\mathrm{d}y \\ &= \int_\Omega \Big(d^{-n(1-\mu)} + \int_{|x-y|}^{d} n(1-\mu) t^{-n(1-\mu)-1}\,\mathrm{d}t \Big) f(y)\,\mathrm{d}y \\ &= d^{-n(1-\mu)} \int_\Omega f(y)\,\mathrm{d}y + \int_0^d \Big(\int_{\Omega\cap B(x,t)} f(y)\,\mathrm{d}y \Big) n(1-\mu) t^{-n(1-\mu)-1}\,\mathrm{d}t, \end{aligned}$$

observing that in the double integral the integration limits are $0 \leq |x-y| < t < d$, and $|x-y| < t$ is the same as $y \in B(x,t)$. Using the definition of the Morrey norm via $\int_{\Omega\cap B(x,t)} f \leq \|f\|_{M^p} t^{n/p'}$ and also via

$$\int_\Omega f = \int_{\Omega\cap B(x,d)} f \leq \|f\|_{M^p} d^{n/p'} \tag{8.15}$$

and substituting back, we find that

$$\begin{aligned} I_\mu f(x) &\leq d^{-n(1-\mu)+n/p'} \|f\|_{M^p} + n(1-\mu) \int_0^d t^{n/p'-n(1-\mu)-1} \|f\|_{M^p}\,\mathrm{d}t \\ &= \|f\|_{M^p} \Big(d^{n(\mu-1/p)} + n(1-\mu) \frac{1}{n(\mu-1/p)} d^{n(\mu-1/p)} \Big) = \|f\|_{M^p} \frac{1-1/p}{\mu-1/p} d^{n(\mu-1/p)}, \end{aligned}$$

as claimed. □

Exercise 8.16. Prove the Sobolev's inequality for $p > n$: if $v \in W_0^{1,p}(\Omega)$ in bounded domain Ω, then $v \in L^\infty(\Omega)$, and estimate $\|v\|_\infty$ in terms of $\|\nabla v\|_p$. Hint: use some of the previous results and exercises.

The following pointwise bound will allow us to 'interpolate' between the L^1 and L^∞ estimates for $I_\mu f$ established above.

Lemma 8.17. *For $f \geq 0$ and any $\mu \in (0,1)$ and $q \in (1,\infty)$ such that $\mu + \mu/q < 1$, we have the pointwise bound*

$$I_\mu f(x) \leq \big(I_{\mu/q} f(x)\big)^{1/q} \big(I_{\mu+\mu/q} f(x)\big)^{1/q'}.$$

While the result is formally valid for any pair of μ and q, we need to impose the restriction $\mu + \mu/q < 1$ in order that also the Riesz potential on the right have their parameters in the correct range $(0,1)$.

Proof. We write

$$n(\mu-1) = n(\frac{\mu}{q}-1)\frac{1}{q} + n\big[(\mu-1) - (\frac{\mu}{q}-1)\frac{1}{q}\big],$$

where

$$\begin{aligned} n\big[(\mu-1) - (\frac{\mu}{q}-1)\frac{1}{q}\big] &= n\big[\mu(1-\frac{1}{q^2}) - (1-\frac{1}{q})\big] \\ &= n\big[\mu(1+\frac{1}{q}) - 1\big](1-\frac{1}{q}) = n\big[(\mu+\frac{\mu}{q}) - 1\big]\frac{1}{q'}. \end{aligned}$$

Hence

$$\begin{aligned} I_\mu f(x) &= \int |x-y|^{n(\mu-1)} f(y)\,\mathrm{d}y \\ &= \int |x-y|^{n(\mu/q-1)/q} f(y)^{1/q}\cdot |x-y|^{n((\mu+\mu/q)-1)/q'} f(y)^{1/q'}\,\mathrm{d}y \\ &\leq \Big(\int |x-y|^{n(\mu/q-1)} f(y)\,\mathrm{d}y\Big)^{1/q}\Big(\int |x-y|^{n((\mu+\mu/q)-1)} f(y)\,\mathrm{d}y\Big)^{1/q'} \\ &= \big(I_{\mu/q}f(x)\big)^{1/q}\big(I_{\mu+\mu/q}f(x)\big)^{1/q'}. \end{aligned}$$

□

Lemma 8.18. *For $p\in[2,\infty)$ and $q\in[1,\infty)$, we have $I_{1/p}: M^p(\Omega)\to L^q(\Omega)$, and more precisely*

$$\|I_{1/p}f\|_{L^q(\Omega)}^q \leq \omega_n p'\big[q(p-1)\|f\|_{M^p(\Omega)}\big]^q(\operatorname{diam}\Omega)^n.$$

Note that here we have taken $\mu=1/p$ as the parameter of the Riesz potential.

Proof. If $p\in[2,\infty)$ and $q\in(1,\infty)$, then $1/p+1/pq<1/2+1/2=1$, so that we can apply Lemma 8.17 to the result that

$$\begin{aligned} \|I_{1/p}f\|_{L^q}^q = \int_\Omega (I_{1/p}f(x))^q\,\mathrm{d}x &\leq \int_\Omega I_{1/pq}f(x)\cdot(I_{1/p+1/pq}f(x))^{q/q'}\,\mathrm{d}x \\ &\leq \|I_{1/p+1/pq}f\|_{L^\infty}^{q/q'}\int_\Omega I_{1/pq}f(x)\,\mathrm{d}x = \|I_{1/p+1/pq}f\|_{L^\infty}^{q-1}\|I_{1/pq}\|_{L^1}. \end{aligned}$$

If $q=1$, this conclusion is still valid with the interpretation that anything to power 0 is just 1.

To the two factors above, we may apply Lemmas 8.14 and 8.8. For the application of Lemma 8.14 with M^p, note that the parameter of the Riesz potential, $\mu=1/p+1/pq$, is indeed strictly bigger than $1/p$. Thus

$$\|I_{1/p+1/pq}f\|_{L^\infty}\leq \frac{1-\frac{1}{p}}{\frac{1}{p}+\frac{1}{pq}-\frac{1}{p}}d^{n(\frac{1}{p}+\frac{1}{pq}-\frac{1}{p})}\|f\|_{M^p} = \frac{pq}{p'}d^{n/pq}\|f\|_{M^p};$$

this bound is unnecessary if $q=1$, since in this case it is raised to power 0 anyway.

For the L^1 norm, Lemma 8.8 with $\mu=1/pq$ and (8.15) gives

$$\|I_{1/pq}\|_{L^1}\leq \omega_n pq\cdot d^{n/pq}\|f\|_{L^1}\leq \omega_n pq\cdot d^{n/pq}d^{n/p'}\|f\|_{M^p}.$$

Substituting back, we have

$$\begin{aligned} \|I_{1/p}f\|_{L^q}^q &\leq \Big(\frac{pq}{p'}\Big)^{q-1}d^{n(q-1)/pq}\|f\|_{M^p}^{q-1}\cdot\omega_n pq\cdot d^{n/pq}d^{n/p'}\|f\|_{M^p} \\ &= \omega_n p'\Big(\frac{pq}{p'}\Big)^q\|f\|_{M^p}^q d^{n[1/p-1/pq+1/pq+1/p']} = \omega_n p'(q[p-1])^q\|f\|_{M^p}^q d^n, \end{aligned}$$

which is the claim. □

Our final preparation is the following simple calculus bound. More precise information could be deduced from Stirling's formula, but this is good enough for our purposes:

Lemma 8.19. *For all $q=1,2,\ldots$, we have*

$$\frac{q^q}{q!}\leq e^{q-1}.$$

Proof.

$$\log\frac{q^q}{q!} = \log\prod_{k=1}^q\frac{q}{k} = \sum_{k=1}^q\big(\log q-\log k\big) = q\log q-\sum_{k=1}^q\log k.$$

Since log is increasing, we have $\log k \geq \int_{k-1}^{k} \log t \, dt$, and hence

$$\sum_{k=1}^{q} \log k = \sum_{k=2}^{q} \log k \geq \sum_{k=2}^{q} \int_{k-1}^{k} \log t \, dt = \int_{1}^{q} \log t \, dt = q \log q - (q-1).$$

Thus

$$\log \frac{q^q}{q!} \leq q \log q - \sum_{k=1}^{q} \log k \leq q \log q - \big(q \log q - (q-1)\big) = q - 1,$$

and taking exponentials of both sides gives the claim. □

We have now all preparations for the following proposition, which contains our goal (8.13) as the special case $p = n$:

Proposition 8.20. *For $p, n \geq 2$, there are constants $c = c_{n,p}$ and $C = C_{n,p}$ such that*

$$\int_\Omega \exp\Big(c \frac{I_{1/p} f}{\|f\|_{M^p(\Omega)}}\Big) \leq C (\operatorname{diam} \Omega)^n$$

for all $f \in M^p(\Omega)$ on any bounded domain $\Omega \subset \mathbb{R}^n$.

