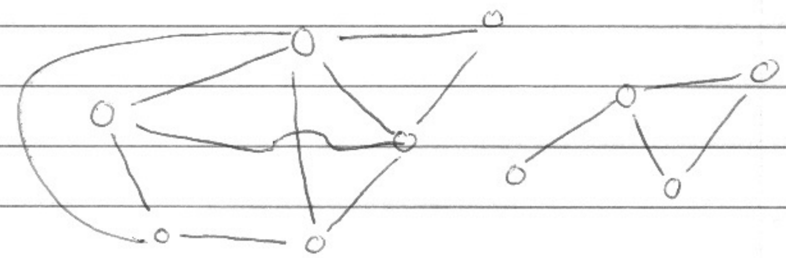
Section 2.1 Definition of Ising model on a graph G

Definition The pair $G=(V,E)$ is called (finite) graph if V is a finite, non-empty set and $E \subset \{ \{v,w\} \subset V : v \neq w \}$.
We call

- $v \in V$ a vertex (point)
- $e = \{v,w\} \in V$ an edge (line)

A vertex w is a neighbor of v if $\{v,w\} \in E$.
A graph is connected if it is connected by nearest-neighbor paths.



Degree of a vertex is $\deg(v) = \#\{w : \{v,w\} \in E\}$

Example (Ising model)
Let $G=(V,E)$ be connected graph.
Let $B \in \mathbb{R}$. For $\underline{\sigma} = (\sigma_v)_{v\in V} \in \{-1,+1\}^V$ define

$$H_B(\underline{\sigma}) = -\sum_{\{v,w\}\in E} \sigma_v \sigma_w - B \sum_{v \in V} \sigma_v$$

and for $\beta \geq 0$

$$\begin{cases} Z(\beta,B) = \sum_{\underline{\sigma}} \exp(-\beta H_B(\underline{\sigma})) \\ \mathbb{P}_{\beta,B}(\underline{\sigma}) = \frac{1}{Z(\beta,B)} \exp(-\beta H_B(\underline{\sigma})) \end{cases}$$

$\beta = 0$: independent coin flips $\sigma_v = \pm 1$

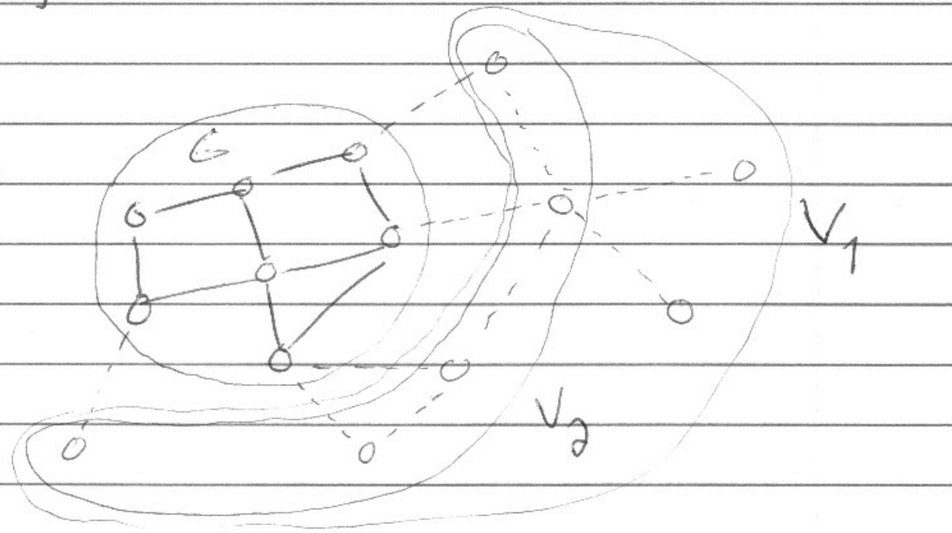
$\beta = \infty$: two minimum energy configurations $\sigma_v \equiv +1$ and $\sigma_v \equiv -1$.

Boundary conditions in Ising model

Example (+1-boundary cond.)

To define constant boundary conditions (b.c.) suppose $G \subset G'$ (i.e. $V(G) \subset V(G')$ and $E(G) \subset E(G')$).

Then $V_1 = V(G') \setminus V(G)$ are the vertices outside G. $V_\partial = \{v \in V_1 : \{v,w\} \in E(G')$ for some $w \in V\}$ are the vertices closest to $V(G)$



Use $\Omega = \{-1,+1\}^{V(G')}$, restrict to $\sigma \in \Omega$ s.t. $\sigma_v = +1 \ \forall v \in V_1$ and use the Hamiltonian of G'. Then Ising model on G with +1-b.c. set

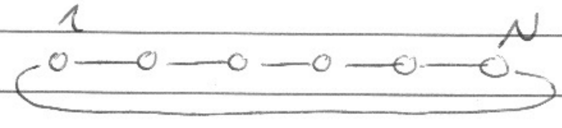
$$\mathbb{P}_{G,\beta,B,+}[\underline{\sigma}] = \frac{1}{Z_{G,+}(\beta,B)} e^{-\beta H_{G',B}(\underline{\sigma})} \mathbb{1}_{\{\sigma_v=+1,\ \forall v \in V_1\}}$$

where $Z_{G,+}(\beta,B) = \sum_{\underline{\sigma}} e^{-\beta H_{G',B}(\underline{\sigma})} \mathbb{1}_{\{\sigma_v=+1,\ \forall v \in V_1\}}$

Similarly define $\underline{\xi}$-b.c. for any $\underline{\xi} \in \{-1,+1\}^{V(G')}$ so that $\sigma_v = \xi_v \ \forall v \in V_1$.

