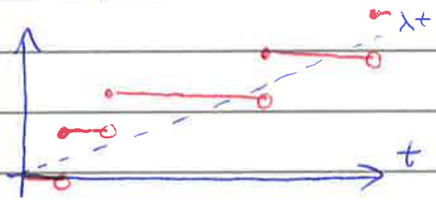
Stochastic processes using exponential clocks

In this section we explain how to construct other stochastic processes from Poisson processes.



Summary: Poisson process

Poisson process $N = (N_t)_{t \geq 0}$ is continuous-time process which takes discrete +1 steps at random times.

A characterization of N: Intensity $\lambda > 0$ is a parameter.

(1) $N_0 = 0$

(2) $t \mapsto N_t(\omega)$ is non-decreasing, right-continuous step function whose jumps are $+1$

$$\left(N_t - \lim_{s \uparrow t} N_t \in \{0, 1\} \quad \forall t \right)$$

(3) Increments are independent and stationary and $N_{t+s} - N_t \sim \text{Poisson}(\lambda s) \quad \forall s, t \geq 0$

Thinning of Poisson process:

Proposition Let S be a finite set and $(p_x)_{x \in S}$ a probability distribution on S. Let $(X_j)_{j>0}$ be i.i.d. with $\mathbb{P}[X_j = x] = p_x \ \forall x \in S$. If $N = (N_t)_{t \geq 0}$ is Poisson process with intesity λ independet of $(X_j)_{j > 0}$ then the processes $N^{(x)} = (N_t^{(x)})_{t \geq 0}$

$$N_t^{(x)} = \# \{ j \leq N_t : X_j = x \}$$

are Poisson process with intesities λp_x. $\{ N_t^{(x)} : x \in S \}$ are independent.

Continuous-time random walk

Let $S = (S_n)_{n \geq 0}$ be a random walk on $\mathbb{Z}$ with parameter p, that is, let $(\xi_k)_{k>0}$ be i.i.d., $\mathbb{P}[\xi_k = +1] = p$, $\mathbb{P}[\xi_k = -1] = 1-p$

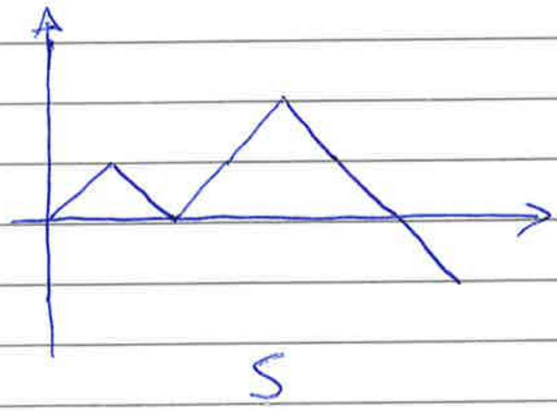
$$S_n = \sum_{k=1}^{n} \xi_k, \quad n > 0, \qquad S_0 = 0$$

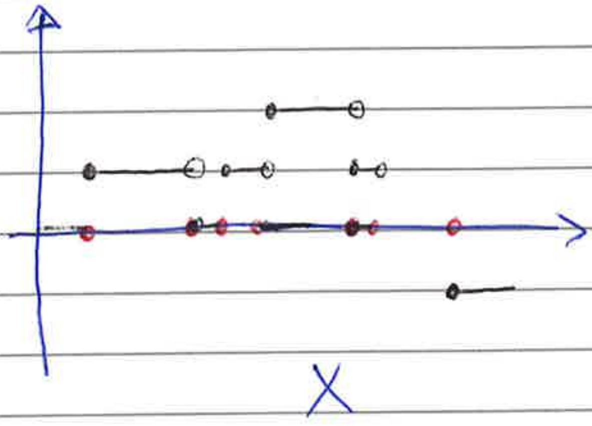
Let $N = (N_t)_{t \geq 0}$ be Poisson process independent of $(\xi_k)_{k>0}$ with intensity $\lambda > 0$.
We define <u>continuous-time random walk</u> as $X = (X_t)_{t \geq 0}$ by

$$X_t = S_{N_t}.$$

<u>Remark</u> X takes random ± 1 steps at random times. Steps are the same as for S.