Proof. We compute with an undetermined c, which we choose below. Using the power series expansion of exp, we have

$$\int_\Omega \exp\Big(c \frac{I_{1/p} f}{\|f\|_{M^p}}\Big) = \int_\Omega \sum_{q=0}^{\infty} \frac{1}{q!} \Big(c \frac{I_{1/p} f}{\|f\|_{M^p}}\Big)^q = |\Omega| + \sum_{q=1}^{\infty} \frac{1}{q!} \frac{c^q \|I_{1/p} f\|_{L^q}^q}{\|f\|_{M^p}^q},$$

where we observed that the zeroth term is just $\int_\Omega 1 = |\Omega|$.

By Lemma 8.18 and Lemma 8.19, we have

$$\frac{1}{q!} \frac{c^q \|I_{1/p} f\|_{L^q}^q}{\|f\|_{M^p}^q} \leq \omega_n p' (\operatorname{diam} \Omega)^n c^q (p-1)^q \frac{q^q}{q!} \leq \omega_n p' (\operatorname{diam} \Omega)^n c^q (p-1)^q e^{q-1},$$

so that

$$\sum_{q=1}^{\infty} \frac{1}{q!} \frac{c^q \|I_{1/p} f\|_{L^q}^q}{\|f\|_{M^p}^q} \leq e^{-1} \omega_n p' (\operatorname{diam} \Omega)^n \sum_{q=1}^{\infty} (c[p-1]e)^q = C_{p,n} (\operatorname{diam} \Omega)^n < \infty,$$

provided that we choose c so small that $c(p-1)e < 1$. With such a choice, we have thus have

$$\int_\Omega \exp\Big(c \frac{I_{1/p} f}{\|f\|_{M^p}}\Big) \leq |\Omega| + C_{p,n} (\operatorname{diam} \Omega)^n \leq C'_{p,n} (\operatorname{diam} \Omega)^n,$$

since clearly $|\Omega| \leq c_n (\operatorname{diam} \Omega)^n$, and this completes the proof. □

Remark 8.21. An exponential integrability property of the type achieved in Theorem 8.4 is also typical for so-called BMO (bounded mean oscillation) functions, defined by the condition that

$$\|f\|_{\mathrm{BMO}} := \sup_B \fint_B |f - \langle f \rangle_B|$$

is finite, where the supremum is over all balls B. This is not a coincidence, since the result that we proved, "$|\nabla v| \in M^n \Rightarrow v$ is exponentially inegrable" could actually be 'factorized' through the implications "$|\nabla v| \in M^n \Rightarrow v \in \mathrm{BMO} \Rightarrow v$ is exponentially integrable". The latter implication is known as the *John–Nirenberg inequality.*

Exercise 8.22. Prove that if $v \in W^1(\mathbb{R}^n)$ satisfies $\nabla v \in M^n(\mathbb{R}^n)$, then $v \in \mathrm{BMO}(\mathbb{R}^n)$. Hint: Lemma 8.5 with $\Omega = B$.

9. Local Hölder-continuity of solutions

With the Harnack inequality as a tool, the local Hölder-continuity of solutions is a relatively easy consequence. However, one should observe that in its time it was one of the great breakthroughs of the theory.

Recall that a function f is called Hölder-continuous with exponent δ (and constant C) if for all x, y, we have

$$|f(x) - f(y)| \leq C|x - y|^\delta.$$

This can be alternatively phrased in terms of the oscillation

$$\operatorname*{osc}_B f := \sup_B f - \inf_B f$$

by saying that

$$\operatorname*{osc}_{B_r} f \leq Cr^\delta$$

for any ball $B_r = B(z, r)$ of radius r. With this notation, we have:

Theorem 9.1 (De Giorgi–Nash). *Let $u \in W^{1,2}(\Omega)$ be a weak solution of $Lu = 0$ in Ω. Then*

$$\operatorname*{osc}_{B_R} u \leq C\Big(\frac{R}{R_0}\Big)^\delta \operatorname*{osc}_{B_{R_0}} u$$

for all balls $B_R = B(z, R) \subseteq B_{R_0} = B(z, R_0) \subseteq \Omega$. Here $C = C(\lambda, \Lambda, n)$ depends only on the ellipticity constants and the dimension.

The depends on the following lemma:

Lemma 9.2. *Under the same assumptions, for some $\eta = \eta(\lambda, \Lambda, n) \in (0, 1)$, we have*

$$\operatorname*{osc}_{B_R} u \leq \eta \operatorname*{osc}_{B_{4R}} u$$

whenever $4R \leq R_0$.

Proof. Let us denote

$$m_1 := \inf_{B_R} u, \quad M_1 := \sup_{B_R} u, \quad m_4 := \inf_{B_{4R}} u, \quad M_4 := \sup_{B_{4R}} u.$$

Thus $m_4 \leq u \leq M_4$ in B_{4R}, and hence both $v = M_4 - u$ and $v = u - m_4$ are nonnegative functions in B_{4R}. They are also solutions of $Lv = 0$, since u is a solution, and L annihilates constants. So both these functions v are admissible for the application of Harnack's inequality

$$\sup_{B_R} v \leq C \inf_{B_R} v, \tag{9.3}$$

where $C = C(\lambda, \Lambda, n)$. For $v = M_4 - u$, this gives

$$\sup_{B_R}(M_4 - u) \leq C \inf_{B_R}(M_4 - u) \quad \Rightarrow \quad M_4 - m_1 \leq C(M_4 - M_1),$$

and for $v = u - m_4$,

$$\sup_{B_R}(u - m_4) \leq C \inf_{B_R}(u - m_4) \quad \Rightarrow \quad M_1 - m_4 \leq C(m_1 - m_4).$$

Adding the two inequalities and rearranging, we arrive at

$$\begin{aligned}(M_4 - m_4) + (M_1 - m_1) &\leq C[(M_4 - m_4) - (M_1 - m_1)] \qquad \Rightarrow \\ (C + 1)(M_1 - m_1) &\leq (C - 1)(M_4 - m_4).\end{aligned}$$

Noting that $M_k - m_k = \operatorname{osc}_{B_{kR}} u$ for $k \in \{1, 4\}$, this is the same as

$$\operatorname*{osc}_{B_R} u \leq \eta \operatorname*{osc}_{B_{4R}} u, \quad \text{where} \quad \eta = \frac{C - 1}{C + 1} \in (0, 1).$$

□

Exercise 9.4. Suppose that, instead of Harnack's inequality (9.3), the two choices of v above are only known to satisfy the *weak Harnack inequality*

$$\Big(\fint_{B_{2R}} v^p\Big)^{1/p} \leq C \inf_{B_R} v$$

with $p = 1$. Show that this is enough to conclude the proof of Lemma 9.2.

Proof of Theorem 9.1. Let $k \in \{0, 1, 2, \ldots\}$ be the largest number such that $4^k R \leq R_0$. (The bound clearly holds for $k = 0$ and cannot hold for arbitrarily large k, so this is well defined.) In particular, $4^{k+1}R > R_0$. Now, we can iterate Lemma 9.2 with $4^j R$ in place of R, as long as $4 \cdot 4^j R \leq R_0$, or $j < k$. This leads to

$$\operatorname*{osc}_{B_R} u \leq \eta \operatorname*{osc}_{B_{4R}} u \leq \cdots \leq \eta^k \operatorname*{osc}_{B_{4^k R}} u \leq \eta^k \operatorname*{osc}_{B_{R_0}} u,$$

where the last step is just the trivial observation that the oscillation is bigger in a bigger ball.

