Lemma If c_m and $c_m^\dagger$ defined as above, then for $m \neq l$

$$[c_m, c_l]_+ = 0 \;,\; [c_m, c_l^\dagger]_+ = 0 \;,\; [c_m^\dagger, c_l^\dagger]_+ = 0$$

$$[c_m, c_m^\dagger]_+ = I \;,\; [c_m, c_m]_+ = 0 \;,\; [c_m^\dagger, c_m^\dagger]_+ = 0$$

The inverse transform from c's to τ's is given by

$$\tau_m^- = \left(\prod_{j=1}^{m-1} (-(2N_j - I)\right) c_m$$

$$\tau_m^+ = \left(\prod_{j=1}^{m-1} (-(2N_j - I)\right) c_m^\dagger$$

where $N_j = c_j^\dagger c_j$.

Proof of the lemma Write $\underline{f}_S$, $S \subset [\![1, M]\!]$, $\underline{f}_{n_1,\dots,n_M} = \underline{f}_{\{m : n_m = 1\}}$

Observe that the operation of $C_m, C_m^\dagger$ to the basis $\{\underline{f}_{n_1,\dots,n_M}\}$ gives

$$C_m^\dagger \underline{f}_S = \begin{cases} (-1)^{\#([\![1,m-1]\!] \cap S)} \underline{f}_{S \cup \{m\}}, & \text{if } m \notin S \\ 0, & \text{if } m \in S \end{cases}$$

$$C_m \underline{f}_S = \begin{cases} (-1)^{\#([\![1,m-1]\!] \cap S)} \underline{f}_{S \setminus \{m\}}, & \text{if } m \in S \\ 0, & \text{if } m \notin S \end{cases}$$

By Exercise 1 of the problem sheet 9

$$[C_m, C_l]_+ = 0 = [C_m^\dagger, C_l^\dagger]_+$$

$$[C_m, C_l^\dagger]_+ = \delta_{m,l} I$$

and hence the claim about anticommutation relations is true. Using that the operators operating on different tensor components commute, we get

$$N_m = C_m^\dagger C_m = \left[\prod_{j=1}^{m-1} (-\tau_j^z)^2\right] \tau_m^+ \tau_m^-$$

$$= \tau_m^+ \tau_m^- = \frac{1}{2}(\tau_m^z + I)$$

$$\Rightarrow \left(\prod_{j=1}^{m-1} (-(2N_j - I))\right) C_m = \left(\prod_{j=1}^{m-1} (-\tau_j^z)\right) C_m$$

$$= \left(\prod_{j=1}^{m-1} (-\tau_j^z)^2\right) \tau_m^- = \tau_m^-$$

Similarly $\left(\prod_{j=1}^{m-1} (-(2N_j - I))\right) C_m^\dagger = \tau_m^+$.

$\square$

Lemma The following relations hold for $\tau_m^-, \tau_m^+, C_m, C_m^+$, $m = 1, 2, \dots, M$

$$\tau_m^+ \tau_m^- = C_m^+ C_m$$

$$\tau_m^+ \tau_{m+1}^- = C_m^+ C_{m+1}$$

$$\tau_m^+ \tau_{m+1}^+ = C_m^+ C_{m+1}^+$$

$$\tau_m^- \tau_{m+1}^+ = - C_m C_{m+1}^+$$

$$\tau_m^- \tau_{m+1}^- = - C_m C_{m+1}$$

where
$$\begin{cases} C_{M+1} = \prod_{j=1}^{M} \left(- \tau_j^z \right) \tau_1^- = \prod_{j=1}^{M} \left(-(2N_j - 1) \right) C_1 \\ C_{M+1}^+ = \prod_{j=1}^{M} \left(- \tau_j^z \right) \tau_1^+ = \prod_{j=1}^{M} \left(-(2N_j - 1) \right) C_1^+ \end{cases}$$

and $\tau_{M+1}^{\pm} = \tau_1^{\pm}$.

