Infinitesimal generators of stochastic processes

Here we restrict to some simple processes such as
- Poisson process
- Jump processes on finite set
- Brownian motion

Infinitesimal generator tells about behaviour of the process during a very short time interval. The full time evolution is recovered by exponentiation of the generator and it satisfies the properties of semigroup.

The generator is
- matrix for jump processes
- differential operator for Brownian motion

More general processes would need more advanced mathematical tools. See Univ. of Helsinki math. courses:
- Semigroups and delay equations (spring 2014)
- Quantum dynamics (fall 2013)

Jump processes

Let $N = (N_t)_{t \geq 0}$ be Poisson process with intensity $\lambda > 0$.
Since $N_t \sim \text{Poisson}(\lambda t)$

$$p_k(t) := \mathbb{P}[N_t = k] = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

Exercise Show that

$$\begin{cases} \frac{d}{dt} p_k(t) = \lambda p_{k-1}(t) - \lambda p_k(t) & \text{when } k = 1, 2, 3, \ldots \\ \frac{d}{dt} p_0(t) = -\lambda p_0(t) \end{cases}$$

Let $X = (X_t)_{t \geq 0}$ be a jump process on a finite set S with intesities $\lambda_{x,y}$. Define

$$p_x(t) := \mathbb{P}[X_t = x]$$

Exercise Show that for any $x \in S$, $t \geq 0$

$$\frac{d}{dt} p_x(t) = \sum_{y \in S} \left(\lambda_{y,x} p_y(t) - \lambda_{x,y} p_x(t) \right)$$

Write $\underline{p}(t) = (p_x(t))_{x \in S}$ and consider it as a row vector. Define a matrix

$$G = \Big(\lambda_{x,y} - \delta_{x,y} \sum_{z \in S} \lambda_{x,z} \Big)_{x,y \in S}$$

We call G the <u>infinitesimal generator</u> of X. By the previous exercise

$$\frac{d}{dt} \underline{p}(t) = \underline{p}(t) G$$

Exponentiation of matrices

To solve equation of type $\frac{d}{dt}\underline{V}(t) = \underline{V}(t) M$ we review some linear algebra.

Let $\|\cdot\|$ be the usual Euclidian norm in $\mathbb{R}^n$, i.e. $\|\underline{v}\| = \left(\sum_{k=1}^{n} v_k^2\right)^{1/2}$. Define the operator norm of a matrix $M \in \mathbb{R}^{n\times n}$ as

$$\|M\|_{op} = \sup\{\|M\underline{v}\| : \underline{v} \in \mathbb{R}^n, \|\underline{v}\| \leq 1\}$$

Then $\|\cdot\|_{op}$ is a norm on the space of linear operators $L(\mathbb{R}^n) = \{T : \mathbb{R}^n \to \mathbb{R}^n \text{ linear}\}$, i.e. it satisfies

- $\|M\|_{op} = 0$ if and only if $M = 0$
- $\|M_1 + M_2\|_{op} \leq \|M_1\|_{op} + \|M_2\|_{op}$
- $\|\lambda M\|_{op} = |\lambda| \, \|M\|_{op}$

Now for any matrix M and any vector v, $\|Mv\| \leq \|M\|_{op} \|v\|$. Hence if $\|v\| \leq 1$

$$\Rightarrow \|M_1 M_2 v\| \leq \|M_1\|_{op} \|M_2 v\| \leq \|M_1\|_{op} \|M_2\|_{op} \|v\| \leq \|M_1\|_{op} \|M_2\|_{op}$$

Therefore $\|M_1 M_2\| \leq \|M_1\|_{op} \cdot \|M_2\|_{op}$.

$L(\mathbb{R}^n) \cong \mathbb{R}^{n^2}$ is finite dimensional vector space. Hence all its norms are equivalent.

$$\sup\{|M_{i,j}| : i,j \in [\![1,n]\!]\} \leq c \|M\|_{op} \qquad (*)$$

$(L(\mathbb{R}^n), \|\cdot\|_{op})$ is <u>complete</u> : if $(M^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence then the entries of $M^{(n)}$ $(M^{(n)}_{ij})_{n\in\mathbb{N}}$ form a Cauchy sequence in $\mathbb{R}$ and hence they are convergent.

<u>Lemma</u> If $M \in \mathbb{R}^{n\times n}$ is a matrix, then the series

$$\exp(M) := \sum_{n=0}^{\infty} \frac{1}{n!} M^n$$

converges in $\|\cdot\|_{op}$.