To estimate the factor η^k, we observe that $4^k > R_0/4R$, so that $k \log 4 > \log(R_0/4R)$ and hence, recalling that $\eta \in (0, 1)$,

$$\begin{aligned} \eta^k < \eta^{\log(R_0/4R)/\log 4} = \Big(\frac{R_0}{4R}\Big)^{\log \eta/\log 4} = \Big(\frac{4R}{R_0}\Big)^{\log \frac{1}{\eta}/\log 4} &= 4^{\log \frac{1}{\eta}/\log 4}\Big(\frac{R}{R_0}\Big)^{\log \frac{1}{\eta}/\log 4} \\ &= \frac{1}{\eta}\Big(\frac{R}{R_0}\Big)^{\delta}, \qquad \delta := \frac{\log \frac{1}{\eta}}{\log 4}, \end{aligned}$$

which is the asserted bound with δ as above and $C = 1/\eta$. □

10. Why we care about subsolutions?

Recall that some results in these lectures did not require that u is an exact solution of $Lu = 0$, only that it is a subsolution with $Lu \geq 0$. (The word *sub*solution is better understood from the bilinear formulation

$$\int_\Omega A\nabla u \cdot \nabla \phi \leq 0 \qquad \text{whenever } 0 \leq \phi \in C_c^1(\Omega);$$

the bilinear form stays below zero, i.e., 'sub-zero'.) This basically depended on the fact that in proving an inequality (like most of the results we have considered), it was enough to have an inequality instead of equality to begin with. But is there any added value from the fact that some results hold for subsolutions? A positive answer is partially due to the following facts:

(1) Sometimes, *positivity* of a function u is more important than it being an exact solution.
(2) If u is a solution, then $|u|$ is a(n obviously positive) subsolution.

We shall shortly prove (2). Concerning (1), we argue by the example of Harnack's inequality. In our proof given earlier, we needed u to be both positive and a solution. As it turns out, part of the argument, namely (7.2) for $p > 1$, is also valid for subsolutions. In combination with (2), this allows us to derive a Harnack-type estimate for all solutions, without any positivity assumption. Namely, we have:

Theorem 10.1. *Let $u \in W^{1,2}(\Omega)$ satisfy $Lu \geq 0$ in Ω. Suppose moreover that $u \geq 0$ on some ball $4B = B(y, 4R) \subset \Omega$. Then*

$$\sup_B u \leq C_p\Big(\fint_{B_{2R}} u^p\Big)^{1/p}, \qquad \forall\ p > 1.$$

Exercise 10.2. Sketch the proof of Theorem 10.1, following the proof of Theorem 7.1. Discuss shortly, why the restriction $p > 1$ arises from this method of proof. Hint: Follow the beginning of the proof of Theorem 7.1 carefully (with the same test function $v = \eta^2\bar{u}^\beta$), but using the condition $0 \geq \int A\nabla u \cdot \nabla v$ instead of equality. With the chosen v, you get an inequality (instead of equality) for two terms, and this lead to a useful consequence if (and only if) $\beta > 0$. Check that this leads to the bound (7.10) for $\gamma = \beta + 1 > 1$, from which the argument is concluded by Moser's iteration, as before. [As in Theorem 7.1, this sketch would need an additional modification (to ensure that v is a valid test function) for $\beta > 1$ (thus $\gamma > 2$), but you are not asked to repeat this here; just proceed formally assuming that v is a valid test function.]

We now discuss some consequences.

Corollary 10.3. *Let $u \in W^{1,2}(\Omega)$ satisfy $Lu = 0$ in Ω, and $4B = B(y, 4R) \subset \Omega$. Then*

$$\sup_B |u| \leq C_p \Big(\fint_{B_{2R}} |u|^p \Big)^{1/p}, \qquad \forall\, p > 1.$$

Proof. Apply Theorem 10.1 to the subsolution $|u|$, assuming (2). □

Corollary 10.3 is an important companion to Theorem 9.1, since it allows to estimate the right side of the Hölder-continuity estimate a quantity that is more naturally adapted to the Sobolev space $W^{1,2}$, especially with $p = 2$:

Corollary 10.4. *Let $u \in W^{1,2}(\Omega)$ be a weak solution of $Lu = 0$ in Ω. Then*

$$\operatorname*{osc}_{B_R} u \leq C_p \Big(\frac{R}{R_0} \Big)^{\delta} \Big(\fint_{B_{2R_0}} |u|^p \Big)^{1/p}, \qquad \forall\, p > 1,$$

for all concentric balls $B_R \subseteq B_{R_0} \subseteq B_{2R_0} \subseteq \Omega$.

Proof. Dominate $\operatorname{osc}_{B_R} u$ by Theorem 9.1, and then estimate the right side by Corollary 10.3. □

All this motivates a justification of the claim (2), and we turn to this now:

Proposition 10.5. *Let $u \in W^{1,2}(\Omega)$ be a solution of $Lu = 0$. Then $|u|$ is a subsolution.*

Proof. By definition, we need to prove that

$$\int_\Omega A\nabla|u| \cdot \nabla\phi \leq 0 \qquad \text{for } 0 \leq \phi \in C_c^1(\Omega). \tag{10.6}$$

Let us first consider the smooth approximation

$$\int_\Omega A\nabla g_\varepsilon(u) \cdot \nabla\phi \leq 0 \qquad \text{for } 0 \leq \phi \in C_c^1(\Omega), \tag{10.7}$$

where

$$g_\varepsilon(t) := (\varepsilon^2 + t^2)^{1/2} - \varepsilon.$$

Then $0 \leq g_\varepsilon(t) \leq |t|$ and $g_\varepsilon(t) \to |t|$ pointwise as $\varepsilon \to 0$. We compute the derivatives

$$g_\varepsilon'(t) = (\varepsilon^2 + t^2)^{-1/2} t$$

and

$$g_\varepsilon''(t) = -(\varepsilon^2 + t^2)^{-3/2} t^2 + (\varepsilon^2 + t^2)^{-1/2} = (\varepsilon^2 + t^2)^{-3/2}\varepsilon^2 \geq 0. \tag{10.8}$$

We have both $g_\varepsilon', g_\varepsilon'' \in L^\infty(\mathbb{R})$, and the positivity of g_ε'' will be critical in the argument.

By the chain rule, we have

$$\int A\nabla g_\varepsilon(u) \cdot \nabla\phi = \int A(g_\varepsilon'(u)\nabla u) \cdot \nabla\phi = \int A\nabla u \cdot g_\varepsilon'(u)\nabla\phi, \tag{10.9}$$

since the scalar factor $g_\varepsilon'(u)$ commutes with both the matrix product and the dot product. The function $f = g_\varepsilon'$ satisfies $f \in C^1$ and $f' = g_\varepsilon'' \in L^\infty$; thus $w := f(u) = g_\varepsilon'(u) \in W^{1,2}(\Omega) \subset W^1(\Omega)$. We then have the product rule $\nabla(w\phi) = (\nabla w)\phi + w\nabla\phi$ (see Exercise 10.10 below), and thus

$$g_\varepsilon'(u)\nabla\phi = \nabla(g_\varepsilon'(u)\phi) - (\nabla g_\varepsilon'(u))\phi = \nabla(g_\varepsilon'(u)\phi) - g_\varepsilon''(u)(\nabla u)\phi.$$

Hence

$$\int A\nabla u \cdot g_\varepsilon'(u)\nabla\phi = \int A\nabla u \cdot \nabla(g_\varepsilon'(u)\phi) - \int (A\nabla u \cdot \nabla u) g_\varepsilon''(u)\phi = I - II.$$

In the second integral, all three factors are nonnegative — the first by ellipticity $A\nabla u \cdot \nabla u \geq \lambda|\nabla u|^2 \geq 0$, the second by (10.8), and the third by the assumption on ϕ in (10.7) —, and thus $II \geq 0$.

For term I, we only need to observe that $g_\varepsilon'(u)\phi = w\phi \in W_0^{1,2}(\Omega)$, to conclude that $I = 0$ from the definition of $Lu = 0$. That the product of $w \in W^{1,2}$ and $\phi \in C_c^1$ belongs to $W_0^{1,2}$, has been checked earlier.

Altogether, we have $I = 0$, $II \geq 0$, and thus $I - II \leq 0$, as required for (10.7). To prove (10.6), it only remains to recall that $v \mapsto \int A\nabla v \cdot \nabla\phi$ is with respect to $v \in W^{1,2}$, and to check that $g_\varepsilon(u) \to |u|$ in the norm of $W^{1,2}$, which follows easily from the pointwise convergence $g_\varepsilon(u) \to |u|$ and $g'_\varepsilon(u) \to \operatorname{sgn}(u) := 1_{\{u>0\}} - 1_{\{u<0\}}$, together with dominated convergence theorem. □

Exercise 10.10. (a) Check the product rule

$$\nabla(wv) = (\nabla w)v + w\nabla v \tag{10.11}$$

for all $w \in W^1(\Omega)$ and $v \in C^1(\Omega)$. Hint: Verify that the right side qualifies for a weak gradient of wv directly from the definition. Use the classical product rule for $v\phi$, where $\phi \in C^1_c(\Omega)$, and note that $v\phi \in C^1_c(\Omega)$ is also an admissible test function.

