I. PROBABILITY THEORY

3 main ingredients:

- Ω : "sample space", "set of possible outcomes"
- $\mathcal{F}$: a collection of "events" $E \subset \Omega$
- $\mathbb{P}$: probability measure $\mathbb{P} : \mathcal{F} \longrightarrow [0,1]$

I.1 FINITE PROBABILITY SPACES

Ω finite set

list the probabilities of outcomes $(p_\omega)_{\omega\in\Omega}$

$$0 \leq p_\omega \leq 1 \qquad \forall \omega \in \Omega \qquad (*)$$

$$\sum_{\omega\in\Omega} p_\omega = 1 \qquad (**)$$

all $E \subset \Omega$ are events: $\mathcal{F} = \mathcal{P}(\Omega) = \{\text{subsets of } \Omega\}$

probability $\mathbb{P}[E] = \sum_{\omega\in E} p_\omega$

Many statistical physics models are

1. first defined on a finite probab. sp. Ω_N (N = "size parameter")
2. then analyzed in the "thermodynamical limit" $N \longrightarrow \infty$

<u>Example</u> Ising model (without external magnetic field, in a d-dim. box)

d-dimensional box $V = [\![1, L]\!]^d \subset \mathbb{Z}^d$

$[\![1,L]\!] = \{1,2,\ldots,L\}$

$$\Omega = \{-1,+1\}^V = \{\text{functions } V \to \{-1,+1\}\}$$

$$= \{\omega = (\omega_v)_{v\in V} \mid \omega_v \in \{-1,+1\} \ \forall v \in V\}, \quad \underbrace{\#\Omega = 2^{\#V} = 2^{L^d}}_{\text{finite}}$$

set of neighbor pairs, i.e. "edges"

$$E = \{\langle v,v'\rangle \mid v,v' \in V, \ \|v-v'\| = 1\}$$

unordered pair of points, $\langle v,v'\rangle = \langle v',v\rangle$

set of disagreeing neighbors of a config. $\omega = (\omega_v)_{v\in V}$

$$E_{dis}(\omega) = \{\langle v,v'\rangle \in E \mid \omega_v \neq \omega_{v'}\}$$

probabilities $p_\omega = \frac{1}{Z} x^{\#E_{dis}(\omega)}$ $\left(Z = \sum_\omega x^{\#E_{dis}(\omega)} \implies (**) \right)$

Example Ehrenfest urn $\quad N \in \mathbb{N}, \; q \in [0,1]$

A toy model of gas in a container consisting of two parts, A and B.

N "particles", $\quad N_A$ in part A, $\quad N - N_A$ in part B

Dynamics (one step, to be repeated)

- choose a particle uniformly at random
- with proba q place it in part A, otherwise in part B

$$N_A \xrightarrow{\text{"step of dynamics"}} \begin{cases} N_A' = N_A + 1 & \text{with proba } \frac{N-N_A}{N} \cdot q \\ N_A' = N_A & \text{(otherwise)} \\ N_A' = N_A - 1 & \text{with proba } \frac{N_A}{N} \cdot (1-q) \end{cases}$$

Stationary distribution: N_A and N_A' have the same distribution iff $N_A \sim \mathrm{Bin}(N, q)$

(Exercise 2/week 1)

Deterministic thermodynamical limit

"density" $\quad \rho_A = \frac{N_A}{N}$

mean $\mathbb{E}[\rho_A] = q$, $\quad$ variance $\mathbb{E}[\rho_A] = \frac{q(1-q)}{N} \xrightarrow[N \to \infty]{} 0$

Uniform probability measure on a finite set

$\# \Omega < \infty \quad \Longrightarrow \quad p_\omega = \frac{1}{\# \Omega} \quad \forall \omega \in \Omega$ satisfies (i) and (ii)

trivial looking case, but often important (and can be complicated)

Example The self-avoiding polymer

d-dimensional (hyper)cubic lattice $\mathbb{Z}^d$

$$\Omega = \{ \text{injective } N \text{ step nearest neighbor paths on } \mathbb{Z}^d, \text{ starting from the origin} \}$$

$$= \{ \omega = (\omega_j)_{j=0}^{N} \mid \omega_0 = \underline{0}, \ \|\omega_j - \omega_{j-1}\| = 1 \ \forall j = 1, 2, \ldots, N, \ \omega_j \neq \omega_k \ \forall j \neq k \}$$

Clearly Ω is finite, $c_N := \# \Omega \leq (2d)^N < \infty$

The uniform measure $p_\omega = \frac{1}{\# \Omega} = \frac{1}{c_N}$ is a good model of the random shape of a d-dimensional polymer.