After a step X waits $\mathrm{Exp}(\lambda)$ random time before it takes the next step.

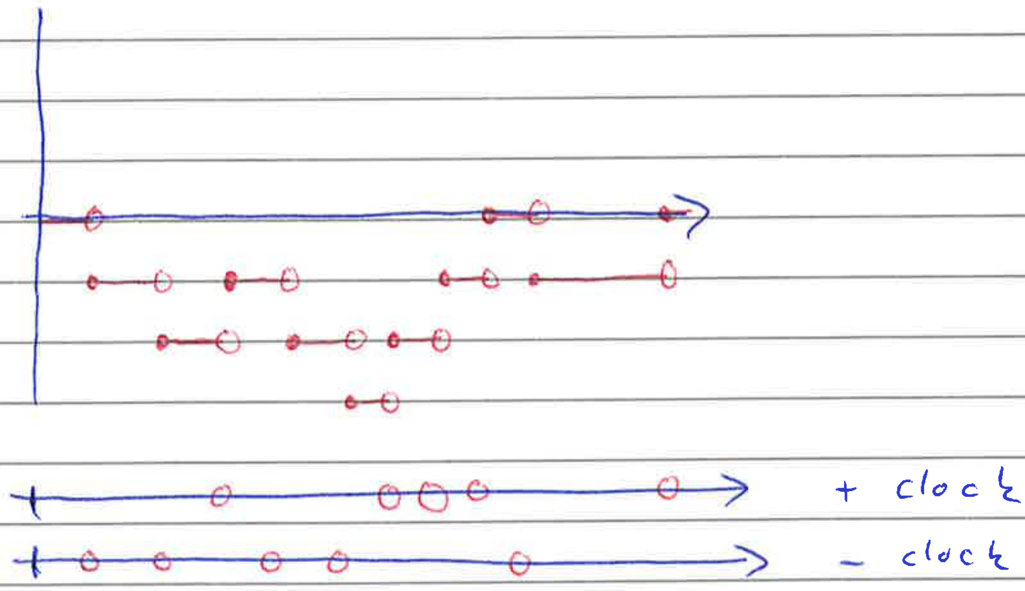




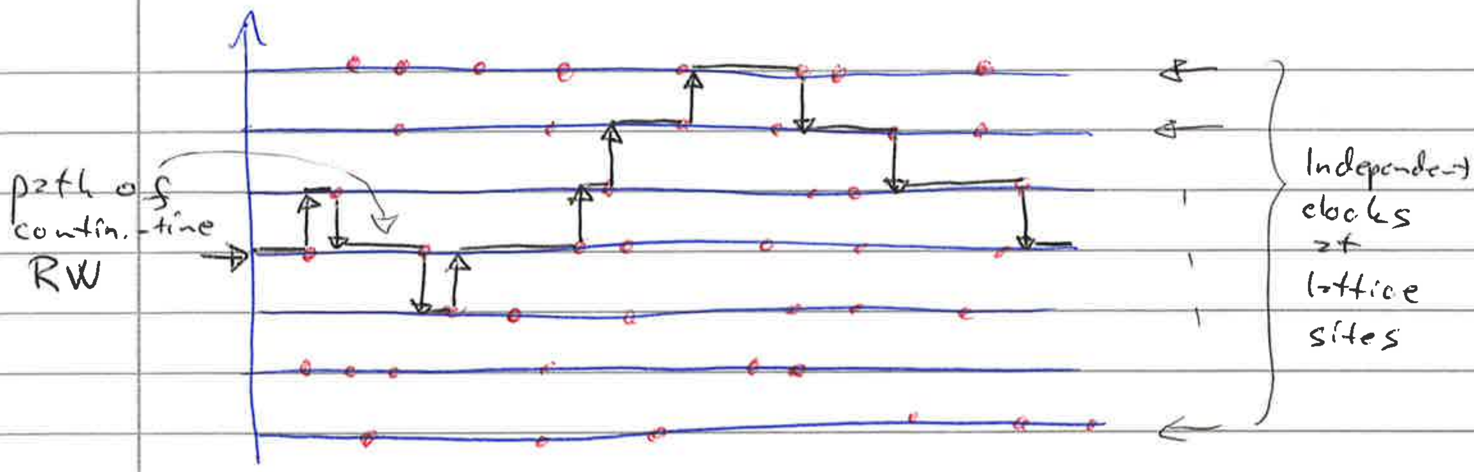
Propositio If $(N_t^+)_{t\geq 0}$ and $(N_t^-)_{t\geq 0}$ are two independent Poisson processes with intesities $p\lambda$ and $(1-p)\lambda$, then $t \mapsto N_t^+ - N_t^-$ is distributed as continuous-time random walk $t \mapsto X_t$.