(b) Prove that (10.11) is also true for $w \in W^{1,p}(\Omega)$ and $v \in W^{1,p'}(\Omega)$ for $1 < p < \infty$. Hint: Argue similarly, exploit case (a), and note that $v\phi \in W^{1,p'}_0(\Omega)$ is in the closure of test functions.

11. Oscillatory upper bounds in terms of the gradient

One drawback of the bound of Corollary 10.4 is that it is not very good if the function u varies only a little around a large average value: even if the oscillation is small, the upper bound involving $|u|$ can be very large. This is easily remedied by noting that, for any constant $\alpha \in \mathbb{R}$, we have that

- $u - c$ is also a solution, since $L = \nabla \cdot A\nabla$ annihilates constants, and
- this function has the same oscillation $\operatorname{osc}_B(u - c) = \operatorname{osc}_B u$.

Thus, in fact, Corollary 10.4 self-improves to

$$\operatorname*{osc}_{B_R} u = \operatorname*{osc}_{B_R}(u - c) \leq C_p\Big(\frac{R}{R_0}\Big)^\delta \Big(\fint_{B_{2R_0}} |u - c|^p\Big)^{1/p}, \qquad \forall\, p > 1,$$

for any $c \in \mathbb{R}$. Choosing $c = \langle u\rangle_{B_{2R_0}}$, we may apply Lemma 8.5 to continue with

$$\operatorname*{osc}_{B_R} u \leq C_p\Big(\frac{R}{R_0}\Big)^\delta \Big(\fint_{B_{2R_0}} |I_{1/n}(|\nabla u| 1_{B_{2R_0}})|^p\Big)^{1/p}, \qquad \forall\, p > 1.$$

Below, we shall prove the following mapping property of $I_{1/n}$:

Lemma 11.1.

$$\|I_{1/n} f\|_{L^p(B_r)} \leq C_p r \|f\|_{L^p(B_r)}.$$

This allows us to deduce the following variant of Corollary 10.4:

Corollary 11.2. *Let $u \in W^{1,2}(\Omega)$ be a weak solution of $Lu = 0$ in Ω. Then*

$$\operatorname*{osc}_{B_R} u \leq C_p\Big(\frac{R}{R_0}\Big)^\delta R_0 \Big(\fint_{B_{2R_0}} |\nabla u|^p\Big)^{1/p}, \qquad \forall\, p > 1,$$

for all concentric balls $B_R \subseteq B_{R_0} \subseteq B_{2R_0} \subseteq \Omega$.

The most important case of both Corollaries 10.4 and 11.2 is $p = 2$, since the L^2 norms of u and ∇u are immediately related to the solution space $W^{1,2}$. These two inequalities are essentially equivalent. Indeed, our approach to Corollary 11.2 is precisely to derive it from Corollary 10.4 via the preceding considerations including Lemma 11.1. The converse direction (with a slight change in the size of the bigger ball on the right) follows from the following result of independent interest:

Proposition 11.3 (Caccioppoli inequality)**.** *If $u \in W^{1,2}(\Omega)$ is a weak solution of $Lu = 0$, then it satisfies*

$$\int_{B_r} |\nabla u|^2 \leq \frac{C}{r^2}\int_{B_{2r}} u^2$$

for all concentric ball $B_r \subset B_{2r} \subset \Omega$.

Proof. Let $\eta \in C_c^1(\Omega)$ be equal to 1 in B_r, equal to 0 outside B_{2r}, and such that $|\nabla\eta| \leq 2/r$. We estimate the following quantity, which is clearly an upper bound for the left side of the claim:

$$\int |\nabla u|^2\eta^2 \leq \frac{1}{\lambda}\int (A\nabla u\cdot\nabla u)\eta^2 = \frac{1}{\lambda}\int [A\nabla u\cdot\nabla(u\eta^2) - A\nabla u\cdot 2\eta(\nabla\eta)u] = -\frac{2}{\lambda}\int (A\nabla u\cdot\nabla\eta)u\eta,$$

where the first term vanished, since $u\eta^2 \in W_0^{1,2}$ (as a product of $u \in W^{1,2}$ and $\eta^2 \in C_c^1$) is a test function, and u is a solution.

We may continue with

$$-\frac{2}{\lambda}\int (A\nabla u\cdot\nabla\eta)u\eta \leq \frac{2\Lambda}{\lambda}\int |\nabla u||\nabla\eta||u||\eta| \leq \frac{2\Lambda}{\lambda}\Big(\int |\nabla u|^2\eta^2\Big)^{1/2}\Big(\int |\nabla\eta|^2|u|^2\Big)^{1/2},$$

so altogther

$$\int |\nabla u|^2\eta^2 \leq \frac{2\Lambda}{\lambda}\Big(\int |\nabla u|^2\eta^2\Big)^{1/2}\Big(\int |\nabla\eta|^2|u|^2\Big)^{1/2} \leq \frac{4\Lambda}{r\lambda}\Big(\int |\nabla u|^2\eta^2\Big)^{1/2}\Big(\int_{B_{2r}} |u|^2\Big)^{1/2}$$

Dividing both sides by $\big(\int |\nabla u|^2\eta^2\big)^{1/2}$, we deduce the claim. □

We then turn to the proof of Lemma 11.1 to complete the proof of Corollary 11.2. This will be accomplished via somewhat more general bounds for integrals of similar type, which are also important elsewhere. Recall that the convolution of two functions is defined by

$$k * f(x) := \int_{\mathbb{R}^n} k(x-y)f(y)\,\mathrm{d}y.$$

Theorem 11.4 (Convolution inequality). *For exponents $p, q, r \in [1,\infty]$ such that*

$$\frac{1}{p} + \frac{1}{q'} + \frac{1}{r} = 2, \tag{11.5}$$

we have

$$\|k * f\|_q \leq \|k\|_r\|f\|_p.$$

Proof. We make use of the duality of L^p spaces via

$$\|k * f\|_q = \sup\Big\{\int k * f(x)g(x)\,\mathrm{d}x : \|g\|_{q'} \leq 1\Big\}.$$

Thus we need to estimate

$$\int k * f(x)g(x)\,\mathrm{d}x = \iint k(x-y)f(y)g(x)\,\mathrm{d}y\,\mathrm{d}x,$$

and we would like to dominate this by

$$\|k\|_r\|f\|_p\|g\|_{q'}. \tag{11.6}$$

With auxiliary numbers $\alpha, \beta, \gamma, \delta, \varepsilon, \theta \in [0,1]$ and $s, t, u \in [1,\infty]$ such that

$$\alpha+\beta = 1, \quad \gamma+\delta = 1, \quad \varepsilon+\theta = 1, \quad \frac{1}{s}+\frac{1}{t}+\frac{1}{u} = 1, \tag{11.7}$$

we split

$$\begin{aligned}\iint k(x-y)f(y)g(x)\,\mathrm{d}y\,\mathrm{d}x &= \iint k(x-y)^{\alpha+\beta}f(y)^{\gamma+\delta}g(x)^{\varepsilon+\theta}\,\mathrm{d}y\,\mathrm{d}x \\ &= \iint k(x-y)^{\alpha}f(y)^{\gamma}\times k(x-y)^{\beta}g(x)^{\varepsilon}\times f(y)^{\delta}g(x)^{\theta}\,\mathrm{d}y\,\mathrm{d}x\end{aligned}$$

and apply Hölder's inequality with these three factors to estimate this by

$$\begin{aligned}&\leq \Big(\iint k(x-y)^{\alpha s}f(y)^{\gamma s}\,\mathrm{d}y\,\mathrm{d}x\Big)^{1/s}\Big(\iint k(x-y)^{\beta t}g(x)^{\varepsilon t}\,\mathrm{d}y\,\mathrm{d}x\Big)^{1/t}\Big(\iint f(y)^{\delta u}g(x)^{\theta u}\,\mathrm{d}y\,\mathrm{d}x\Big)^{1/u} \\ &= I\times II\times III.\end{aligned}$$

It is immediate that $III = \|f\|_{\delta u}^{\delta}\|g\|_{\theta u}^{\theta}$. But a similar splitting also takes place for the first two factors, observing by iterated integration and change of variable that,

$$I = \Big(\int\Big[\int k(x-y)^{\alpha s}\,\mathrm{d}x\Big]f(y)^{\gamma s}\,\mathrm{d}y\Big)^{1/s} = \Big(\int\Big[\int k(z)^{\alpha s}\,\mathrm{d}z\Big]f(y)^{\gamma s}\,\mathrm{d}y\Big)^{1/s} = \|k\|_{\alpha s}^{\alpha}\|f\|_{\gamma s}^{\gamma},$$

and analogously $II = \|k\|_{\beta t}^{\beta}\|g\|_{\varepsilon t}^{\varepsilon}$. So altogether we check that

$$I\times II\times III = \|k\|_{\alpha s}^{\alpha}\|k\|_{\beta t}^{\beta}\|f\|_{\gamma s}^{\gamma}\|f\|_{\delta u}^{\delta}\|g\|_{\varepsilon t}^{\varepsilon}\|g\|_{\theta u}^{\theta}.$$

To arrive at the desired bound (11.6), we need that

$$\alpha s = \beta t = r, \quad \gamma s = \delta u = p, \quad \varepsilon t = \theta u = q'. \tag{11.8}$$

If this can be satisfied, we indeed arrive at

$$I\times II\times III = \|k\|_{r}^{\alpha+\beta}\|f\|_{p}^{\gamma+\delta}\|g\|_{q'}^{\varepsilon+\theta} = \|k\|_r\|f\|_p\|g\|_{q'},$$

as desired.