<u>Proof</u> Let $T_m = \sum_{n=0}^{m} \frac{1}{n!} M^n$. Let $m_0 \leq m_1 < m_2$

$$\| T_{m_1} - T_{m_2} \|_{op} = \Big\| \sum_{n=m_1+1}^{m_2} \frac{1}{n!} M^n \Big\|_{op}$$

$$\leq \sum_{n=m_1+1}^{m_2} \frac{1}{n!} \|M^n\|_{op} \leq \sum_{n=m_1+1}^{m_2} \frac{1}{n!} \|M\|_{op}^n$$

$$\leq \sum_{n=m_0+1}^{\infty} \frac{1}{n!} \|M\|_{op}^n \longrightarrow 0, \text{ as } m_0 \to \infty$$

Hence T_m is Cauchy and converges. $\square$

From the generator to a operator semigroup

Define for each $t \in \mathbb{R}$

$$E(t) = \exp(tG) = \sum_{n=0}^{\infty} \frac{1}{n!} t^n G^n$$

Proposition Matrices $(E(t))_{t\in\mathbb{R}}$ form a operator semigroup, whose generator is G, that is

(i) $E(0) = I$

(ii) $E(t+s) = E(t)E(s) \quad \forall s,t \in \mathbb{R}$

(iii) $t \mapsto E(t)$ is continuous

(iv) $E'(t) = \lim\limits_{h\to 0} \frac{1}{h}\left(E(t+h) - E(t)\right) = E(t)G = GE(t)$

Proof (i) $E(0) = \exp(0G) = I$

(ii) $E(t+s) = \sum_{n=0}^{\infty} \frac{1}{n!}(t+s)^n G^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!(n-k)!} t^k s^{n-k} G^n$

$= E(t)E(s)$, the same reindexing of sums as for the real exp function

(iii) $\| E(h) - I \|_{op} = \| \sum_{n=1}^{\infty} \frac{1}{n!} h^n G^n \|_{op} \le \sum_{n=1}^{\infty} \frac{1}{n!} |h|^n \|G\|_{op}^n$

$= \exp(|h| \|G\|_{op}) - 1 \longrightarrow 0$ as $h \to 0$

$\Rightarrow \| E(t+h) - E(t) \|_{op} = \| (E(h) - I) E(t) \|_{op}$

$\le \| E(h) - I \|_{op} \| E(t) \|_{op} \longrightarrow 0$ as $h \to 0$

(iv) $\| \frac{1}{h}(E(h)-I) - G \|_{op} = \| \sum_{n=2}^{\infty} \frac{h^{n-1}}{n!} G \|_{op}$

$$\le \sum_{n=2}^{\infty} \frac{|h|^{n-1}}{n!} \|G\|_{op} = \frac{1}{|h|}\left(e^{|h| \|G\|_{op}} - |h| \|G\|_{op} - 1 \right)$$

$$= \mathcal{O}(|h|) \to 0 \quad \text{as } h \to 0$$

Now similarly as in (iii)

$$\| \frac{1}{h}(E(t+h)-E(t)) - GE(t) \|_{op} \le \| \frac{1}{h}(E(h)-I) - G \|_{op} \; \|E(t)\|_{op}$$

$$\| \frac{1}{h}(E(t+h)-E(t)) - E(t)G \|_{op} \le \| \frac{1}{h}(E(h)-I) - G \|_{op} \; \|E(t)\|_{op}$$

$\square$

Remark $(E(t))_{t\in\mathbb{R}}$ is a group since $E(-t)E(t) = I$.
Often only $E(t)$, $t \ge 0$ defined and hence we call it semigroup.

Theorem The unique solution of $\frac{d}{dt} \underline{p}(t) = \underline{p}(t) G$ with initial value $\underline{p}(0)$ is

$$\underline{p}(t) = \underline{p}(0) E(t) \qquad (*)$$

Proof If $\underline{p}(t)$ is given by $(*)$, then by changing the order of differentiation and summation

$$\frac{d}{dt} \underline{p}(t) = \underline{p}(0) \left(\frac{d}{dt} E(t) \right) = \underline{p}(0) E(t) G$$

$$= \underline{p}(t) G$$

The uniqueness follows from the general theory of ODE (ordinary differential equations). $\square$

Generator and expectation values

Write a function $f : S \to \mathbb{R}$ as $\underline{f} = (f(x))_{x \in S}$ and consider it as a column vector.

$$\mathbb{E}[f(X_t)] = \sum_{y \in S} p_y(t) f(y)$$

$$= \sum_{x,y \in S} p_x(0) \left(E(t) \right)_{x,y} f(y)$$

$$= \underline{p}(0) E(t) \underline{f}$$

$$\Rightarrow \frac{d}{dt} \mathbb{E}[f(X_t)] = \underline{p}(0) E(t) G \underline{f}$$

$$= \mathbb{E}[(G f)(X_t)]$$

Infinitesimal generators of diffusions

Here we consider Brownian motion $B = (B_t)_{t \in [0,\infty)}$ and its generalizations such as $\beta B_t + \alpha t$ or $h(t, B_t)$.