Proof This follows directly from thinning: we can take $(\xi_t)_{t>0}$ as the thinning random variables. $\square$

Interpretation of the proposition: we can construct the walk so that the walker carries two independent exponential clocks with parameters λp and $\lambda(1-p)$, one for $+1$ steps and the other for -1 steps.



Clocks on each lattice site: alternatively we can place clocks on each lattice site. When the clock rings, if the walker is on that site, it jumps. If the walker is not there nothing happens.



Notice that in this approach it is not clear that the process is well-defined. Since the number of clocks ringing during any non-empty time interval is ∞, it might be possible that X escapes to ∞ in finite time.

Limit theorems of contin.-time RW

To calculate $\mathbb{E}[X_t]$ and $Var[X_t]$, we recall the following elementary lemma.

Lemma ((simple versions of) Wald's identities)
Suppose that $(\xi_k)_{k>0}$ is i.i.d and $N \geq 0$ is integer valued random variable independent of $(\xi_k)_{k>0}$. Then

(i) If $\mathbb{E}[N] < \infty$ and $\mathbb{E}[|\xi_k|] < \infty$, then

$$\mathbb{E}\left[\sum_{k=1}^{N} \xi_k\right] = \mathbb{E}[N]\,\mathbb{E}[\xi_1]$$

(ii) If $\mathbb{E}[N^2] < \infty$ and $\mathbb{E}[\xi_k^2] < \infty$, then

$$Var\left[\sum_{k=1}^{N} \xi_k\right] = \mathbb{E}[N]\,Var[\xi_1] + Var[N]\,\mathbb{E}[\xi_1]^2$$

Proof $\mathbb{E}\left[\sum_{k=1}^{N} \xi_k\right] = \mathbb{E}\left[\sum_{n=0}^{\infty} \mathbb{1}_{N=n} \sum_{k=1}^{n} \xi_k\right]$

$$\overset{(*)}{=} \sum_{n=0}^{\infty} \mathbb{P}[N=n] \sum_{k=1}^{n} \mathbb{E}[\xi_k]$$

$$= \sum_{n=0}^{\infty} \mathbb{P}[N=n] \cdot n\,\mathbb{E}[\xi_1] = \mathbb{E}[N]\,\mathbb{E}[\xi_1]$$

In $(*)$ we used Fubini. We needed $\mathbb{E}[N] < \infty$ and $\mathbb{E}[|\xi_k|] < \infty$. Hence (i) follows.

The proof of (ii) is similar. $\square$

Now if $\mathbb{E}[\xi_i = +1] = p$, $\mathbb{E}[\xi_i = -1] = 1-p$, $S_n = \sum_{i=1}^{n} \xi_i$ and $N \sim \text{Poisson}(\lambda)$, then

$$\mathbb{E}[\xi_1] = 2p-1 \qquad \mathbb{E}[N] = \lambda$$
$$\text{Var}[\xi_1] = 4p(1-p) \qquad \text{Var}[N] = \lambda$$

and hence by Wald's identities

$$\mathbb{E}[S_N] = \lambda(2p-1)$$
$$\text{Var}[S_N] = \lambda \text{Var}[\xi_1] + \lambda \mathbb{E}[\xi_1]^2 = \lambda \mathbb{E}[\xi_1^2] = \lambda$$