So it remains to see that the conditions (11.7) and (11.8) can be satisfied. These impose ten conditions on the nine numbers $\alpha,\beta,\gamma,\delta,\varepsilon,\theta,s,t,u$, so this seems somewhat tricky at first.

Solving from (11.8), we have

$$\alpha = r/s, \quad \beta = r/t, \quad \gamma = p/s, \quad \delta = p/u, \quad \varepsilon = q'/t, \quad \theta = q'/u,$$

and using the first three equations in (11.7), we need that

$$\frac{1}{r} = \frac{1}{s}+\frac{1}{t}, \quad \frac{1}{p} = \frac{1}{s}+\frac{1}{u}, \quad \frac{1}{q'} = \frac{1}{t}+\frac{1}{u}. \tag{11.9}$$

Note that, if we add up these three, we arrive at $1/r+1/p+1/q' = 2(1/s+1/t+1/u) = 2$, which explains the condition (11.5), and shows that in the presence of (11.5), the ten equations (11.7) and (11.8) are actually dependent; thus there is some hope for solving them for the nine unknowns.

Adding the first two equations in (11.9) and subtracting the third one, we arrive at

$$\frac{1}{r}+\frac{1}{p}-\frac{1}{q'} = \frac{2}{s}$$

thus

$$\frac{1}{s} = \frac{1}{2}\Big(\frac{1}{r}+\frac{1}{p}-\frac{1}{q'}\Big) = \frac{1}{2}\Big(\frac{1}{r}+\frac{1}{p}+\frac{1}{q'}-\frac{2}{q'}\Big) = \frac{1}{2}\Big(2-\frac{2}{q'}\Big) = 1-\frac{1}{q'} = \frac{1}{q}.$$

In a similar way, we also solve for $1/t$ and $1/u$, to the result that

$$s = q, \quad t = p', \quad u = r',$$

which belong to the required range $[1,\infty]$, since $p,q,r\in[1,\infty]$. The numbers $\alpha,\beta,\gamma,\delta,\varepsilon,\theta$ can then be solved from (11.7), and this shows that all the conditions (11.7) and (11.8) can indeed be satisfied, which proves the result. □

Remark 11.10. There are different ways both to prove the convolution inequality and to guess the correct range of exponents. It is relatively easy to check the two cases $\|k*f\|_p \le \|k\|_1\|f\|_p$ and $\|k*f\|_\infty \le \|k\|_{p'}\|f\|_p$ for all $p\in[1,\infty]$. A reader familiar with interpolation of L^p spaces can easily deduce the full Theorem 11.4 from these cases.

We then apply the same technique to the Riesz potentials:

Lemma 11.11. *For a bounded domain Ω, the fractional integral I_μ satisfies*

$$\|I_\mu f\|_{L^q(\Omega)} \le \sup_{y\in\Omega}\Big\|x\mapsto\frac{1}{|x-y|^{n(1-\mu)}}\Big\|_{L^r(\Omega)}\|f\|_{L^p(\Omega)} \quad \textit{for} \quad \frac{1}{p}+\frac{1}{q'}+\frac{1}{r} = 2. \tag{11.12}$$

Proof. This is a slight modification of the convolution theorem, but not an immediate consequence. Indeed, note that

$$\|I_\mu f\|_{L^q(\Omega)} = \sup\Big\{\int_\Omega I_\mu f(x)g(x)\,\mathrm{d}x = \iint_{\Omega\times\Omega} k(x-y)f(y)g(x)\,\mathrm{d}y\,\mathrm{d}x : \|g\|_{L^{q'}(\Omega)} \leq 1\Big\},$$

where

$$k(x-y) = \frac{1}{|x-y|^{n(1-\mu)}}.$$

Repeating the initial steps of the previous proof, we would estimate this by

$$\begin{aligned} I\times II\times III = \Big(&\iint_{\Omega\times\Omega} k(x-y)^{\alpha s} f(y)^{\gamma s}\,\mathrm{d}y\,\mathrm{d}x\Big)^{1/s} \\ &\times\Big(\iint_{\Omega\times\Omega} k(x-y)^{\beta t} g(x)^{\varepsilon t}\,\mathrm{d}y\,\mathrm{d}x\Big)^{1/t}\Big(\iint_{\Omega\times\Omega} f(y)^{\delta u} g(x)^{\theta u}\,\mathrm{d}y\,\mathrm{d}x\Big)^{1/u}. \end{aligned}$$

Factor III splits as $\|f\|_{\delta u}^{\delta}\|g\|_{\theta u}^{\theta}$ as before. However, the factors I and II do not split quite as directly, since for instance the change of variable $z = x - y$ in I would result in the integration domain for z being $\Omega - y$, which depends on y. Nevertheless, we can make the following slight modifications:

$$\begin{aligned} I &= \Big(\int_\Omega\Big[\int_\Omega k(x-y)^{\alpha s}\,\mathrm{d}x\Big]f(y)^{\gamma s}\,\mathrm{d}x\Big)^{1/s} \\ &\leq \Big(\int_\Omega \sup_{z\in\Omega}\Big[\int_\Omega k(x-z)^{\alpha s}\,\mathrm{d}x\Big]f(y)^{\gamma s}\,\mathrm{d}x\Big)^{1/s} = \sup_{z\in\Omega}\|x\mapsto k(x-z)\|_{\alpha s}^{\alpha}\|f\|_{\gamma s}^{\gamma}, \end{aligned}$$

and simiarly

$$II \leq \sup_{z\in\Omega}\|y\mapsto k(z-y)\|_{\beta t}^{\beta}\|g\|_{\varepsilon t}^{\varepsilon}.$$

Noting that $k(x-y) = k(y-x)$, we conclude the argument by choosing $\alpha,\beta,\gamma,\delta,\varepsilon,\theta,s,t,u$ exactly as before. □

The bound (11.12) is only interesting if the L^q norm on the right is finite. This is estimated in the following:

Lemma 11.13.

$$\sup_{y\in\Omega}\Big\|x\mapsto\frac{1}{|x-y|^{n(1-\mu)}}\Big\|_{L^r(\Omega)} \leq \frac{\omega_n^{1-\mu}|\Omega|^{\mu-1/r'}}{(1-(1-\mu)r)^{1/r}} \qquad \text{if}\quad \frac{1}{r'} < \mu < 1.$$

Proof. Clearly the function $x\mapsto 1/|x-y|^{n(1-\mu)r}$ decreases radially away from y. One can then check, varying Ω over all domains of fixed measure $|\Omega|$ for a fixed $y\in\mathbb{R}^n$, that the integral

$$\int_\Omega\frac{1}{|x-y|^{n(1-\mu)r}}\,\mathrm{d}x$$

is maximized by the ball $B(y,r_\Omega)$ with $|\Omega| = |B(y,r_\Omega)| = \omega_n r_\Omega^n$. Thus, integrating in polar coordinates centred at y, we have

$$\begin{aligned} \int_\Omega\frac{1}{|x-y|^{n(1-\mu)r}}\,\mathrm{d}x &\leq \int_{B(y,r_\Omega)}\frac{1}{|x-y|^{n(1-\mu)r}}\,\mathrm{d}x = \int_0^{r_\Omega}\int_{S^{n-1}}\mathrm{d}\sigma t^{-n(1-\mu)r}t^{n-1}\,\mathrm{d}t \\ &= n\omega_n\frac{r_\Omega^{-n(1-\mu)r+n}}{-n(1-\mu)r+n} = \frac{\omega_n r_\Omega^{n[1-(1-\mu)r]}}{1-(1-\mu)r} = \frac{\omega_n^{(1-\mu)r}|\Omega|^{1-(1-\mu)r}}{1-(1-\mu)r}, \end{aligned}$$

provided that $n-n(1-\mu)r = n[1-(1-\mu)r] > 0$, and the integral diverges to ∞ otherwise. Taking the rth root, we deduce the claim about the L^r norm, observing that $1/r-(1-\mu) = \mu-1/r'$. □

A combination of the previous lemmas shows that:

Proposition 11.14.

$$\|I_\mu f\|_{L^q(\Omega)} \leq C|\Omega|^{\mu+1/q-1/p}\|f\|_{L^p(\Omega)}$$

for all $p,q\in[1,\infty]$ *such that* $\mu+1/q-1/p > 0$.