Example (Periodic boundary conditions)

For $[\![1, N]\!]$ add edge between 1 and N



Similarly in $[\![1, N]\!]^d$ connect corresponding points in opposite faces of the hypercube.

$\boxed{\text{Section 2.2}}$ One dimensional Ising

Boundary cond. $s, t \in \{\pm 1, 0\}$

$$\underset{\sigma_0 = s}{\circ} - \underset{\sigma_1}{\circ} - \underset{\sigma_2}{\circ} - \cdots - \circ - \underset{\sigma_N}{\circ} - \underset{\sigma_{N+1} = t}{\circ}$$

$$Z = \sum_{\underline{\sigma}} \exp\left(\beta \sum_{k=0}^{N} \sigma_k \sigma_{k+1} + \beta B \sum_{k=1}^{N} \sigma_k\right)$$

$$= e^{-\beta B s} \sum_{\underline{\sigma}} \exp\left(\beta \sum_{k=0}^{N} \left(\sigma_k \sigma_{k+1} + B \sigma_k\right)\right)$$

$$= e^{-\beta B s} \sum_{\sigma_1 = \pm 1} \sum_{\sigma_2 = \pm 1} \cdots \sum_{\sigma_N = \pm 1} \prod_{k=0}^{N} V_{\sigma_k \sigma_{k+1}}$$

$$V_{\sigma \sigma'} = \exp\left(\beta\left(\sigma \sigma' + B \sigma\right)\right)$$

> Method of transfer matrix:
>
> Z_N can be written as a product of $\sim N$ identical matrices

Notice: 1D is a fake example of the method: $V \in \mathbb{R}^{2 \times 2}$ where as for $d > 1$ $V \in \mathbb{R}^{2^d \times 2^d}$.

$h = \beta B$

$$\begin{array}{cccc} & \sigma = +1 & \sigma = -1 & \\ V = & \left(\begin{matrix} e^{\beta + h} & e^{-\beta - h} \\ e^{-\beta + h} & e^{\beta - h} \end{matrix}\right) & & \begin{matrix} \sigma' = +1 \\ \sigma' = -1 \end{matrix} \end{array}$$

$$\underline{e}_{\sigma=+1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \underline{e}_{\sigma=-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Characteristic polynomial of V:

$$\det(V - \lambda I) = (e^{\beta+h} - \lambda)(e^{\beta-h} - \lambda) - e^{-2\beta}$$

$$= \lambda^2 - 2e^{\beta}\cosh(h)\,\lambda + 2\sinh(2\beta)$$

$$\lambda_{\pm} = e^{\beta}\cosh(h) \pm \sqrt{e^{2\beta}\cosh^2(h) - 2\sinh(2\beta)}$$

Notice that $e^{2\beta}\cosh^2(h) \geq e^{2\beta} > 2\sinh(2\beta) \quad \forall \beta > 0$

$\Rightarrow \lambda_+ > \lambda_- \quad \forall \beta > 0$

For this reason there is no phase transition in 1D Ising model!

Example For $\underline{\sigma} = (\sigma_k)_{k=1}^{N}$ with $\sigma_{N+1} = \sigma_1$ (periodic b.c.)

$$Z_N(\beta, B) = \mathrm{Tr}(V^N) = \lambda_+^N + \lambda_-^N$$

$$\Rightarrow F_N(\beta, B) = -\frac{1}{\beta}\log Z_N(\beta, B) = \left(-\frac{N}{\beta}\log\lambda_+\right)\left(1 + O\left(\left(\frac{\lambda_-}{\lambda_+}\right)^N\right)\right)$$

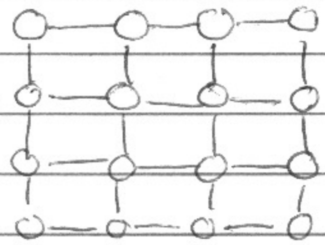
$$\partial_B\left(-\frac{1}{N}F_N(\beta, B)\right) = \mathbb{E}_{\beta,B}\left[\frac{1}{N}\sum_{k=1}^{N}\sigma_k\right]$$

$\boxed{\text{Remember } h = \beta B}$

$$\approx \frac{1}{\beta}\partial_B \log\lambda_+ = \partial_h \log\lambda_+$$

Section 2.3 | Ising model on $\mathbb{Z}^d$ and thermodynamical limit of F

$$L_N = [\![1, N]\!]^d$$



$$\underline{\sigma} = (\sigma_v)_{v \in L_N} \in \{-1,+1\}^{L_N}$$

Denote neighbors by $v \sim w$,
that is, $v \sim w$ if and only if $|v-w| = 1$.