Example Random permutation $\quad n \in \mathbb{N}$

denote $[\![1, n]\!] = \{1, 2, \ldots, n\}$

$$S_n = \{ \sigma : [\![1,n]\!] \to [\![1,n]\!] \text{ bijective} \}$$ "symmetric group" "permutation group"

$$\# S_n = n!$$

Take $\Omega = S_n$, $\quad p_\omega = \frac{1}{n!} \quad$ "uniform random permutation"

- a shuffled deck of cards (with $n = 52$)
- natural model of (random) input for a sorting algorithm
- example questions:
    - $\mathbb{P}[\exists j \in [\![1,n]\!] \text{ s.t. } \sigma(j) = j]$ "rencontre problem"
    - the distribution of the length of the cycle that contains 1 ?
    - probability that 1 and 2 are in the same cycle ?
    - expected number of cycles ?
    - length of the longest increasing subsequence ? "last passage percolation" "polynuclear growth"

1.2 KOLMOGOROV AXIOMS AND GENERAL THEORY OF PROBABILITY

- Ω a set "sample space"
- $\mathcal{F}$ σ-algebra on Ω : a collection of subsets of Ω s.t.

  (σ-1) : $\phi \in \mathcal{F}$

  (σ-2) : $E \in \mathcal{F} \implies \Omega \setminus E \in \mathcal{F}$

  (σ-3) : $(E_n)_{n\in\mathbb{N}}$, $E_n \in \mathcal{F} \implies \bigcup_{n\in\mathbb{N}} E_n \in \mathcal{F}$

- $\mathbb{P}$ probability measure on Ω : $\mathbb{P}: \mathcal{F} \to [0,1]$ such that

  ($\mathbb{P}$-1) : $\mathbb{P}[\Omega] = 1$

  ($\mathbb{P}$-2) : $E_n \in \mathcal{F}$ (for $n\in\mathbb{N}$) disjoint : $E_n \cap E_m = \phi \;\; \forall n \neq m$

  $$\implies \mathbb{P}\Big[\bigcup_{n\in\mathbb{N}} E_n\Big] = \sum_{n\in\mathbb{N}} \mathbb{P}[E_n]$$

Random variable $\quad X : \Omega \to \mathcal{X}$

"measurable space", i.e a set equipped with a sigma-algebra of measurable sets

s.t. for all measurable $A \subset \mathcal{X}$

$\{\omega \in \Omega \mid X(\omega) \in A\}$

is an event

i.e. $X^{-1}(A) \in \mathcal{F}$

often we take $\mathcal{X} = \mathbb{R}$ with the Borel σ-algebra: the smallest σ-algebra that contains all open sets.

Expected value (integral with respect to the measure $\mathbb{P}$)

$$X : \Omega \to \mathbb{R} \quad \rightsquigarrow \quad \mathbb{E}[X] = \int X \, d\mathbb{P} = \int_\Omega X(\omega)\, d\mathbb{P}(\omega)$$

construction:

1. indicator: $\mathbb{1}_E(\omega) = \begin{cases} 1 & \text{if } \omega \in E \\ 0 & \text{if } \omega \notin E \end{cases}$, $\quad \mathbb{E}[\mathbb{1}_E] = \mathbb{P}[E]$
2. elementary random var. $\quad X = \sum_{j=1}^n x_j \cdot \mathbb{1}_{E_j}$, $\quad \mathbb{E}[X] = \sum_{j=1}^n x_j \, \mathbb{P}[E_j]$
3. positive random var. $X \geq 0$, approximate from below by elementary
4. real valued integrable random var.

   $X \in L^1(\mathbb{P})$, i.e., $\mathbb{E}[|X|] < \infty$: split to positive and negative parts

Example Finite probability spaces

Ω finite set, $\quad \mathcal{F} = \mathcal{P}(\Omega) = \{E \subset \Omega\}$, $\quad \mathbb{P}[E] = \sum_{\omega \in E} p_\omega$

Example Probability on integers $\mathbb{Z}$ (or $\mathbb{N}$)

$\Omega = \mathbb{Z} \quad (\text{or } \Omega = \mathbb{N})$

$\mathcal{F} = \mathcal{P}(\mathbb{Z}) = \{E \subset \mathbb{Z}\}$ still ok, because Ω countable

$\mathbb{P}[E] = \sum_{n \in E} p_n \qquad (p_n)_{n \in \mathbb{Z}}, \; p_n \geq 0, \; \sum_{n \in \mathbb{Z}} p_n = 1$

- ▶ Poisson (λ) : $\quad p_n = \frac{1}{n!} \lambda^n e^{-\lambda} \quad n \in \mathbb{Z}_{\geq 0} \quad (p_n = 0 \; \forall n < 0)$
- ▶ Geom(r) : $\quad p_n = r^n (1-r) \quad n \in \mathbb{Z}_{\geq 0} \quad$ (— " —)

$f : \mathbb{Z} \to \mathbb{R}, \quad \mathbb{E}[f] = \sum_{n \in \mathbb{Z}} p_n \cdot f(n)$

