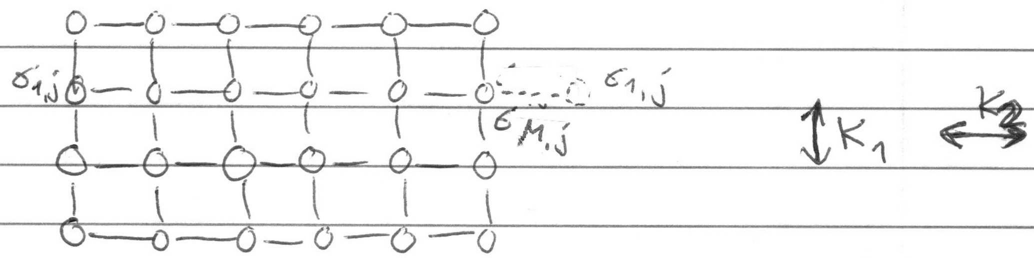
Description of 2D Ising model in terms of fermions

Setting

We look Ising model on a rectangle $L = I \times J$, $I = [\![1, M]\!]$, $J = [\![1, N]\!]$ with periodic b.c., that is $(\sigma_{i,j})_{(i,j)\in L} \in \{-1,+1\}^L$ is extended by setting

$$\sigma_{M+1,j} = \sigma_{1,j}, \quad \sigma_{i,N+1} = \sigma_{i,1}$$



We consider anisotropic Ising model on L with coupling constants K_1, K_2, h. Let

$$H\left((\sigma_{i,j})_{(i,j)\in L}\right) = -\beta^{-1} \sum_{m=1}^{M} \sum_{n=1}^{N} \left(K_1 \sigma_{m,n}\sigma_{m,n+1} + K_2 \sigma_{m,n}\sigma_{m+1,n} + h\sigma_{m,n} \right)$$

$B = \beta^{-1} h$ is the external magnetic field and we can always specialize to $K_1 = K_2 = \beta$ to return to the usual definition of Ising model.

We will assume $K_1 > 0$, $K_2 > 0$ and at some point we will set $h = 0$.

Pauli spin matrices

Define $\mathbb{C}^{2\times 2}$ matrices

won't be used here

$$\tau^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \left[\tau^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right], \tau^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\tau^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \tau^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and vectors $\underline{e}_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\underline{e}_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Then e.g. $\tau^z \underline{e}_\uparrow = \underline{e}_\uparrow$, $\tau^z \underline{e}_\downarrow = -\underline{e}_\downarrow$, $\tau^- \underline{e}_\uparrow = \underline{e}_\downarrow$, $\tau^+ \underline{e}_\downarrow = \underline{e}_\uparrow$, $(\tau^+)^2 = 0 = (\tau^-)^2$ and

$$[\tau^+, \tau^-]_+ = \tau^+\tau^- + \tau^-\tau^+ = I.$$

Lemma For any $\alpha \in \{x, y, z\}$ and for all $\theta \in \mathbb{R}$, we have

$$\exp(\theta \tau^\alpha) = \cosh(\theta) I + \sinh(\theta) \tau^\alpha$$

$$= \cosh(\theta)\left(I + \tanh(\theta) \tau^\alpha \right)$$

Proof Notice that $(\tau^\alpha)^2 = I$, $\alpha \in \{x, y, z\}$

$$\Rightarrow \exp(\theta\tau^\alpha) = \sum_{n=0}^{\infty} \frac{1}{n!} \theta^n (\tau^\alpha)^n$$

$$= \left(\sum_{n=0}^{\infty} \frac{1}{(2n)!} \theta^{2n}\right) I + \left(\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \theta^{2n+1}\right) \tau^\alpha$$

$$= (\cosh\theta) I + (\sinh\theta) \tau^\alpha$$

$\square$

Example (1D Ising model and the transfer matrix)

$$(\sigma_i)_{i\in[\![1,N]\!]} \in \{-1,+1\}^N \quad , \sigma_{N+1}=\sigma_1$$

Partition function is given by

$$Z=\sum_{\sigma_1=\pm1}\sum_{\sigma_2=\pm1}\cdots\sum_{\sigma_N=\pm1}\prod_{n=1}^{N}\exp(K_1\sigma_n\sigma_{n+1})$$