Proof All the identities follow from $(\tau^z)^2 = I$ and

$$\tau^+ \tau^z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -\tau^+$$

$$\tau^- \tau^z = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \tau^-$$

For example, the fifth relations follows from

$$- C_m C_{m+1} = \Bigg(\prod_{j=1}^{m-1} \underbrace{\left((-\tau_j^z)^2 \right)}_{=I} \Bigg) \underbrace{\tau_m^- \tau_m^z}_{=\tau_m^-} \tau_{m+1}^-$$

$$= \tau_m^- \tau_{m+1}^-$$

The others follow similarly. $\square$

Therefore we can write

$$V_1 = (2\sinh(2K_1))^{\frac{M}{2}} \exp\left(-2K_1^* \sum_{m=1}^{M} (\tau_m^+ \tau_m^- - \tfrac{1}{2})\right)$$

$$= (2\sinh(2K_1))^{\frac{M}{2}} \exp\left(-2K_1^* \sum_{m=1}^{M} (C_m^+ C_m - \tfrac{1}{2})\right)$$

$$V_2 = \exp\left(K_2 \sum_{m=1}^{M} (\tau_m^+ + \tau_m^-)(\tau_{m+1}^+ + \tau_{m+1})\right)$$

$$= \exp\left(K_2 \sum_{m=1}^{M} (C_m^+ - C_m)(C_{m+1}^+ + C_{m+1})\right)$$

Unfortunately we have to now set $h = 0$.
Now on $V_3 = I$ and

$$T = (V_2)^{\frac{1}{2}} V_1 (V_2)^{\frac{1}{2}}$$

<u>Remark</u> The quadratic expressions of $\tau_{m+}^+ \tau_m^-$ where transformed nicely to quadratic expressions of C_m^+, C_m, but the transformation of linear expressions takes no nice form.

Parity of the fermion states

Now $N = \sum_{m=1}^{M} N_m = \sum_{m=1}^{M} C_m^+ C_m$ tells the number of fermions in state $\underline{f}_{n_1,\dots,n_M}$

$$N \underline{f}_{n_1,\dots,n_M} = \sum_{m=1}^{M} \underbrace{C_m^+ C_m}\, (C_1^+)^{n_1} \dots (C_M^+)^{n_M} \underline{f}_\phi$$

commute if $m \neq 1$

$$\overset{(*)}{=} \left(\sum_{m=1}^{M} n_m \right) \underline{f}_{n_1,\dots,n_M}$$

In $(*)$ we used (1) $C_m \underline{f}_\phi = \underline{0}$ and (2) $C_m^+ C_m C_m^+ = C_m^+ (I - C_m^+ C_m) = C_m^+$.

Similarly

$$\underbrace{\left(\prod_{m=1}^{M} \left(-(2N_m - I) \right) \right)}_{=:P} \underset{\sim}{f}_{n_1,\dots,n_M} = (-1)^{\sum_{n=1}^{M} n_m} \underline{f}_{n_1,\dots,n_M}$$

The eigenvalue of P tells the parity of $f_{n_1,\dots,n_M}$.
Since V_1, V_2 are of the form $\exp(A)$ where A is quadratic in C_m, C_m^+, $m \in [\![1, M]\!]$

We can diagonalize T separately in spaces $\mathcal{V}_0, \mathcal{V}_1$ given by

$$\mathcal{V}_k = \text{span}\left\{ \underline{f}_{n_1,\dots,n_M} : \sum_{m=1}^{M} n_M \equiv k \pmod 2 \right\}$$

Then $(\mathbb{C}^2)^{\otimes M} = \mathcal{V}_0 \oplus \mathcal{V}_1$.