Proof. The condition 11.5 implies that

$$\frac{1}{r'} = 1 - \frac{1}{r} = \frac{1}{p} + \frac{1}{q'} - 1 = \frac{1}{p} - \frac{1}{q},$$

so that $\mu - 1/r' = \mu + 1/q - 1/p$. □

In particular, with $p = q$, the condition $\mu + 1/q - 1/p = \mu > 0$ is always satisfied, and we get

$$\|I_\mu f\|_{L^p(\Omega)} \leq C|\Omega|^\mu \|f\|_{L^p(\Omega)}.$$

If $\mu = 1/n$ and $\Omega = B_r$, then $|\Omega|^\mu = |B_r|^{1/n} = cr$ is essentially the radius of the ball, proving Lemma 11.1, and thereby completing the proof of Corollary 11.2.

Other instances of Proposition 11.14 have useful applications as well, as illustrated by the following:

Exercise 11.15 (Sobolev's inequality for $p > n$)**.** Prove the following versions of Sobolev's inequality for $u \in W_0^{1,p}(\Omega)$, where $p > n$:

(1) $\|u\|_\infty \leq C|\Omega|^{1/n-1/p}\|\nabla u\|_p$.
(2) $|u(x) - u(y)| \leq C|x - y|^\gamma \|\nabla u\|_p$ for almost every $x, y \in \Omega$, where $\gamma = 1 - n/p \in (0, 1)$. (That is, u is Hölder-continuous with exponent γ, up to a null set.)

Hint: Recall Exercise 8.12. For part (2), consider *Lebesgue points* x, y, which means that we have the limit $\lim_{r\to 0}\langle u\rangle_{B(z,r)} = u(z)$ for $z \in \{x, y\}$, and recall from Real Analysis that almost every point has this property. Then expand, with $r = \frac{1}{2}|x - y|$,

$$u(x) = \sum_{k=0}^{\infty}(\langle u\rangle_{B(x,2^{-k-1}r)} - \langle u\rangle_{B(x,2^{-k}r)}) + \langle u\rangle_{B(x,r)}$$

and similarly with $u(y)$, and write

$$\langle u\rangle_{B(x,r)} - \langle u\rangle_{B(y,r)} = (\langle u\rangle_{B(x,r)} - \langle u\rangle_{B(\frac{1}{2}(x+y),2r)}) - (\langle u\rangle_{B(y,r)} - \langle u\rangle_{B(\frac{1}{2}(x+y),2r)}).$$

Observe that all terms of the form $\langle u\rangle_B - \langle u\rangle_{B^*}$, where $B^* \supset B$ with $|B^*| \leq c|B|$, can be estimated as

$$|\langle u\rangle_B - \langle u\rangle_{B^*}| = \Big|\fint_B (u - \langle u\rangle_{B^*})\Big| \leq C\fint_{B^*}|u - \langle u\rangle_{B^*}|,$$

which you can estimate with the help of Lemma 8.5 and a suitable bound for $I_{1/n}$.

It is interesting to compare Exercise 11.15 with Theorem 9.1: The Hölder-continuity of a Sobolev function may also be deduced without any relation to solutions of elliptic equations, but in this case we need a much higher integrability exponent p in $W^{1,p}$. In Theorem 9.1, we only needed $W^{1,2}$, while in Exercise 11.15 we need $p > n$, which is larger already in dimension $n = 2$. Indeed, solutions of elliptic equations are much better than just arbitrary Sobolev functions!

12. Continuous boundary values and the elliptic measure

In this final section, which is based on the book [Ken94], we return to the treatment of the basic boundary value problem

$$\begin{cases} Lu = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{12.1}$$

Recall that previously we interpreted the boundary condition $u = g$ (i.e., $u - g = 0$) on $\partial\Omega$ in the sense that $u - g \in W_0^{1,2}(\Omega)$, the Sobolev space of zero boundary values. Denoting $v = u - g$, the problem was transformed into $Lv = -Lg = -\nabla\cdot A\nabla g$, or, in the weak formulation,

$$\mathscr{L}(v, \phi) := \int A\nabla v\cdot\nabla\phi = -\int A\nabla g\cdot\nabla\phi =: F(\phi), \qquad \forall\ \phi \in C_c^1(\Omega).$$

This, in turn, was uniquely solvable in $W_0^{1,2}(\Omega)$ by the Lax–Milgram lemma, provided that $g \in W^{1,2}(\Omega)$, which makes the functional F above bounded on $W_0^{1,2}(\Omega)$.

A drawback of this approach is that it requires the boundary function g to be defined on all of Ω, whereas it would seem more natural that the boundary condition is only defined on the boundary $\partial\Omega$.

12.A. **Lipschitz boundary values.** We now develop a theory for the boundary value problem (12.1) in the case that we are given $g \in \mathrm{Lip}(\partial\Omega)$, where

$$\mathrm{Lip}(E) := \{f : E \to \mathbb{R}; |f(x) - f(y)| \leq M|x - y| \;\forall\; x, y \in E\}$$

is the space of Lipschitz-continuous functions on a set E. For $f \in \mathrm{Lip}(E)$, the smallest possible M is denoted by $\|f\|_{\mathrm{Lip}(E)}$.

In order to reduce (12.1) with $g \in \mathrm{Lip}(\partial\Omega)$ to the case already known to us, we want to extend the function g from the boundary to the domain. In fact, we have the following:

Proposition 12.2. *Let $E \subset \mathbb{R}^n$ be any set and $f \in \mathrm{Lip}(E)$. Then there exists $\tilde{f} \in \mathrm{Lip}(\mathbb{R}^n)$ such that*

(i) *$\tilde{f}(x) = f(x)$ for all $x \in E$, and*
(ii) *$\|\tilde{f}\|_{\mathrm{Lip}(\mathbb{R}^n)} = \|f\|_{\mathrm{Lip}(E)}$.*

Proof. Let $M := \|f\|_{\mathrm{Lip}(E)}$. We define

$$\tilde{f}(x) := \inf_{y \in E} \big(f(y) + M|x - y|\big)$$

and check that it verifies the claimed properties.

(i). Let $x \in E$. Clearly $\tilde{f}(x) \leq f(x)$, simply by choosing $y = x$ in the formula, where the infimum is taken. On the other hand, by the Lipschitz-condition, we have $f(y) \geq f(x) - M|x - y|$ for all $x, y \in E$, thus $f(y) + M|x - y| \geq f(x)$, and taking the infimum over the left side we see that $\tilde{f}(x) \geq f(x)$.