Define $V_1 \in \mathbb{C}^{2\times2}$

$$V_1 = \begin{array}{cc} \begin{pmatrix} e^{K_1} & e^{-K_1} \\ e^{-K_1} & e^{K_1} \end{pmatrix} & \begin{array}{l} \sigma'=+1 \\ \sigma'=-1 \end{array} \\ \begin{array}{cc} \sigma=+1 & \sigma=-1 \end{array} & \end{array}$$

Then $(V_1)_{\sigma,\sigma'} = \exp(K_1\sigma\sigma')$ and

$$Z=\sum_{\sigma_1=\pm1}\sum_{\sigma_2=\pm1}\cdots\sum_{\sigma_N=\pm1}(V_1)_{\sigma_1\sigma_2}\cdot(V_1)_{\sigma_2\sigma_3}\cdot\ldots\cdot(V_1)_{\sigma_N\sigma_1}$$

$$=\sum_{\sigma_1=\pm1}(V_1^N)_{\sigma_1,\sigma_1} = \mathrm{Tr}(V_1^N)$$

All this we saw in Part II of the course, already.

Now we use the Pauli matrices to write

$$V_1 = e^{K_1}(I+e^{-2K_1}\tau^x)$$

Since $K_1>0$, $0<e^{-2K_1}<1$ and we can find $K_1^*>0$ s.t. $e^{-2K_1}=\tanh(K_1^*)$.

Exercise (a) Show that $K_1^* > 0$ exists and is unique.

(b) Show that $K_1 \mapsto K_1^*$ is decreasing.

(c) Show that the equation $e^{-2K_1} = \tanh(K_1)$ has an unique solution and solve K_1.

Lemma When $K > 0$, let K^* be s.t. $e^{-2K^*} = \tanh(K)$. Then $e^{-2K} = \tanh(K^*)$ and $\sinh(2K)\sinh(2K^*) = 1$.

Proof

$$e^{-2K^*} = \tanh(K) = \frac{1-e^{-2K}}{1+e^{-2K}}$$

$$\Rightarrow \quad e^{-2K^*}(1+e^{-2K}) = 1-e^{-2K}$$

$$\Rightarrow \quad (1+e^{-2K^*})e^{-2K} = 1-e^{-2K^*}$$

$$\Rightarrow \quad e^{-2K} = \frac{1-e^{-2K^*}}{1+e^{-2K^*}} = \tanh(K^*)$$

The second claim is left to the reader. $\square$

Now V_1 of the example can be written as

$$V_1 = e^{K_1}(I + e^{-2K_1}\tau^x) = e^{K_1}(I + \tanh(K_1^*)\tau^x)$$

$$= \frac{e^{K_1}}{\cosh(K_1^*)} e^{K_1^*\tau^x} = \frac{1}{\sqrt{\cosh^2(K_1^*)\tanh(K_1^*)}} e^{K_1^*\tau^x}$$

$$= \sqrt{\frac{2}{\sinh(2K_1^*)}}\, e^{K_1^*\tau^x} = \sqrt{2\sinh(2K_1)}\, e^{K_1^*\tau^x}$$

In 1D, the basis of $\mathbb{C}^2$ was given by $\underline{e}_{+1} = \binom{1}{0}$ and $\underline{e}_{-1} = \binom{0}{1}$.

In 2D, the basis of $(\mathbb{C}^2)^{\otimes M} \cong \mathbb{C}^{2^M}$ is indexed by the spin states of a row of the Ising model:

- States: $\Sigma = \{ (\sigma_i)_{i \in [\![1,M]\!]} : \sigma_i \in \{-1,+1\},\ i \in [\![1,M]\!] \}$
- Basis: $\{ \underline{e}_{\underline{\sigma}} : \underline{\sigma} \in \Sigma \}$

$$\underline{e}_{\underline{\sigma}} = \underbrace{\underline{e}_{\sigma_1} \otimes \underline{e}_{\sigma_2} \otimes \dots \otimes \underline{e}_{\sigma_M}}_{\substack{\text{to us, } (\mathbb{C}^2)^{\otimes M} \text{ is defined} \\ \text{as the vector space with this basis}}}$$