Notice that

anticommute $[C_1, 2N_1 - I]_+ = 2C_1 - 2C_1 = 0$

- If $\underline{v} \in \mathcal{V}_0$, then $C_{M+1} \underline{v} = P C_1 \underline{v} = -C_1 P \underline{v} = -C_1 \underline{v}$
- If $\underline{v} \in \mathcal{V}_1$, then $C_{M+1} \underline{v} = P C_1 \underline{v} = C_1 \underline{v}$

Therefore we consider two transfer matrices of the form

$$T_k = (V_2)^{\frac{1}{2}} V_1 (V_2)^{\frac{1}{2}} \quad , k = 0, 1$$

where

- In T_0 , $c_{M+1} = -c_1$
- In T_1 , $c_{M+1} = c_1$

<u>Proposition</u> T is a block matrix such that $T : \mathcal{V}_k \to \mathcal{V}_k$, $k = 0, 1$ and the blocks are $T|_{\mathcal{V}_k} = T_k|_{\mathcal{V}_k}$, $k = 0, 1$.

A linear map $A: V \to V$ is said to be <u>diagonalizable</u> if there exists a basis $\{\underline{v}_k\}$ of V s.t. for all k, $\exists \lambda_k \in \mathbb{C}$ s.t. $A\underline{v}_k = \lambda_k \underline{v}_k$

Theorem If $[A, B] = 0$ and A and B are diagonalizable, then A and B are <u>jointly diagonalizable</u>, i.e., there exists a basis $\{\underline{u}_k\}$ s.t. $\exists \mu_k, \lambda_k$ s.t.

$$A\underline{u}_k = \lambda_k \underline{u}_k \quad \text{and} \quad B\underline{u}_k = \mu_k \underline{u}_k$$

for all k.

Proof Let $\{\underline{v}_n\}$ and $\{\lambda_n\}$ be a set of eigenvectors of A that forms a basis of V and the corresponding eigenvalues and similarly $\{\underline{w}_m\}$, $\{\mu_m\}$ for B.

For fixed n, write $\underline{v}_n$ in basis $\{\underline{w}_m\}$

$$\underline{v}_n = \sum_{m=1}^{d} c_m \underline{w}_m$$

Let $\tilde{\mu}_1, \tilde{\mu}_2, \ldots, \tilde{\mu}_e$, $e \leq d$, be the set of distinct eigenvalues of B. Define

$$\underline{u}_k^{(n)} = \sum_{\substack{m \in [\![1, d]\!] \\ \text{s.t. } \mu_m = \tilde{\mu}_k}} c_m \underline{w}_m$$

$$\Rightarrow \underline{v}_n = \sum_{k=1}^{e} \underline{u}_k^{(n)}$$

Observe that

$$(*) \qquad \underline{0} = (A - \lambda_n)\underline{v}_n = \sum_{k=1}^{e} (A - \lambda_n)\underline{u}_k^{(n)}$$

Now use the fact that A and B commute.

$$\Rightarrow \quad B\left((A-\lambda_n)\, \underline{u}_k^{(n)} \right) = (A-\lambda_n)\, B\, \underline{u}_k^{(n)}$$

$$= \tilde{\mu}_k\, (A-\lambda_n)\, \underline{u}_k^{(n)}$$

That is, $(A-\lambda_n)\,\underline{u}_k^{(n)}$ is an eigenvector of B, if it is $\neq \underline{0}$, and $\tilde{\mu}_k$ are distinct.

Therefore $(*)$ can only hold if $(A-\lambda_n)\,\underline{u}_k^{(n)} = \underline{0}$ $\forall k$. Now if $\underline{u}_k^{(n)} \neq \underline{0}$ it is simultaneous eigenvector for both A and B.

Since $\{\underline{v}_n\}$ is a basis, the set $\{\underline{u}_k^{(n)} ; n=1,\dots,d,\ k=1,\dots,e\}$ spans V. We can choose a maximal linearly independent subset of vectors $\{\underline{u}_1,\dots,\underline{u}_d\}$ which then satisfies the desired properties, i.e., it is a basis and $\underline{u}_k$ are simultaneously eigenvectors for A and B.

$\square$

Diagonalization with plane wave ansatz (/ Fourier transform)

This is the most important step of our attempt to diagonalize $T \in \mathbb{C}^{2^M \times 2^M}$. It reduces the problem with huge, varying dimension to many problems of the same type with low, fixed dimension.

