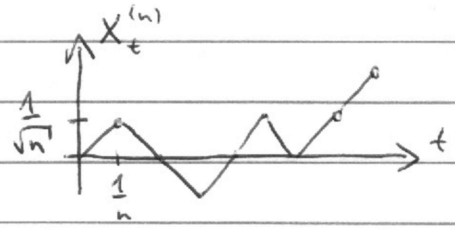
Part III Continuous-time processes

In this part of the course we look at some stochastic processes $X=(X_t)_{t\in\mathbb{R}_{\geq 0}}$ which are defined in continuous time, i.e. $t\in\mathbb{R}_{\geq 0}=[0,\infty)$.

Section 1 Brownian motion



As before we look random sums $S=(S_n)_{n\in\mathbb{Z}_{\geq 0}}$

$$S_n=\sum_{k=1}^{n}\xi_k \quad , \quad S_0=0$$

Define linear interpolation so that discrete time step corresponds to $\frac{1}{n}$ time in the cont. time and space is scaled by $\frac{1}{\sqrt{n}}$.

$$X_t^{(n)}=\frac{1}{\sqrt{n}}\left(S_{\lfloor nt\rfloor}+(nt-\lfloor nt\rfloor)\,\xi_{\lfloor nt\rfloor+1}\right)$$

Notation $\mathbb{R}\ni x=\lfloor x\rfloor+r \quad , \quad \lfloor x\rfloor\in\mathbb{Z},\ r\in[0,1)$

We aim to show the following result

Theorem (Donsker) If $(\xi_k)_{k\geq 1}$ are i.i.d and $\mathbb{P}[\xi_k=+1]=\mathbb{P}[\xi_k=-1]=\frac{1}{2}$, then $X^{(n)}=(X_t^{(n)})_{t\in\mathbb{R}_{\geq 0}}$ converges to (a) Brownian motion in the sense of (b) weak convergence of probability measures on the (c) space of continuous functions.

Before the proof we discuss topics (a) – (c).

Section 1.1 Multivariate Gaussian

<u>Definition</u> A random vector $\underline{\xi} = (\xi_1, \dots, \xi_n)$ is a <u>multivariate Gaussian</u> (Gaussian vector) if for all $\underline{a} \in \mathbb{R}^n$, the r.v.

$$\underline{a} \cdot \underline{\xi} = \sum_{j=1}^{n} a_j \xi_j$$

is a Gaussian random variable.

The law of multivariate Gaussian is determined by its mean vector $\underline{M} \in \mathbb{R}^n$ and covariance matrix $C \in \mathbb{R}^{n \times n}$.

$$\underline{M} := (\mathbb{E}[\xi_1], \dots, \mathbb{E}[\xi_n]), \quad C_{ij} = \mathrm{Cov}[\xi_i, \xi_j]$$

To see this, let $\underline{\theta} \in \mathbb{R}^n$ and $X = \sum_{j=1}^{n} \theta_j \xi_j$. Then X is $N(m, \sigma^2)$ with

$$m = \sum_{j=1}^{n} \theta_j M_j, \quad \sigma^2 = \sum_{j,k=1}^{n} \theta_j \theta_k C_{jk}$$

$$\begin{aligned}
\Rightarrow \chi_{\underline{\xi}}(\underline{\theta}) &= \chi_{\underline{\xi}}(\theta_1, \dots, \theta_n) = \mathbb{E}\left[e^{i \underline{\theta} \cdot \underline{\xi}} \right] \\
&= \chi_X(1) = e^{im - \frac{1}{2}\sigma^2} \\
&= \exp\left(\sum_{j=1}^{n} \theta_j M_j - \frac{1}{2} \sum_{j,k=1}^{n} \theta_j \theta_k C_{jk} \right)
\end{aligned}$$

<u>Proposition</u> If $\underline{\xi} = (\xi_1, \dots, \xi_n)$ is multivariate Gaussian, then the following are equivalent

(1) $\xi_1, \dots, \xi_n$ are independent

(2) C_{jk} is diagonal, that is, $\mathrm{Cov}[\xi_j, \xi_k] = 0 \quad \forall j \neq k$