<u>Theorem</u>

(Law of large numbers) $\frac{X_t}{t} \xrightarrow[t\to\infty]{a.s.} (2p-1)\lambda$

(Central limit theorem) $\frac{X_t - (2p-1)\lambda t}{\sqrt{t}} \xrightarrow[t\to\infty]{w} N(0, \lambda)$

(Donsker) $X^{(n)} \xrightarrow[n\to\infty]{w} B$ where B is a standard Brownian motion and $X^{(n)}_t = \frac{1}{\sqrt{\lambda n}} X_{nt}$, $p = \frac{1}{2}$.
(We can use linear interpolation here.)

<u>Remark</u> The limit theorem are the same as for the process $S^{sym.}_{\lambda t} + (2p-1)\lambda t$ where S^{sym} is a symmetric random walk. We call $(2p-1)\lambda t$ the <u>drift</u>.

Example (Distribution of X_t and probability generating functions)

Define probability generating functions

$$\begin{cases} G_{N_t}(z) = \sum_{n=0}^{\infty} \mathbb{P}[N_t = n]\, z^n = \mathbb{E}[z^{N_t}] \\ G_{\xi}(z) = \sum_{x \in \mathbb{Z}} \mathbb{P}[\xi = x]\, z^x = \mathbb{E}[z^{\xi}] \end{cases}$$

Which can be easily calculated

$$\begin{cases} G_{N_t}(z) = \sum_{n=0}^{\infty} \frac{(\lambda t)^n e^{-\lambda t}}{n!} z^n = \exp(\lambda t(z-1)) \\ G_{\xi}(z) = pz + (1-p) z^{-1} \end{cases}$$

Therefore

$$G_{S_n}(z) = \mathbb{E}\left[\prod_{k=1}^{n} z^{\xi_k}\right] \overset{\text{independence}}{=} \prod_{k=1}^{n} \mathbb{E}[z^{\xi_k}]$$

$$= G_{\xi}(z)^n$$

Similarly as before we will use $1 = \sum_{n=0}^{\infty} \mathbb{1}_{N_t = n}$ to calculate

$$G_{X_t}(z) = \mathbb{E}[z^{X_t}] = \mathbb{E}\left[\sum_{n=0}^{\infty} \mathbb{1}_{N_t=n} \prod_{k=1}^{n} z^{\xi_k}\right]$$

$$\overset{\text{Fubini}}{=} \sum_{n=0}^{\infty} \mathbb{P}[N_t = n]\, \mathbb{E}\left[\prod_{k=1}^{n} z^{\xi_k}\right]$$

$$\overset{\text{independence}}{=} \sum_{n=0}^{\infty} \mathbb{P}[N_t = n] \left(\mathbb{E}[z^{\xi_k}]\right)^n$$

$$= G_{N_t}(G_{\xi}(z))$$

We got an expicit formula

$$G_{X_t}(z) = \exp\left(\left(pz + (1-p)z^{-1} - 1\right)\lambda t\right)$$

Example (Continuous-time RW on $\mathbb{Z}^d$)

Let $(\xi_k)_{k>0}$ be i.i.d sequence of uniform random variables on $\{-\underline{e}_1, \underline{e}_1, -\underline{e}_2, \underline{e}_2, \ldots, -\underline{e}_d, \underline{e}_d\} \subset \mathbb{Z}^d$ and let $N=(N_t)_{t\geq 0}$ be independent Poisson process with intensity $\lambda>0$.

Set $S_n = \sum_{k=1}^{n} \xi_k$. Continuous-time RW on $\mathbb{Z}^d$ is defined as $X=(X_t)_{t\geq 0}$

$$X_t = S_{N_t}$$

Write X in terms of the components

$$X_t = \underline{e}_1 X_t^{(1)} + \underline{e}_2 X_t^{(2)} + \ldots + \underline{e}_d X_t^{(d)}$$

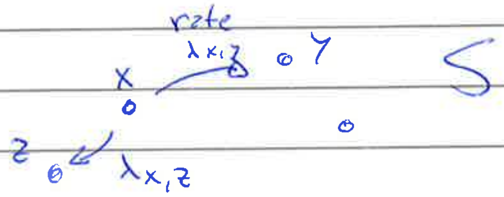
Then by thinning the processes $X^{(k)} = (X_t^{(k)})_{t\geq 0}$ are independent Poisson processes with intensities λ/d.