For $A : \mathbb{C}^2 \to \mathbb{C}^2$ linear define $A_m : (\mathbb{C}^2)^{\otimes M} \to (\mathbb{C}^2)^{\otimes M}$ as the linear map

$$A_m = I_2 \otimes I_2 \otimes \dots \otimes I_2 \otimes A \otimes I_2 \otimes I_2 \otimes \dots \otimes I_2$$

where $I_2 = I$ is the identity map in $\mathbb{C}^2$ and A operates to the m'th copy of $\mathbb{C}^2$, i.e.,

$$A_m(\underline{e}_{\sigma_1} \otimes \underline{e}_{\sigma_2} \otimes \dots \otimes \underline{e}_{\sigma_M}) = \underline{e}_{\sigma_1} \otimes \dots \otimes \underline{e}_{\sigma_{m-1}} \otimes (A\underline{e}_{\sigma_m}) \otimes \otimes \underline{e}_{\sigma_{m+1}} \otimes \dots \otimes \underline{e}_{\sigma_M}.$$

To return the right-hand side back to basis vectors use linearity of the tensor product ($\otimes$) in all the components. Finally to see how A_m operates to the whole space $(\mathbb{C}^2)^{\otimes M}$, use linearity.

Some observations on matrices

Let $A, B, C \in \mathbb{C}^{n \times n}$, $\underline{v} \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$. Then the following properties hold. (Remember that the commuter is defined as $[A,B] = AB - BA$ and A and B are said to commute if $AB = BA$ $\Leftrightarrow [A,B] = 0$)

(a) $[AB, C] = A[B, C] + [A, C]B$

(b) $[A,B] = 0 \Rightarrow [\exp(A), B] = 0$

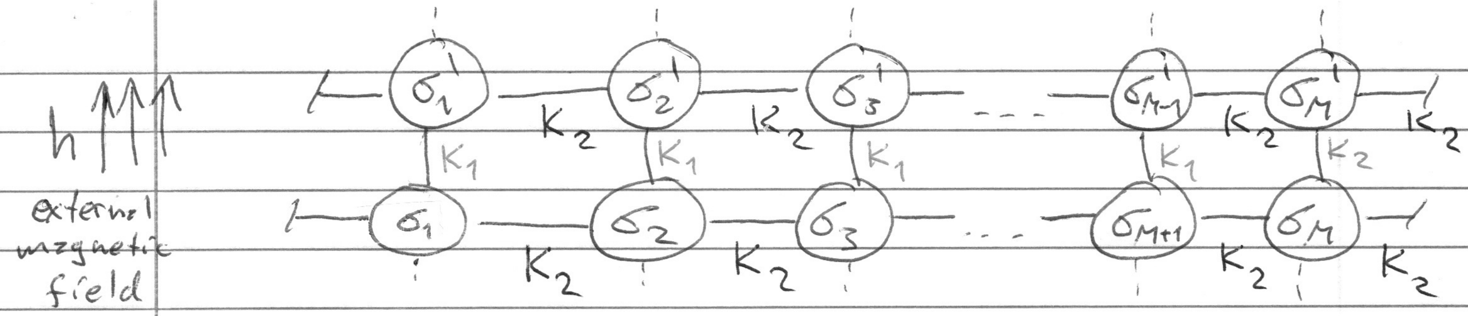
(c) $[A,B] = 0 \Rightarrow \exp(A+B) = \exp(A)\exp(B)$

(d) $A\underline{v} = \lambda \underline{v} \Rightarrow \exp(A)\underline{v} = e^{\lambda} \underline{v}$

(e) If C is invertible and $B = C^{-1}AC$, then $\exp(B) = C^{-1}\exp(A)C$

Proofs are left to the reader.

Remark If we are considering tensor product $V \otimes W$, ~~then~~ the linear operators $A \otimes id_W$ and $id_V \otimes B$ commute.