From the collection of fermions c_m, c_m^+, $m \in [\![1, M]\!]$ that satisfy the canonical anticommutation relations

$$[c_m, c_l]_+ = 0 = [c_m^+, c_l^+] \quad , \quad [c_m, c_l^+] = \delta_{ml} I$$

we will construct two sets of fermions $\vartheta_m, \vartheta_m^+$, $m \in I_0(M)$, and $\vartheta_m, \vartheta_m^+$, $m \in I_1(M)$, which naturally handle the anticyclic $(c_{M+1} = -c_1)$ and the cyclic $(c_{M+1} = c_1)$ conditions, respectively.

The ansatz is inspired by the translational invariance of the form of T. We write

$$\begin{cases} c_m = \dfrac{c}{\sqrt{M}} \displaystyle\sum_{l \in I_k(M)} e^{i\frac{\pi m l}{M}} \vartheta_l \\ c_m^+ = \dfrac{\bar{c}}{\sqrt{M}} \displaystyle\sum_{l \in I_k(M)} e^{-i\frac{\pi m l}{M}} \vartheta_l \end{cases}$$

where $I_k(M) = \{ l \in [\![-(M-1), M]\!] : l \equiv 1-k \pmod 2 \}$ and $c \in \mathbb{C}$, $|c| = 1$, which we choose later.

$$\Rightarrow c_{M+1} = \frac{c}{\sqrt{M}} \sum_{l \in I_k(M)} \underbrace{e^{i\pi l}}_{=(-1)^{1-k}} e^{i\frac{\pi l}{M}} \vartheta_l$$

$$= (-1)^{1-k} \frac{c}{\sqrt{M}} \sum_l e^{i\frac{\pi l}{M}} \vartheta_l = (-1)^{1-k} c_1$$

and similarly: $C^{+}_{M+1} = (-1)^{1-k} C_1$.

Now to define $\mathcal{D}_l, \mathcal{D}_l^+$, set

$$U_{m,l} = \frac{c}{\sqrt{M}} e^{i\frac{\pi m l}{M}}$$

$$\Rightarrow \sum_{l \in I_k(M)} U_{m,l} \overline{U}_{m',l} = \frac{1}{M} \sum_{l \in I_k(M)} e^{i\frac{\pi(m-m')l}{M}}$$

$$= \begin{cases} 1 & , m = m' \\ \frac{D_{m,m',k}}{M} \underbrace{\sum_{j=0}^{M-1} q^j_{m,m'}}_{= \frac{1 - q^M_{m,m'}}{1 - q_{m,m'}} = 0} & , m \neq m' \end{cases}$$

($q^M_{m,m'} = 1$, $1 - q_{m,m'} \neq 0$)

$$= \delta_{m,m'}$$

Where $q_{m,m'} = e^{i\frac{2\pi(m-m')}{M}} \neq 1$, when $m \neq m'$, and $D_{m,m',k}$ is some constant that we could write explicitely. Therefore if we set

$$\mathcal{D}_l = \sum_{m=1}^{M} \overline{U}_{m,l} C_m , \quad \mathcal{D}_l^+ = \sum_{m=1}^{M} U_{m,l} C_m^+$$

They satisfy the requirements. And one can check that they satisfy the canonical anticommutation relations

$$[\mathcal{D}_l, \mathcal{D}_{l'}]_+ = 0 = [\mathcal{D}_l^+, \mathcal{D}_{l'}^+] , \quad [\mathcal{D}_l, \mathcal{D}_{l'}^+] = \delta_{l l'} I$$

$l, l' \in I_k(M)$. (An exercise)