(Compare to Exercise 5 of the problem set 3)

Jump processes

Let S be a finite set which will be the <u>state space</u> of $X = (X_t)_{t \geq 0}$, $X_t \in S$.

Let $\lambda_{x,y} \geq 0$ be <u>jump intensities</u> from state $x \in S$ to state $y \in S$.



The process X is constructed from Poisson processes $N^{(x,y)} = (N_t^{(x,y)})_{t \geq 0}$ which are assumed to be independent and have intensities $\lambda^{(x,y)}$.

If $N_{t^-}^{(x,y)} = \lim_{s \nearrow t} N_t^{(x,y)}$, then we say that clock $N^{(x,y)}$ rings at time t, if $N_t^{(x,y)} = N_{t^-}^{(x,y)} + 1$.

Properties:

(1) During a finite time interval only finite number of clocks ring: for any $t \geq 0$

$$\mathbb{P}\left[\sum_{x,y} N_t^{(x,y)} < \infty \right] = 1$$

(2) There is no time such that two or more clocks ring at the same time

$$\mathbb{P}\left[\exists\, (x,y) \neq (x',y') \text{ and } t \text{ s.t. } N_t^{(x,y)} = N_{t^-}^{(x,y)} + 1,\ N_t^{(x',y')} = N_{t^-}^{(x',y')} + 1 \right] = 0$$

Both (1) & (2) follow from the fact that $N_t = \sum_{x,y} N_t^{(x,y)}$ is Poisson process. (1) $\Leftrightarrow$ N_t is finite. (2) $\Leftrightarrow$ $P[N_t > N_{t^-} + 1 \text{ for some } t \geq 0] = 0$

Given the paths of the Poisson counters $t \mapsto N_t^{(x,y)}(\omega)$ define

$$A(\omega) = \{ (t,x,y) \in [0,\infty) \times S \times S : N_t^{(x,y)} = N_{t^-}^{(x,y)} + 1 \}$$

Define the jump process $X = (X_t)_{t \in [0,\infty)}$ with intensities $\lambda^{(x,y)}$ by the following algorithm:

(i) Set $X_0 = x_0$ where $x_0 \in S$ is a fixed initial value. Put values $\xi = x_0$ and $\tau = 0$ to memory where τ is the time up to which have constructed the process so far and ξ is the current state.

(ii) Find smallest $t > \tau$ s.t. $(t, \xi, y) \in A$ for some $y \in S$. Add the values $X_s = \xi$, $\tau < s < t$, $X_t = y$ to the path of X. Set $\xi = y$ and $\tau = t$.

(iii) Repeat (ii) indefinitely.

By properties (1) & (2) of $N^{(x,y)}$, X is well-defined: on stage (ii) the point y is unique and to determine the value of X_t we need to repeat (ii) only finite (but random) number of times.

The behavior of X is summerized:

- Sleeping: $t \mapsto X_t$ is constant on any interval where all the counter processes $t \mapsto N_t^{(x,y)}$ are constant.
- Jumping: If $N_t^{(x,y)} = N_{t^-}^{(x,y)} + 1$, $X_{t^-} = x$, then $X_t = y$.
- Paths $t \mapsto X_t$ are piece-wise constant and right-continuous

Example (Glauber dynamics of Ising model)

This is a natural dynamics for Ising model, also called heat bath dynamics, where we flip one spin at the time and the rate is proportional to the Boltzmann weight $\exp(-\beta H(\underline{\sigma}'))$ of the state $\underline{\sigma}'$ where we are jumping to.