We will define three matrices V_k, $k=1,2,3$, in the space $(\mathbb{C}^2)^{\otimes M}$ in the basis $\{e_{\underline{\sigma}} : \underline{\sigma} \in \{+1,-1\}^{\{1,M\}}\}$ corresponding to interactions K_1, K_2 and h, respectively.

$$(V_1)_{\underline{\sigma},\underline{\sigma}'} = \exp\left(K_1 \sum_{m=1}^{M} \sigma_m \sigma_m'\right)$$

$$\overset{(*)}{\Rightarrow} V_1 = \left(2\sinh(2K_1)\right)^{\frac{M}{2}} \exp\left(K_1^* \sum_{m=1}^{M} \tau_m^x\right)$$

$$= \left(2\sinh(2K_1)\right)^{\frac{M}{2}} \prod_{m=1}^{M} \underbrace{\exp\left(K_1^* \tau_m^x\right)}_{\text{these matrices commute. Hence the order of the product is irrelevant}}$$

One can verify $(*)$ by the 1D calculation and by operating with the latter form to a basis vector.

$$(V_2)_{\underline{\sigma},\underline{\sigma}'} = \delta_{\underline{\sigma},\underline{\sigma}'} \exp\left(K_2 \sum_{m=1}^{M} \sigma_m \sigma_{m+1}\right)$$

$$\overset{(**)}{\Rightarrow} V_2 = \exp\left(K_2 \sum_{m=1}^{M} \tau_m^z \tau_{m+1}^z\right)$$

$$= \prod_{m=1}^{M} \underbrace{\exp\left(K_2 \tau_m^z \tau_{m+1}^z\right)}_{\text{again, these operators commomute}}$$

Note that the basis vectors are eigenvectors of τ_m^z and $(**)$ is easy. See (d) of the previous page.

$$(V_3)_{\underline{\sigma},\underline{\sigma}'} = \delta_{\underline{\sigma},\underline{\sigma}'} \exp\left(h \sum_{m=1}^{M} \sigma_m \right)$$

$$\Rightarrow V_3 = \exp\left(h \sum_{m=1}^{M} \tau_m^z \right) = \prod_{m=1}^{M} \underbrace{\exp(h \tau_m^z)}_{\text{these operators commute}}$$

Now

$$(V_1 V_2 V_3)_{\underline{\sigma},\underline{\sigma}'}$$

$$= \sum_{\underline{\sigma}''} \sum_{\underline{\sigma}'''} \delta_{\underline{\sigma}'',\underline{\sigma}'''} \delta_{\underline{\sigma}''',\underline{\sigma}'} \exp\Big(K_1 \sum_{m=1}^{M} \sigma_m \sigma_m'' + K_2 \sum_{m=1}^{M} \sigma_m'' \sigma_{m+1}''$$
$$+ h \sum_{m=1}^{M} \sigma_m''' \Big)$$

$$= \exp\left(\sum_{m=1}^{M} K_1 \sigma_m \sigma_m' + K_2 \sigma_m' \sigma_{m+1}' + h \sigma_m' \right)$$

$$\Rightarrow \quad Z = \sum_{\underline{\sigma}_1} \sum_{\underline{\sigma}_2} \dots \sum_{\underline{\sigma}_N} \prod_{n=1}^{N} (V_1 V_2 V_3)_{\underline{\sigma}_n, \underline{\sigma}_{n+1}}$$

$$= \mathrm{Tr}\left((V_1 V_2 V_3)^N \right)$$

$$= \mathrm{Tr}\left(T^N \right)$$

Where we defined the <u>transfer matrix</u> T as a symmetric matrix

$$T = (V_2 V_3)^{\frac{1}{2}} \, V_1 \, (V_2 V_3)^{\frac{1}{2}}$$

Here $(\exp(A))^{\frac{1}{2}} = \exp(\frac{1}{2} A)$. We will find during the next lectures the largest eigenvalue λ_0 of T and show an explicit formula for the free energy

$$f(K_1, K_2, h=0) = -\frac{1}{\beta} \lim_{M,N\to\infty} \frac{1}{NM} \mathrm{Tr}(T^N) \Big|_{h=0}$$

A change of basis $\tau^x \to -\tau^z, \ \tau^z \to \tau^x$

Notice that $V_k = \exp(A_k)$ where A_k is linear or quadratic in $\tau^x_m, \tau^z_m, m \in [\![1, M]\!]$. We want to keep the low order of polynomials, but move to $\tau^+_m, \tau^-_m, m \in [\![1, m]\!]$

Relations we want to use are

$$\tau^x = \tau^+ + \tau^- \ , \ \tau^z = \tau^+\tau^- - \tau^-\tau^+ = 2\tau^+\tau^- - I_2$$