Also $\underline{f}_{0,0,\dots,0}$ is a vacuum for $\vartheta_l, \vartheta_l^\dagger, l \in I_\varepsilon(M)$
since $\vartheta_l \underline{f}_{0,0,\dots,0} = \sum_m \bar{U}_{m,l} C_m \underline{f}_{0,\dots,0} = \underline{0}$
and

$$\operatorname{span}\{ (\vartheta_{l_1}^\dagger)^{\tilde{n}_1} \dots (\vartheta_{l_M}^\dagger)^{\tilde{n}_M} \underline{f}_{0,\dots,0} \}$$

$$= \operatorname{span}\{ (C_1^\dagger)^{n_1} \dots (C_M^\dagger)^{n_M} \underline{f}_{0,\dots,0} \} = (\mathbb{C}^2)^{\otimes M}$$

Transfer matrices T_0 and T_1 in terms of $\psi_l, \psi_l^\dagger$

Suppose that M is <u>even</u> for simplicity.
Then

$$I_0(M) = \{\pm 1, \pm 3, \dots, \pm (M-1)\}$$

$$I_1(M) = \{0, \pm 2, \pm 4, \dots, \pm (M-2), M\}$$

Now by a direct calculation we get

$$\sum_{m=1}^{M} c_m^\dagger c_m = \frac{1}{M} \sum_{m=1}^{M} \sum_{l \in I_k(M)} \sum_{l' \in I_k(M)} e^{i\pi \frac{m(l-l')}{M}} \psi_l^\dagger \psi_{l'}$$

$$= \sum_{l} \sum_{l'} \underbrace{\left(\frac{1}{M} \sum_{m=1}^{M} e^{i\pi \frac{m(l-l')}{M}} \right)}_{\delta_{l,l'}} \psi_l^\dagger \psi_{l'}$$

$$= \sum_{l \in I_k(M)} \psi_l^\dagger \psi_l$$

When $k = 0$ (the parity of l)

$$\sum_{m=1}^{M} (c_m^\dagger - c_m)(c_{m+1}^\dagger + c_{m+1})$$

$$= \sum_{l \in I_0} \sum_{l' \in I_0} \Bigg[\bar{c}^2 \underbrace{\left(\frac{1}{M} \sum_{m=1}^{M} e^{i\pi \frac{-ml-(m+1)l'}{M}} \right)}_{e^{i\pi \frac{l}{M}} \delta_{l,-l'}} \psi_l^\dagger \psi_{l'}^\dagger$$

$$+ \underbrace{\left(\frac{1}{M} \sum_{m=1}^{M} e^{i\pi \frac{-ml+(m+1)l'}{M}} \right)}_{= e^{i\pi \frac{l}{M}} \delta_{l,l'}} \psi_l^\dagger \psi_{l'} - \underbrace{\left(\frac{1}{M} \sum_{m=1}^{M} e^{i\pi \frac{ml-(m+1)l'}{M}} \right)}_{= e^{-i\pi \frac{l}{M}} \delta_{l,l'}} \psi_l \psi_{l'}^\dagger$$

$$- c^2 \underbrace{\left(\frac{1}{M} \sum_{m=1}^{M} e^{i\pi \frac{ml+(m+1)l'}{M}} \right)}_{= e^{i\pi \frac{l}{M}} \delta_{l,-l'}} \psi_l \psi_{l'} \Bigg]$$

Choose $c = e^{-i\frac{\pi}{4}}$. Then for $k=0$

$$\sum_{m=1}^{M} (C_m^+ - C_m)(C_{m+1}^+ + C_{m+1})$$

$$= 2 \sum_{\substack{l \in I_0(M) \\ l>0}} \Big[\cos\left(\frac{\pi l}{M}\right) (\vartheta_l^+ \vartheta_l + \vartheta_{-l}^+ \vartheta_{-l}) + i \sin\left(\frac{\pi l}{M}\right) (\bar{c}^2 \vartheta_l^+ \vartheta_{-l}^+ + c^2 \vartheta_l \vartheta_{-l}) \Big]$$

$$= 2 \sum_{\substack{l \in I_0(M) \\ l>0}} \Big[\cos\left(\frac{\pi l}{M}\right) (\vartheta_l^+ \vartheta_l + \vartheta_{-l}^+ \vartheta_{-l}) + \sin\left(\frac{\pi l}{M}\right) (\vartheta_{-l}^+ \vartheta_l + \vartheta_l \vartheta_{-l}) \Big]$$