The definition of multivariate Gaussian doesn't require existence of probability density in $\mathbb{R}^n$.
But if the density exists, it's of the form

$$p(\underline{x}) = \frac{1}{(2\pi)^{n/2} \det(C)^{1/2}} \exp\left(-\frac{1}{2}(\underline{x}-\underline{\mu})^T C^{-1} (\underline{x}-\underline{\mu})\right)$$

Extend the definition to $X = (X_t)_{t \in \mathbb{R}_{\geq 0}}$

Definition A stochastic process $X = (X_t)_{t \in \mathbb{R}_{\geq 0}}$ is a Gaussian process if for all n and $0 \leq t_1 < t_2 < \dots < t_n$, the random vector

$$(X_{t_1}, \dots, X_{t_n})$$

is multivariate Gaussian.

Section 1.2 | Definition of Brownian motion

Definition $X=(X_t)_{t\in\mathbb{R}_+}$ has *stationary increments* if the law of $X_{t+s}-X_t$ doesn't depend on t.
X has *independent increments* is the random variables $(X_{t_2}-X_{t_1}, X_{t_3}-X_{t_2}, \dots, X_{t_n}-X_{t_{n-1}})$ are independent for all n and $0\le t_1<t_2<t_3<\dots<t_n$.

Proposition For $X=(X_t)_{t\in\mathbb{R}_+}$ the following are equivalent

(a) X has independent and stationary increments and $X_t\sim N(0,t)$

(b) X is Gaussian and $\mathbb{E}[X_t]=0$, $\mathrm{Cov}[X_s,X_t]=\min\{s,t\}$

Notation $s\wedge t=\min\{s,t\}$, $s\vee t=\max\{s,t\}$

Proof Assume (a). Let $0=t_0<t_1<\dots<t_n$, $a_k\in\mathbb{R}$. Then there is $b_k\in\mathbb{R}$ (by partial summation) s.t.

$$\sum_{k=1}^n a_k X_{t_k}=\sum_{k=1}^n b_k(X_{t_k}-X_{t_{k-1}})$$

Since $X_{t_k}-X_{t_{k-1}}\sim N(0,t_k-t_{k-1})$ are independent, this is Gaussian and X is Gaussian process. Now $\mathrm{Cov}[X_s,X_t]=\mathbb{E}[X_tX_s]=\mathbb{E}[X_s^2]+\mathbb{E}[(X_t-X_s)X_s]=s$ when $s<t$. Thus (b) follows

Assume now (b). For any $0\le s<t$, X_t-X_s is normally distributed and $\mathbb{E}[X_t-X_s]=\mathbb{E}[X_t]-\mathbb{E}[X_s]=0$

$$Var[X_t - X_s] = Var[X_t] - 2\,Cov[X_t, X_s] + Var[X_s]$$
$$= t - 2s + s = t - s$$

$\Rightarrow X$ has stationary increments, $X_t \sim N(0, t)$

We need to show that the increments are independent. We show that their covariances vanish which is enough for Gaussian vectors. Let $0 \le s < t < u$

$$Cov[X_t - X_s, X_u - X_t]$$
$$= Cov[X_t, X_u] - Cov[X_t, X_t] - Cov[X_s, X_u] + Cov[X_s, X_t]$$
$$= t - t - s + s = 0$$

Hence $(X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}})$ are independent for $t_1 < t_2 < \dots < t_n$. Thus (a) follows.

$\square$

<u>Definition</u> (0) Stoch. process $X = (X_t)_{t \ge 0}$ is <u>continuous process</u> if

$$\mathbb{P}[\{\omega : t \mapsto X_t(\omega) \text{ is continuous}\}] = 1$$

(1) Standard Brownian motion $B = (B_t)_{t \in \mathbb{R}_+}$ is a stoch. process which satisfies

(I) B is continuous process

(II) B satisfies either one of equivalent conditions (a) or (b) of the previous proposition.

<u>Remark</u> Requirement (I) is non-trivial. There are discontinuous processes satisfying (II).