We move to basis of eigenvectors of τ^x

$$\underline{f}_1 = \frac{1}{\sqrt{2}}(\underline{e}_1 - \underline{e}_{-1}) \ , \ \underline{f}_0 = \frac{1}{\sqrt{2}}(\underline{e}_{+1} + \underline{e}_{-1})$$

The change of basis matrix is

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

and the matrices τ^x and τ^z are transformed as

$$U^T \tau^x U = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= -\tau^z$$

$$U^T \tau^z U = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \tau^x$$

In the new basis

$$\begin{cases} V_1 = (2\sinh(2K_1))^{\frac{M}{2}} \exp\left(-K_1^* \sum_{m=1}^{M} \tau_m^z\right) \\ \qquad = (2\sinh(2K_1))^{\frac{M}{2}} \exp\left(-2K_1^* \sum_{m=1}^{M} \left(\tau_m^+ \tau_m^- - \frac{1}{2}\right)\right) \\ V_2 = \exp\left(K_2 \sum_{m=1}^{M} \tau_m^x \tau_{m+1}^x\right) = \exp\left(K_2 \sum_{m=1}^{M} (\tau_m^+ + \tau_m^-)(\tau_{m+1}^+ + \tau_{m+1}^-)\right) \\ V_3 = \exp\left(h \sum_{m=1}^{M} \tau_m^x\right) = \exp\left(h \sum_{m=1}^{M} (\tau_m^+ + \tau_m^-)\right) \end{cases}$$

quadratic in τ^+, τ^-

linear in τ^+, τ^-

Because of the form of V_3 we soon have to set $h = 0$.

Fermionic creation and anhihilation operators $C_m^{\dagger}, C_m$

The operators $\tau_m^{\pm} : (\mathbb{C}^2)^{\otimes M} \to (\mathbb{C}^2)^{\otimes M}$ satisfy

(Bosonic) $\quad [\tau_m^{\pm}, \tau_l^{\pm}] = 0 \qquad m \neq l$

(Fermionic) $\quad [\tau_m^{+}, \tau_m^{-}]_+ = I \quad , [\tau_m^{+}, \tau_m^{+}]_+ = 0 = [\tau_m^{-}, \tau_m^{-}]_+$

These relations are partially bosonic and partially fermionic. We want to have purely fermionic relations. Hence we set

$$\begin{cases} C_m = \left(\prod_{j=1}^{m-1} (-\tau_j^z) \right) \tau_m^- \\ C_m^{\dagger} = \left(\prod_{j=1}^{m-1} (-\tau_j^z) \right) \tau_m^+ \end{cases}$$

The currently used basis of $(\mathbb{C}^2)^{\otimes M}$ is $\{ \underline{f}_{n_1, n_2, \dots, n_M} = \underline{f}_{n_1} \otimes \underline{f}_{n_2} \otimes \dots \otimes \underline{f}_{n_M} : n_k \in \{0,1\} \quad k = 1, 2, \dots, M \}$

Now $\underline{f}_{0,0,\dots,0}$ is the vacuum vector for $C_m, C_m^{\dagger}$ since

$$C_m \underline{f}_{0,0,\dots,0} = 0$$

$$\underline{f}_{n_1, n_2, \dots, n_M} = (C_1^{\dagger})^{n_1} (C_2^{\dagger})^{n_2} \dots (C_M^{\dagger})^{n_M} \underline{f}_{0,0,\dots,0}$$

We will start next time by showing the following lemma.

Lemma If C_m and $C_m^\dagger$ defined as above, then for $m \neq l$

$$[C_m, C_l]_+ = 0 \ , \ [C_m, C_l^\dagger]_+ = 0, \ [C_m^\dagger, C_l^\dagger]_+ = 0$$

$$[C_m, C_m^\dagger]_+ = I \ , \ [C_m, C_m]_+ = 0, \ [C_m^\dagger, C_m^\dagger]_+ = 0$$

The inverse transform from C's to τ's is given by

$$\tau_m^- = \left(\prod_{j=1}^{m-1} (-(2N_j - I)) C_m \right.$$

$$\tau_m^+ = \left(\prod_{j=1}^{m-1} (-(2N_j - I) \right) C_m^\dagger$$

where $N_j = C_j^\dagger C_j$.