Example Probability on the real line $\mathbb{R}$

$\Omega = \mathbb{R}$ (or a subset of $\mathbb{R}$)

$\mathcal{F} = \mathcal{B} = \{\text{Borel subsets of } \mathbb{R}\}$

$\quad$ = smallest σ-algebra that contains all open sets

$\mathbb{P}$ for example

- ▶ Gaussian $N(\mu, \sigma^2)$: $\mathbb{P}[E] = \int_E \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx$
- ▶ Unif$([a,b])$: $\mathbb{P}[E] = \frac{|E \cap [a,b]|}{b-a}$ where $|\cdot|$ = Lebesgue measure
- ▶ Exp(λ) : $\mathbb{P}[E] = \int_{E \cap [0,\infty)} \lambda \cdot e^{-\lambda x} dx$
- ▶ delta measure at x_0 : $\mathbb{P} = \delta_{x_0} \qquad \mathbb{P}[E] = \begin{cases} 1 & \text{if } x_0 \in E \\ 0 & \text{if } x_0 \notin E \end{cases}$

If $\mathbb{P}[E] = \int_E p(x)\, dx$ with some density p, then the expected value takes the form $\mathbb{E}[f] = \int_{\mathbb{R}} p(x) f(x)\, dx$

Example Brownian motion (see Part III of the course)

$\Omega = C([0,T]) = \{\text{continuous functions } [0,T] \to \mathbb{R}\}$

$\mathcal{F}$ = Borel σ-algebra of the metric space $(C([0,T]), \underbrace{\|\cdot\|_\infty}_{\text{sup-norm}})$

$\mathbb{P}$ Wiener measure (see Part III)

Example The Maxwell-Boltzmann classical ideal gas

"container" $\quad V = [0,L]\times[0,L]\times[0,L] \subset \mathbb{R}^3$

N "particles" with "positions" $\underline{x}_j \in V \quad j=1,2,\dots,N$

and "velocities" $\underline{v}_j \in \mathbb{R}^3 \quad j=1,2,\dots,N$

$$\Omega = \underbrace{V\times V\times\cdots\times V}_{N\ \text{times}} \times \underbrace{\mathbb{R}^3\times\mathbb{R}^3\times\cdots\times\mathbb{R}^3}_{N\ \text{times}} \subset (\mathbb{R}^3)^{2N}$$

$\mathcal{F} = \{\text{Borel sets}\}$ $\quad$ note: $\Omega \subset \mathbb{R}^{6N}$

$\mathbb{P}$ has a density $(\mathbb{R}^3)^{2N} \to [0,\infty)$

$$\mathbb{P}[\underline{x}_1\in A_1,\dots,\underline{x}_N\in A_N,\ \underline{v}_1\in U_1,\dots,\underline{v}_N\in U_N]$$
$$= \frac{|A_1|}{|V|}\cdot\frac{|A_2|}{|V|}\cdot\ldots\cdot\frac{|A_N|}{V}\prod_{j=1}^{N}\left(\left(\frac{\beta}{2\pi}\right)^{3/2}\cdot\int_{U_j} d^3\underline{u}\ e^{-\frac{\beta}{2}\|\underline{u}\|^2}\right)$$

"factorized" $\longleftrightarrow$ "all coordinates and velocities independent"

position $\quad \underline{x}_j \sim \text{Uniform}(V) = \frac{1}{|V|}\cdot\text{Lebesgue}|_V$

velocity $\quad \underline{v}_j \sim N(0, {}^1\!/\!_\beta\, \mathbb{1}_{3\times3})$ $\quad$ 3-dim. Gaussian

Let us consider a random variable on this probability space.

Let $S = \|\underline{v}_1\|$ be the speed of the 1^{st} particle,
(absolute value of velocity)

What is the distribution of S?

$$\mathbb{P}[s_{min} \le S \le s_{max}] = \mathbb{E}\left[\mathbb{1}_{\{S\in[s_{min},s_{max}]\}}\right]$$

$$= \left(\frac{\beta}{2\pi}\right)^{3/2}\int_{\mathbb{R}^3} d^3\underline{u}\ e^{-\frac{\beta}{2}\|\underline{u}\|^2}\cdot\mathbb{1}_{\{\|\underline{u}\|\in[s_{min},s_{max}]\}} \leftarrow \left\{\begin{array}{l}\text{integrated out}\\ \text{all the other}\\ \text{independent variables}\end{array}\right.$$

$$= \left(\frac{\beta}{2\pi}\right)^{3/2}\int_{s_{min}}^{s_{max}} ds\ 4\pi s^2\ e^{-\frac{\beta}{2}s^2} \leftarrow \left\{\begin{array}{l}\text{changed variables}\\ \text{to radial coordinates}\end{array}\right.$$

$\therefore$ The distribution of S has density $\sqrt{\frac{2}{\pi}}\,\beta^{3/2}\, s^2 e^{-\frac{\beta}{2}s^2}$.