(ii). It is clear that $\|\tilde{f}\|_{\mathrm{Lip}(\mathbb{R}^n)} \geq M$, so we only need to show the opposite. Let $x, z \in \mathbb{R}^n$. By definition of $\tilde{f}(z)$ as an infimum, for every $\varepsilon > 0$, we can find some $\bar{y} \in E$ where the infimum is almost reached, i.e., where $\tilde{f}(z) > f(\bar{y}) + M|x - \bar{y}| - \varepsilon$. Thus

$$\tilde{f}(x) - \tilde{f}(z) < \tilde{f}(x) - f(\bar{y}) - M|z - \bar{y}| + \varepsilon.$$

Since $\tilde{f}(x)$ is the infimum over all y, it is certainly dominated by the value of the same expression at the particular value $y = \bar{y}$; thus

$$\begin{aligned}\tilde{f}(x) - \tilde{f}(z) &\leq f(\bar{y}) + M|x - \bar{y}| - f(\bar{y}) - M|z - \bar{y}| + \varepsilon \\ &= M\big(|x - \bar{y}| - |z - \bar{y}|\big) + \varepsilon \leq M|x - z| + \varepsilon,\end{aligned}$$

where the last step was simply the triangle inequality. Since the above is true for any $\varepsilon > 0$, we have $\tilde{f}(x) - \tilde{f}(z) \leq M|x - z|$, and by symmetry of x and z, also $\tilde{f}(z) - \tilde{f}(x) \leq M|x - z|$. Thus $|\tilde{f}(x) - \tilde{f}(z)| \leq M|x - z|$, as required. □

We take for granted the following property of Lipschitz functions on a domain Ω: each $f \in \mathrm{Lip}(\Omega)$ is weakly differentiable and $|\nabla f| \in L^\infty(\Omega)$ with $\|\nabla f\|_\infty \leq \|f\|_{\mathrm{Lip}}$. (This latter bound should be at least intuitively plausible, observing that the difference quotients approximating the derivative are uniformyl bounded,

$$\frac{|f(x) - f(y)|}{|x - y|} \leq M,$$

by the very definition of Lipschitz functions.) In a bounded domain Ω, we also have $|f(x)| \leq |f(x_0)| + M|x - x_0| \leq |f(x_0)| + M \operatorname{diam} \Omega$ for any fixed $x_0 \in \Omega$, and hence $\mathrm{Lip}(\Omega) \subset W^{1,\infty}(\Omega) \subset W^{1,2}(\Omega)$. This is the class of boundary functions g for which our earlier solvability theory of (12.1) applies. We now define:

Definition 12.3. A function $u \in W^{1,2}(\Omega)$ is called a solution of (12.1) with $g \in \mathrm{Lip}(\partial\Omega)$, if u is a solution of (12.1) in the old sense with a Lipschitz-extension $\tilde{g}$ of g in place of g.

Lemma 12.4. *The previous definition is consistent in the following sense: If $\tilde{g}_1$ and $\tilde{g}_2$ are two extensions of g, then the corresponding solutions u_1 and u_2 of* (12.1)*, with g replaced by $\tilde{g}_1$ or $\tilde{g}_2$, are equal.*

Proof. We already argued that $\tilde{g}_i \in \mathrm{Lip}(\Omega) \subset W^{1,2}(\Omega)$ for $i = 1, 2$. We claim that $\tilde{g}_0 := \tilde{g}_1 - \tilde{g}_2 \in W_0^{1,2}(\Omega)$. This is intuitively clear, since $\tilde{g}_0 = g - g = 0$ on $\partial\Omega$ in the pointwise sense; however, we should recall that the zero-boundary value in $W_0^{1,2}(\Omega)$ was not defined pointwise, but in the sense of closure of $C_c^1(\Omega)$ in $W^{1,2}(\Omega)$. So we want to check that $\tilde{g}_0$ can be approximated by $C_c^1(\Omega)$ functions in the $W^{1,2}(\Omega)$ norm.

For every $\varepsilon > 0$, let $\chi^\varepsilon \in C_c^1(\Omega)$ be a function such that $\chi^\varepsilon(x) = 1$ if $\mathrm{dist}(x, \partial\Omega) \geq 2\varepsilon$ and $\chi^\varepsilon(x) = 0$ if $\mathrm{dist}(x, \partial\Omega) \leq \varepsilon$; moreover, let $0 \leq \chi^\varepsilon(x) \leq 1$ and $|\nabla\chi^\varepsilon(x)| \leq 2/\varepsilon$ everywhere. The function $\chi^\varepsilon\tilde{g}_0$ is clearly compactly supported in Ω (although not necessarily C^1); we check that it approximates $\tilde{g}_0$.

Clearly $\chi^\varepsilon \to 1$ pointwise in Ω, and this function is dominated by 1, so that $\chi^\varepsilon\tilde{g}_0 \to \tilde{g}_0$ in $L^2(\Omega)$ by dominated convergence. For the L^2 convergence of gradients, note that

$$\nabla(\chi^\varepsilon\tilde{g}_0) = (\nabla\chi^\varepsilon)\tilde{g}_0 + \chi^\varepsilon(\nabla\tilde{g}_0),$$

and the second term converges to $\nabla\tilde{g}_0$ in L^2, by the same reasoning as $\chi^\varepsilon\tilde{g}_0 \to \tilde{g}_0$ above. It remains to check that the first term converges to zero, and this is also immediate in the pointwise sense, since $\nabla\chi^\varepsilon$ is supported in the set $\{x : 0 < \mathrm{dist}(x, \partial\Omega) < 2\varepsilon\}$, which converges to $\varnothing$. To apply dominated convergence, consider any $x \in \Omega$. If $\nabla\chi^\varepsilon(x) = 0$, we have a trivial upper bound. Otherwise, we have $\mathrm{dist}(x, \partial\Omega) < 2\varepsilon$, and hence there exists $\bar{x} \in \partial\Omega$ such that $|x - \bar{x}| < 2\varepsilon$. But then, using that $\tilde{g}_0 = 0$ on $\partial\Omega$, and the Lipschitz-continuity, we have

$$|\tilde{g}_0(x)| = |\tilde{g}_0(x) - \tilde{g}_0(\bar{x})| \leq M|x - \bar{x}| \leq M \cdot 2\varepsilon,$$

and also that $|\nabla\chi^\varepsilon(x)| \leq 2/\varepsilon$. So altogether we have $|\nabla\chi^\varepsilon(x)\tilde{g}_0(x)| \leq 4M \in L^\infty(\Omega) \subset L^2(\Omega)$, and so we can use dominated convergence again to conclude that $(\nabla\chi^\varepsilon)\tilde{g}_0 \to 0$. This completes the claim that $\chi^\varepsilon\tilde{g}_0 \to \tilde{g}_0$ in $W^{1,2}(\Omega)$.

It remains to approximate $\chi^\varepsilon\tilde{g}_0$ by a standard mollification $\phi_\delta * (\chi^\varepsilon\tilde{g}_0)$. This belongs to $C_c^1(\Omega)$ (for small enough $\delta > 0$), and tends to $\chi^\varepsilon\tilde{g}_0$ as $\delta \to 0$. So altogether we see that $\tilde{g}_0$ can be approximated by $C_c^1(\Omega)$ functions, showing that $\tilde{g}_0 \in W_0^{1,2}(\Omega)$.

It is now easy to complete the proof of the lemma. Let u_i be the solution of (12.1) with the boundary data $\tilde{g}_i$. Then $u_0 := u_1 - u_2$ solves (12.1) with the boundary data $\tilde{g}_0 \in W_0^{1,2}(\Omega)$. But, using that $W_0^{1,2}(\Omega)$ is a linear space, the condition that $u_0 - \tilde{g}_0 \in W_0^{1,2}(\Omega)$ is equivalent to $u_0 \in W_0^{1,2}(\Omega)$, meaning that u_0 solves (12.1) with the boundary data 0 in the Sobolev sense. Then $u_0 = 0$ follows from the uniqueness of the solution of the Dirichlet problem, and hence $u_1 = u_2$, as claimed. □

12.B. **Continuous boundary values.** We now want to go one step further and discuss a theory of solutions to (12.1) for more general boundary values that are only required to be continuous on the compact set $\partial\Omega$, i.e., $g \in C(\partial\Omega)$. For this, we need a condition on the domain Ω under consideration:

Definition 12.5. We say that the domain Ω is *regular* for L if for every $g \in \mathrm{Lip}(\partial\Omega)$, the unique solution u of (12.1), in the sense described above, satisfies $u \in C(\bar{\Omega})$.

Remark 12.6. The De Giorgi–Nash theorem shows that the solution u is locally Hölder continuous in the interior of the domain Ω without any extra conditions, but the condition of regularity above requires that the continuity can be extended up to the boundary. It is possible to provide more concrete sufficient conditions on when this is possible (indeed, to extend the De Giorgi–Nash estimates of Hölder continuity up to the boundary in some situations), but we do not discuss this any further in these notes.

Suppose henceforth that Ω is a bounded, regular domain, and let $g \in C(\partial\Omega)$. We take for granted the density of Lipschitz-continuous functions in this space, which allows us to choose a sequence $g_j \in \mathrm{Lip}(\partial\Omega)$ such that $\|g_j - g\|_\infty \to 0$. Let u_j be the solution of (12.1) with boundary data g_j. By the assumed regularity, we have $u_j \in C(\bar{\Omega})$.

By linearity, we see that $u_j - u_k$ is a solution of (12.1) with boundary data $g_j - g_k$. By the maximum principle, it follows that

$$\|u_j - u_k\|_{C(\bar{\Omega})} = \sup_{\Omega} |u_j - u_k| \leq \sup_{\partial\Omega} |g_j - g_k| = \|g_j - g_k\|_{C(\partial\Omega)} \to 0.$$

Thus u_j is a Cauchy sequence in the space $C(\bar{\Omega})$, which is a complete space, and it follows that u_j converges uniformly to some function $u \in C(\bar{\Omega})$.