If $B = (B_t)_{t \in \mathbb{R}}$ is standard Brownian motion, then the probability density of B_t is

$$p_t(x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right)$$

Exercise (a) Show that $\frac{\partial}{\partial t} p_t(x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} p_t(x)$

(b) If d-dimensional Brownian motion $\underline{B} = (\underline{B}_t)_{t \in [0,\infty)}$ is defined as

$$\underline{B}_t = \underline{e}_1 B_t^{(1)} + \dots + \underline{e}_d B_t^{(d)}$$

where $B^{(j)}$ are independent, one-dimensional standard Brownian motions and if $p^{(n)}$ is the prob. density of $\underline{B}_t$, show that

$$\frac{\partial}{\partial t} p_t(\underline{x}) = \frac{1}{2} \Delta p_t(\underline{x}), \quad \Delta = \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2}$$

Write the expected value of $f(B_t)$ when f is twice continuously differentiable and with compact support as

$$\mathbb{E}[f(B_t)] = \int_{\mathbb{R}} p_t(x) f(x)\, dx$$

Then for any $t > 0$, by dominated convergence theorem

$$\frac{d}{dt}\mathbb{E}[f(B_t)] = \lim_{h\to 0} \frac{1}{h}\left(\mathbb{E}[f(B_{t+h})] - \mathbb{E}[f(B_t)]\right)$$

$$= \int_{\mathbb{R}} \lim_{h\to 0} \frac{1}{h}\left(p_{t+h}(x) - p_t(x)\right) f(x)\,dx$$

$$= \int_{\mathbb{R}} \left(\frac{1}{2}\partial_{xx} p_t(x)\right) f(x)\,dx$$

The generator of B is $\frac{1}{2}\partial_{xx}$ when it acts on the probability density. Its adjoint is $L = \frac{1}{2}\partial_{xx}$. By twice partially integrating

$$\frac{d}{dt}\mathbb{E}[f(B_t)] = \frac{1}{2}\int_{\mathbb{R}} p_t(x) f''(x)\,dx$$

$$= \mathbb{E}[(Lf)(B_t)]$$

Therefore the operator L is the generator acting on a function. Easily, the mapping $t \mapsto \mathbb{E}[f''(B_t)]$ is continuous hence the previous equation extends to $t=0$ and for example, we have

$$\mathbb{E}[f(B_t)] = \mathbb{E}[f(B_s)] + \int_s^t \mathbb{E}[(Lf)(B_u)]\,du$$

for any $0 \le s \le t$.

We give here more general result without going to the details of the proof.

Theorem If $f : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable and there exist constants $C, \varepsilon \geq 0$ such that $|f''(x)| \leq C e^{\varepsilon |x|}$, then $t \mapsto \mathbb{E}[f(B_t)]$ is continuously differentiable and

$$\partial_t \mathbb{E}[f(B_t)] = \mathbb{E}[Lf(B_t)]$$

where $Lf = \frac{1}{2} f''$.

In the same sense,

- The generator of $X_t = \beta B_t + \alpha t$ is

$$L = \frac{\beta^2}{2} \frac{\partial^2}{\partial x^2} + \alpha \frac{\partial}{\partial x}$$

- The generator of $X_t = h(t, B_t)$, where $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable once in the 1st variable and twice in the 2nd one, is

$$L_t = \frac{1}{2} (\partial_t h(t,x))^2 \frac{\partial^2}{\partial x^2} + \left(\partial_t h(t,x) + \frac{1}{2} \partial_{xx} h(t,x) \right) \frac{\partial}{\partial x}$$

Remark
- αt in $X_t = \beta B_t + \alpha t$ is called drift of the process
- βB_t is the diffusion part of X_t
- We can also interpret that $X_t = h(t, B_t)$ has diffusion and drift parts. You can learn more about this in 'Stochastic Analysis' course (Univ. of Helsinki).