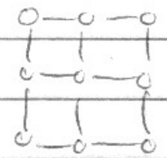
The most general spin model is defined by

$$H(\underline{\sigma}) = -\sum_v B_v \sigma_v - \sum_{v,w} J_{vw} \sigma_v \sigma_w - \sum_{u,v,w} K_{uvw} \sigma_u \sigma_v \sigma_w + \ldots$$

Ising model

$$H(\underline{\sigma}) = -\sum_{v \sim w} \sigma_v \sigma_w - B \sum_v \sigma_v$$

$L_n = [\![1, N]\!]^d$



Thermodynamical limit of F

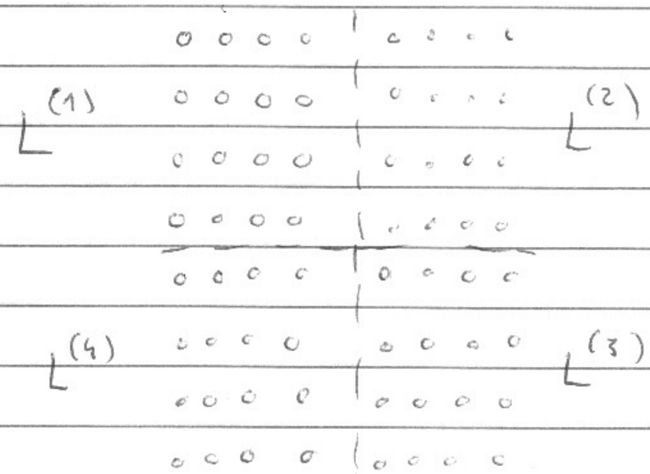
$$F_N(\beta, B) = -\frac{1}{\beta} \log Z_N(\beta, B)$$

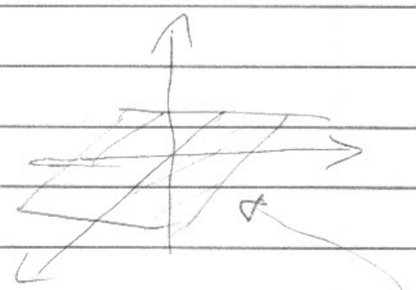
We expect that F_N grows linearly in $\# V_N$
Define free energy density

$$f_N(\beta, B) = \frac{1}{\# V_N} F_N(\beta, B)$$

<u>Theorem</u> Limit $f := \lim_{N \to \infty} f_N(\beta, B)$ exist for $V_N = L_{2^N}$. The limit doesn't depend on boundary conditions.

<u>Proof</u> Divide L_{2^N} into 2^d boxes $L^{(k)}_{2^{N-1}}$





contribution to R: d planes each has $2^{N(d-1)}$ edges crossing

Write $H_N = \sum_{k=1}^{2^d} H^{(k)}_{N-1} + R_N$

$$|R_N| \leq d\, 2^{N(d-1)}$$

$$(Z_{N-1})^{2^d} e^{-\beta d 2^{N(d-1)}} \leq Z_N \leq (Z_{N-1})^{2^d} e^{\beta d 2^{N(d-1)}}$$

$$|f_N - f_{N-1}| = \left|\frac{1}{\beta}\frac{1}{2^{Nd}}\log Z_N - \frac{1}{\beta}\frac{1}{2^{(N-1)d}}\log Z_{N-1}\right|$$

$$= \left|\frac{1}{\beta}\frac{1}{2^{Nd}}\log\left(Z_N / (Z_{N-1})^{2^d}\right)\right|$$

$$\le \frac{d\,2^{N(d-1)}}{2^{Nd}} = d\,2^{-N}$$

If $N_0 \le M < N$, then

$$|f_N - f_M| \le \sum_{k=M+1}^{N} |f_k - f_{k-1}|$$

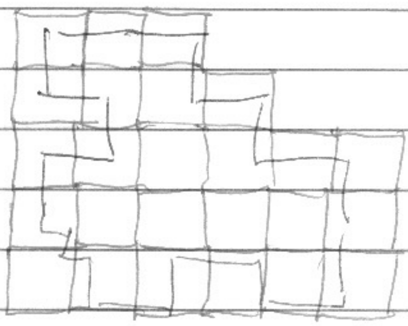
$$< \sum_{k=N_0+1}^{\infty} d\,2^{-k} = d\,2^{-N_0}$$

Hence $(f_N)_{N\ge 1}$ is a Cauchy sequence, and it converges as $N \to \infty$.

Other boundary conditions than free can be handled with an error term $\tilde{R}_N$ s.t. $|\tilde{R}_N| \le 2d\,2^{N(d-1)}$ □

Notice that the limit exists for any $\Lambda_n \nearrow \mathbb{Z}^d$ as soon as

$$\lim_{n\to\infty} \frac{|\partial\Lambda_n|}{|\Lambda_n|} = 0$$



idea: approximate each Λ_n by $\mathbb{L}_m^{(k)}$ with fixed m. By the above result and similar control of error terms

$$|f_{\Lambda_n} - f| < \varepsilon \quad \text{for large } n$$