If we pick another Lipschitz sequence $\tilde{g}_j \to g$ in $C(\partial\Omega)$ with corresponding solutions $\tilde{u}_j$, then also the combined sequence $g'_{2j} = g_j$, $g'_{2j+1} := \tilde{g}_j$ satisfies $g'_j \to g$, and hence the corresponding solutions $u_j \to u$ in $C(\bar{\Omega})$, by what was said above. Since each subsequence of a convergent sequence has the same limit, this shows that the solutions $\tilde{u}_j = u'_{2j+1}$ converge to the same limit u as the first $u_j = u'_{2j}$, and hence this limit functions u is independent of the approximating Cauchy sequence.

Lemma 12.7. *This limit function u belongs to $W^{1,2}_{\mathrm{loc}}(\Omega)$, and satisfies $Lu = 0$ in the sense that*

$$\int A\nabla u \cdot \nabla \phi = 0 \qquad \forall\ \phi \in C^1_c(\Omega).$$

Proof. Since $u_j - u_k$ is a solution, we may apply the Caccioppoli inequality to see that

$$\int_{B_r} |\nabla(u_j - u_k)|^2 \leq \frac{C}{r^2} \int_{B_{2r}} |u_j - u_k|^2 \underset{j,k\to\infty}{\longrightarrow} 0 \qquad \forall B_{2r} \subset \Omega.$$

Hence ∇u_j is a Cauchy sequence in $L^2(K)$ for any compact $K \subset \Omega$ (since it can be covered by finitely many such B_r), and therefore ∇u_j converges to some $w_K \in L^2(K)$ by completeness. From the characterizations of the weak derivative it follows that $u \in W^1(\Omega)$, and $1_K \nabla u = w_K \in L^2(K)$ for every compact set $K \subset \Omega$. The continuity of the $v \mapsto \int A\nabla u_j \cdot \nabla\phi$ with respect to the relevant $W^{1,2}$ norm shows that

$$\int A\nabla u \cdot \nabla\phi = \lim_{j\to\infty} \int A\nabla u_j \cdot \nabla\phi = \lim_{j\to\infty} 0 = 0,$$

for each $\phi \in C^1_c(\Omega)$, since each u_j is a solution. □

Accordingly, we will refer to the above-constructed $u \in C(\bar{\Omega})$ as the solution of (12.1) with boundary values $g \in C(\partial\Omega)$. We take for granted that these generalized solutions still satisfy the maximum principle, which we proved under a somewhat different notion of a solution.

12.C. **The elliptic measure.** We have hence constructed a mapping

$$g \in C(\partial\Omega) \mapsto u \in C(\bar{\Omega}), \tag{12.8}$$

where u is the solution of (12.1). We observe that this mapping is:

uniquely defined: Namely, there is only one solution of (12.1), since if there were two, say u_1 and u_2, then $u := u_1 - u_2$ would be a solution of (12.1) with boundary function $g = 0$. But then we have $\sup_\Omega |u| \leq 0$ by the maximum principle, hence $u = 0$ and thus $u_1 = u_2$.

linear: If u_1 and u_2 are the solutions with boundary values g_1 and g_2, then $\alpha_1 u_1 + \alpha_2 u_2$ is a solution with boundary value $\alpha_1 g_1 + \alpha_2 g_2$, and by uniqueness, it is *the* solution.

positive: If $g \geq 0$, then $u \geq 0$. This is another application of the maximum principle: $-u$ is the solution of (12.1) with boundary function $-g$, and hence

$$\sup_{\Omega}(-u) \leq \sup_{\partial\Omega}(-u) = \sup_{\partial\Omega}(-g) \leq 0;$$

thus $u \geq 0$ in Ω.

unital: The constant function $g \equiv 1$ is mapped into the constant function $u \equiv 1$. Indeed, since L annihilates constants, if u is a solution with boundary value 1, then $u - 1$ is a solution with boundary value $1 - 1 = 0$; hence $u - 1 = 0$ by uniqueness, and $u = 1$.

Let us now compose the solution map (12.8) with the point-evaluation map $u \mapsto u(x)$ at some $x \in \Omega$. It is immediate to check that this map also has the same properties as (12.8), and that these properties are then inherited by the composition

$$g \in C(\partial\Omega) \mapsto u(x) \in \mathbb{R} \qquad (x \in \Omega \text{ fixed}). \tag{12.9}$$

In other words, the mapping (12.9) has all the properties needed to apply the following Riesz representation theorem for the space of continuous functions (not to be confused with the Riesz representation theorem for a Hilbert space, which we used in the context of the Lax–Milgram lemma):

Theorem 12.10 (Riesz representation theorem for $C(K)$)**.** *Let $\Lambda : C(K) \to \mathbb{R}$ be a positive, linear, unital functional on the space of continuous functions on a compact Hausdorff space K. Then there is a regular Borel measure μ on K such that Λ has a representation*

$$\Lambda f = \int_K f(y)\,\mathrm{d}\mu(y).$$

(Regularity of the Borel measure means that

$$\mu(E) = \sup\{\mu(C) : C \subset E \text{ compact}\} = \inf\{\mu(G) : G \supset E \text{ open}\}$$

for every measurable set E.)

Note that the topological boundary $\partial\Omega$ of any set is closed by definition. If Ω is a bounded domain, then $\partial\Omega$ is also bounded, and hence it is a compact subset of $\mathbb{R}^n$. Hence we can apply Theorem 12.10 to the map (12.9) and the compact space $K = \partial\Omega$. The resulting measure μ is then denoted by ω_L^x (since it depends both on the operator L and the point x). In different sources, it is either called the *elliptic measure* (to emphasize the connection to an elliptic operator), or the *harmonic measure* (although this name is often reserved for the classical case when $L = \triangle$ is the Laplace operator). In any case, we have the representation formula for solutions,

$$u(x) = \int_{\partial\Omega} g(y)\,\mathrm{d}\omega_L^x(y),$$

and many further properties of solutions can be established by studying the properties of the elliptic measure ω_L^x. However, our lectures finish here.

The end

References

[GT77] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Springer-Verlag, Berlin-New York, 1977. Grundlehren der Mathematischen Wissenschaften, Vol. 224.

[Ken94] Carlos E. Kenig. *Harmonic analysis techniques for second order elliptic boundary value problems*, volume 83 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1994.

Appendix A. Different function spaces

$$C^1(\Omega) := \text{continuously differentiable functions in } \Omega$$

$$C^1_c(\Omega) := \text{continuously differentiable compactly supported functions in } \Omega$$

$$\begin{aligned} L^1_{\mathrm{loc}}(\Omega) &:= \text{locally integrable functions on } \Omega \\ &:= \Big\{ u : \Omega \to \mathbb{R} \text{ measurable} \Big| \int_K |u| < \infty \ \forall K \subset \Omega \text{ compact} \Big\} \end{aligned}$$

$$\begin{aligned} \mathrm{Lip}(E) &:= \text{Lipschitz-continuous functions on } E \\ &:= \{ f : E \to \mathbb{R}; |f(x) - f(y)| \leq M|x-y| \ \forall \ x, y \in E \} \end{aligned}$$

$$M^p(\Omega) := \text{Morrey space} := \Big\{ f \in L^1_{\mathrm{loc}}(\Omega) : \int_{B_r} |f| \leq K r^{n/p'} \ \forall \text{ balls } B_r \Big\}$$

$$\begin{aligned} W^1(\Omega) &:= \text{weakly differentiable functions on } \Omega \\ &:= \{ u \in L^1_{\mathrm{loc}}(\Omega) : \partial_i u \in L^1_{\mathrm{loc}}(\Omega) \text{ exists for all } i = 1, \ldots, n \} \end{aligned}$$

$$W^{1,p}(\Omega) := \{ u \in W^1(\Omega) : u, \partial_i u \in L^p(\Omega) \ \forall i = 1, \ldots, n \} \quad \text{(Sobolev space)}$$

$$\begin{aligned} W^{1,p}_0(\Omega) &:= \text{closure of } C^1_c(\Omega) \text{ in } W^{1,p}(\Omega) \\ &:= \{ u \in W^{1,p}(\Omega) | \ \exists \ u_k \in C^1_c(\Omega) : \|u_k - u\|_{W^{1,p}} \to 0 \} \end{aligned}$$