Part I

Section 6

Random walk in $\mathbb{Z}^d$, $d=1,2,3$

Set of neighbors of $\underline{0}$ is $\{\underline{v} \in \mathbb{Z}^d : \|\underline{v}\| = 1\}$
$= \{ \pm \underline{e}_i : i=1,2,\dots,d\}$ where $\underline{e}_i = [0\ 0 \dots 0\ \underset{i'th}{1}\ 0 \dots 0]^T$

Let ξ be uniformly random in this set.

$$\mathbb{P}[\xi = \underline{e}_i] = \mathbb{P}[\xi = -\underline{e}_i] = \frac{1}{2d}$$

Definition Simple random walk $S = (S_t)_{t\in\mathbb{N}}$ in $\mathbb{Z}^d$
is the random sum $S_t = x_0 + \sum_{k=1}^{t} \xi_k$
where $(\xi_k)_{k\geq 1}$ are i.i.d and $\xi_k \sim \xi$.

Suppose $x_0 = 0$.
Let $p = \mathbb{P}[S_t = \underline{0}$ for some $t>0]$. We say that

- RW $S=(S_t)_{t\in\mathbb{N}}$ is recurrent if $p=1$
- RW $S=(S_t)_{t\in\mathbb{N}}$ is transient if $p<1$

Theorem Let $S = (S_t)_{t\in\mathbb{N}}$ be a simple random walk on $\mathbb{Z}^d$. ($x_0 = \underline{0}$, say)
- If $d \leq 2$, S is recurrent
- If $d > 2$, S is transient

Remark In some sense, the transition occurs at $d=2$, which is barely recurrent. $d = 2+\varepsilon$ would be transient (whatever that means)

Proof Suppose [redacted] that $X_0 = \underline{0}$. [redacted]

Define

$$L = \#\{t \geq 0 : S_t = \underline{0}\} = \sum_{t=0}^{\infty} \mathbb{1}_{\{S_t = \underline{0}\}}$$

which is the # of visits to $\underline{0}$.

Notice that $\mathbb{P}[L \geq 1] = 1$ and, (since if the walk is at $\underline{0}$ for n^{th} time, then the remaining walk is an independent RW start from $\underline{0}$), for any $n \geq 1$

$$\mathbb{P}[L \geq n+1 \mid L \geq n] = p$$

$$\Rightarrow \mathbb{P}[L \geq n] = \mathbb{P}[L \geq n \mid L \geq n-1]\,\mathbb{P}[L \geq n-1 \mid L \geq n-2] \cdot \ldots \cdot \mathbb{P}[L \geq 2 \mid L \geq 1] \cdot \mathbb{P}[L \geq 1]$$

$$= p^{n-1}$$

$$\Rightarrow \mathbb{E}[L] = \sum_{n=1}^{\infty} \mathbb{P}[L \geq n] = \sum_{n=0}^{\infty} p^n$$

$$= \begin{cases} \frac{1}{1-p}, & p < 1 \\ \infty, & p = 1 \end{cases}$$

Hence S is
- recurrent if $\mathbb{E}[L] = \infty$
- transient if $\mathbb{E}[L] < \infty$

Let's define for $0<\lambda<1$ and for $\underline{x} \in \mathbb{Z}^d$

$$L_\lambda(\underline{x}) = \sum_{t=0}^{\infty} \lambda^t \, \mathbb{1}_{\{S_t = \underline{x}\}}$$

$$G_\lambda(\underline{x}) = \sum_{t=0}^{\infty} \lambda^t \, \mathbb{P}[S_t = \underline{x}]$$

G_λ is called Green's generating function. Note that $G_\lambda < \infty$ for all $0<\lambda<1$.

By monotone convergence theorem

$$\lim_{\lambda \nearrow 1} G_\lambda(0) = \mathbb{E}[L]$$

We'll use the first step analysis to G_λ and then solve the difference equation using Fourier analysis.

Remember that

- $S_0 = 0$
- $S_1 = \xi_1$, $\mathbb{P}[\xi_1 = \pm \underline{e}_j] = \frac{1}{2d}$
- $\tilde{S} := (S_{t+1} - \xi_1)_{t \in \mathbb{N}}$

  $\left[\tilde{S}_t = \sum_{k=1}^{t} \xi_{k+1}\right]$ is RW sent from $\underline{0}$

  which is independent of ξ_1.

Therefore if $t \geq 1$

$$\mathbb{P}[S_t = \underline{x}] = \mathbb{P}[\tilde{S}_{t-1} + \xi_1 = \underline{x}]$$

$$= \sum_{\underline{v} \in \{\|\underline{v}\|=1\}} \mathbb{P}[\xi_1 = \underline{v}] \, \mathbb{P}[\tilde{S}_{t-1} = \underline{x} - \underline{v} \mid \xi_1 = v]$$

$$= \frac{1}{2d} \sum_{\underline{v} \in \{\|\underline{v}\|=1\}} \mathbb{P}[S_{t-1} = \underline{x} - \underline{v}]$$

The equation for G_λ:

$$(\ast) \quad G_\lambda(\underline{x}) = \delta_{\underline{x},0} + \sum_{t=1}^{\infty} \lambda^t \, \mathbb{P}[S_{\cancel{t}} = \underline{x}]$$

$$= \delta_{\underline{x},0} + \frac{\lambda}{2d}\left[\sum_{j=1}^{d} G_\lambda(\underline{x}-\underline{e}_j) + G_\lambda(\underline{x}+\underline{e}_j)\right]$$