When $k=1$, similarly

$$\sum_{m=1}^{M} (C_m^+ - C_m)(C_{m+1}^+ + C_{m+1})$$

$$= \sum_{l \in I_1} \sum_{l' \in I_1} \Big[\bar{c}^2 e^{i\pi \frac{l}{M}} (\delta_{l',-l} + \underbrace{\delta_{l,M}\delta_{l',M}) \vartheta_l^+ \vartheta_{l'}^+}_{=0} + e^{i\frac{\pi l}{M}} \delta_{l',l} \vartheta_l^+ \vartheta_{l'} - e^{-i\pi\frac{l}{M}} \delta_{l',l} \vartheta_l \vartheta_{l'}^+ - c^2 e^{-i\pi\frac{l}{M}} (\delta_{l',-l} + \underbrace{\delta_{l,M}\delta_{l',M}) \vartheta_l \vartheta_{l'}}_{=0} \Big]$$

$$= (2\vartheta_0^+ \vartheta_0 - I) - (2\vartheta_M^+ \vartheta_M - I) + 2 \sum_{\substack{l \in I_1(M) \\ 0<l<M}} \Big[\cos\left(\frac{\pi l}{M}\right) (\vartheta_l^+ \vartheta_l + \vartheta_{-l}^+ \vartheta_l) + \sin\left(\frac{\pi l}{M}\right) (\vartheta_{-l}^+ \vartheta_l^+ + \vartheta_l \vartheta_{-l}) \Big]$$

Define for $l \in [\![1, M-1]\!]$

$$
\begin{cases}
V_1^{(l)} = \exp\left(-2K_1^* \left(\psi_l^+ \psi_l + \psi_{-l}^+ \psi_{-l} - I\right)\right) \\
V_1^{(0)} = \exp\left(-2K_1^* \left(\psi_0^+ \psi_0 - \frac{1}{2} I\right)\right) \\
V_1^{(M)} = \exp\left(-2K_1^* \left(\psi_M^+ \psi_M - \frac{1}{2} I\right)\right) \\
V_2^{(l)} = \exp\left[K_2 \left[\cos\left(\frac{\pi l}{M}\right)\left(\psi_l^+ \psi_l + \psi_{-l}^+ \psi_{-l}\right) + \sin\left(\frac{\pi l}{M}\right)\left(\psi_{-l}^+ \psi_l^+ + \psi_l \psi_{-l}\right)\right]\right] \\
V_2^{(0)} = \exp\left(K_2 \left(\psi_0^+ \psi_0 - \frac{1}{2} I\right)\right) \\
V_2^{(M)} = \exp\left(-K_2 \left(\psi_M^+ \psi_M - \frac{1}{2} I\right)\right)
\end{cases}
$$

__Proposition__ The operators T_0 and T_1 can be written as

$$T_0 = \left(2\sinh(2K_1)\right)^{\frac{M}{2}} \prod_{l=1,3,5,\dots,M-1} V^{(l)} \quad \leftarrow \text{commute}$$

$$T_1 = \left(2\sinh(2K_1)\right)^{\frac{M}{2}} \prod_{l=0,2,4,\dots,M} V^{(l)} \quad \leftarrow \text{commute}$$

Where $V^{(l)} = V_2^{(l)} V_1^{(l)} V_2^{(l)}$. The operators in the products commute and hence the order of the product is irrelevant.

__Proof__ The above calculations + that we notice that if $j \neq l$, then $\psi_l^+ \psi_l \psi_j^+ \psi_j = -\psi_l^+ \psi_j^+ \psi_l \psi_j = \psi_j^+ \psi_l^+ \psi_l \psi_j = \dots = \psi_j^+ \psi_j \psi_l^+ \psi_l$

$\Rightarrow [\psi_l^+ \psi_l, \psi_j^+ \psi_j] = 0$ etc. $\square$