Let $G=(V,E)$ be a finite graph, $S=\{-1,+1\}^V$

For fixed $\beta>0$, $B\in\mathbb{R}$, Hamiltonian is

$$H(\underline{\sigma}) = -\sum_{\{v,w\}\in E} \sigma_v\sigma_w - B\sum_{v\in V}\sigma_v$$

and we define the Glauber dynamics of Ising model as a jump process on S with intensities

$$\lambda_{\underline{\sigma},\underline{\sigma}'} = \frac{\exp(-\beta H(\underline{\sigma}'))}{\exp(-\beta H(\underline{\sigma}))+\exp(-\beta H(\underline{\sigma}'))}$$

when $\#\{v\in V : \sigma_v\neq\sigma'_v\}=1$ and $\lambda_{\underline{\sigma},\underline{\sigma}'}=0$ otherwise.

Example (Alternative formulation of Glauber dynamics)

One can show that (an excersise), if $\mathbb{P}_\beta$ is the Boltzmann distribution of the Ising model, then

$$(*) \qquad \mathbb{P}_\beta\left[\sigma_v = \xi_v \mid \sigma_w = \xi_w \ \forall w \in V \setminus \{v\} \right] = \frac{\exp(-\beta H(\underline{\xi}))}{\exp(-\beta H(\underline{\xi})) + \exp(-\beta H(\underline{\xi}'))}$$

for any $\underline{\xi} \in \{-1,+1\}^V$ and for any $v \in V$. Here $\underline{\xi}'$ is the configuration for which $\xi'_w = \xi_w$ $w \neq v$ and $\xi'_v = -\xi_v$.

Glauber dynamics is equivalently given by (an exercise)

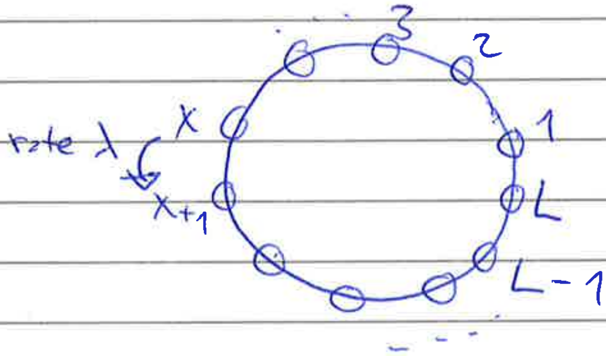
- Introduce Poisson clocks to every site which are independent and of unit intensity.
- When a clock rings at site v, resample the spin σ_v from the conditional distribution $(*)$.

Example (TASEP, totally asymmetric simple exclusion process)

$(n=1)$ $S = [\![1, L]\!]$, $\lambda > 0$

Define $X = (X_t)_{t \geq 0}$ on S with transition intensities

$$\lambda_{x,y} = \begin{cases} \lambda & , \text{ if } y - x \equiv 1 \pmod{L} \\ 0 & , \text{ otherwise} \end{cases}$$



We can define $N = (N_t)_{t \geq 0}$ Poisson process so that $X_t \equiv X_0 + N_t \pmod{L}$.

Average speed : $\lim_{t \to \infty} \frac{N_t}{t} = \lambda$

$(n=2)$ $S = \{ (x_1, x_2) \in [\![1, L]\!]^2 : x_1 \neq x_2 \}$, $\lambda > 0$

Define $X = (X_t)_{t \geq 0}$ on S with intensities

$$\lambda_{(x_1,x_2),(x_1',x_2')} = \begin{cases} \lambda & , \text{ when } x_1' - x_1 \equiv 1 \pmod{L},\ x_2' = x_2 \\ \lambda & , \text{ when } x_2' - x_2 \equiv 1 \pmod{L},\ x_1' = x_1 \\ 0 & , \text{ otherwise} \end{cases}$$

for any $(x_1, x_2), (x_1', x_2') \in S$.

Average speed of particle 1 should be lower than in the $n=1$ case.

In general) $n \leq L$

$$S = \{ (x_1, x_2, \dots, x_n) \in [\![1,L]\!]^n : x_j \neq x_k \ \forall j \neq k \}$$

Similarly as above $\lambda_{(x_1,\dots,x_n),(x_1',\dots,x_n')}$ is equal to λ if one particle moves and the configuration remains in S.