Let's use discrete space Fourier transform (1) to solve the equation. Define for $\underline{k} \in \mathbb{R}^d$

$$\hat{G}_\lambda(\underline{k}) = \sum_{\underline{x}\in\mathbb{Z}^d} G_\lambda(\underline{x}) e^{-i\underline{k}\cdot\underline{x}}$$

The inverse transform is

$$G_\lambda(\underline{x}) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} e^{i\underline{k}\cdot\underline{x}} \hat{G}_\lambda(\underline{k}) dk_1 \dots dk_d$$

Apply to $(\ast)$

$$\Rightarrow \hat{G}_\lambda(\underline{k}) = 1 + \frac{\lambda}{2d}\sum_{j=1}^{d}\left(e^{-i\underline{k}\cdot\underline{e}_j} + e^{i\underline{k}\cdot\underline{e}_j}\right)\hat{G}_\lambda(\underline{k})$$

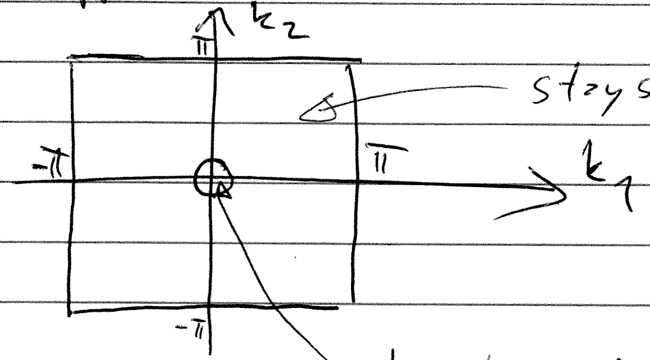
$$\Rightarrow \hat{G}_\lambda(\underline{k}) = \frac{1}{1 - \frac{\lambda}{d}\sum_{j=1}^{d}\cos(k_j)}$$

$$\Rightarrow G_\lambda(\underline{x}) = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} \frac{e^{i\underline{k}\cdot\underline{x}} dk_1 \dots dk_d}{1 - \frac{\lambda}{d}\sum_{j=1}^{d}\cos(k_j)}$$

$$\boxed{\mathbb{E}[L] = \lim_{\lambda\nearrow 1} G_\lambda(0) = \lim_{\lambda\nearrow 1}\frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}\frac{dk_1 \dots dk_d}{1 - \frac{\lambda}{d}\sum_{j=1}^{d}\cos(k_j)}}$$

(1): We need $\sum_{\underline{x}\in\mathbb{Z}^d} |G_\lambda(\underline{x})| < \infty$ which follows from $G_\lambda(\underline{x}) \le \lambda^{\|\underline{x}\|}/(1-\lambda)$

What happens to the integrand as $\lambda \nearrow 1$?



stays bounded away from $\underline{0}$

develops singularity at $\underline{0}$
Is it integrable?

Let $R_0 = [-\frac{\pi}{2}, \frac{\pi}{2}]^d$, $R_1 = [-\pi, \pi]^d \setminus R_0$.

If $\underline{k} \in R_1$, $\cos(k_j) < 0$ for some j.

$$\Rightarrow \quad 0 \leq \frac{1}{1 - \frac{\lambda}{d}\sum_j \cos(k_j)} \leq d$$

Dominated convergence thm.

$$\Rightarrow 0 \leq \lim_{\lambda \nearrow 1} \frac{1}{(2\pi)^d} \int_{R_1} \frac{dk_1 \dots dk_d}{1 - \frac{\lambda}{d}\sum_{j=1}^{d} \cos(k_j)} \leq d < \infty$$

If $\underline{k} \in R_0$, $\cos(k_j) \geq 0 \ \forall j$.

$$\Rightarrow \quad \frac{1}{1 - \frac{\lambda}{d}\sum_j \cos(k_j)} \nearrow \frac{1}{1 - \frac{1}{d}\sum_j \cos(k_j)}$$

Monotone convergence thm

$$\Rightarrow \quad \lim_{\lambda \nearrow 1} \frac{1}{(2\pi)^d} \int_{R_0} \frac{dk_1 \dots dk_d}{1 - \frac{\lambda}{d}\sum_{j=1}^{d} \cos(k_j)}$$

$$= \frac{1}{(2\pi)^d} \int_{R_0} \frac{dk_1 \dots dk_d}{1 - \frac{1}{d}\sum_{j=1}^{d} \cos(k_j)}$$

Let's use $\frac{4}{\pi^2} t^2 \leq 1 - \cos(t^2) \leq \frac{1}{2} t^2$ which holds for $t \in [-\pi/2, \pi/2]$. Suppose

$$\int_{R_0} \frac{dt_1 \cdots dt_d}{1 - \frac{1}{d} \sum_{j=1}^{d} \cos(t_j)} < \infty$$

$$\Longleftrightarrow \int_{R_0} \frac{dt_1 \cdots dt_d}{\sum_{j=1}^{d} t_j^2} < \infty$$

$$\Longleftrightarrow \int_{\{\underline{t} \in \mathbb{R}^d : \|\underline{t}\| \leq 1\}} \frac{dt_1 \cdots dt_d}{\|\underline{t}\|^2} < \infty$$

Now if C_d is the surface area of unit sphere $\{\|\underline{t}\| = 1\}$, then "in spherical coordinates"

$$\int_{\{\underline{t} \in \mathbb{R}^d : \|\underline{t}\| \leq 1\}} \frac{dt_1 \cdots dt_d}{\|\underline{t}\|^2} = \int_0^1 C_d \, r^{d-3} \, dr$$

which is finite if and only if $d > 2$. $\square$