Stochastic processes in physics and biology
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These notes are intended to cover the topics of the exercise sessions which are not included in the lecture notes. Please, don't hesitate to point out possible errors and misprints.

1 Stochastic calculus with Hermite polynomials

Proposition 1. *The transition probability of the Wiener process admits the expansion*

$$\frac{e^{-\frac{(x-y)^2}{2\,t}}}{\sqrt{2\,\pi\,t}} = \sum_{n=0}^{\infty} \frac{e^{-\frac{x^2}{2\,t}}}{\sqrt{2\,\pi\,t}} \left(\frac{y}{t}\right)^n h_n\,(x,t)$$

where the h_n's are Hermite polynomials

$$h_n\,(x\,,t) = \frac{(-t)^n}{n!}\, e^{\frac{x^2}{2\,t}}\, \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-\frac{x^2}{2\,t}} \tag{1}$$

Proof. Consider the Taylor expansion of the transition probability as a function of y at $y = 0$:

$$\frac{e^{-\frac{(x-y)^2}{2\,t}}}{\sqrt{2\,\pi\,t}} = \sum_{n=0}^{\infty} \frac{y^n}{n!}\, \frac{\mathrm{d}^n}{\mathrm{d}z^n}\, \frac{e^{-\frac{(x-z)^2}{2\,t}}}{\sqrt{2\,\pi\,t}}\Bigg|_{z=0} \tag{2}$$

Let us postulate for the n-th order of the Taylor expansion the form

$$\frac{y^n}{n!}\, \frac{\mathrm{d}^n}{\mathrm{d}z^n}\Bigg|_{z=0} \frac{e^{-\frac{(x-z)^2}{2\,t}}}{\sqrt{2\,\pi\,t}} := \frac{e^{-\frac{x^2}{2\,t}}}{\sqrt{2\,\pi\,t}} \left(\frac{y}{t}\right)^n h_n\,(x,t) \tag{3}$$

We will show that the h_n's in 3 are exactly the Hermite polynomials defined in 1. So, let us start by manipulating the left hand side of 3:

$$\frac{e^{-\frac{x^2}{2\,t}}}{\sqrt{2\,\pi\,t}} \left(\frac{y}{t}\right)^n h_n\,(x\,,t) = \frac{y^n}{n!}\, \frac{\mathrm{d}^n}{\mathrm{d}z^n}\Bigg|_{z=0} \int_{\mathbb{R}} \frac{\mathrm{d}p}{2\pi} e^{-ipz} \mathbb{E}(e^{ip\xi}) \tag{4}$$

where ξ is a random variable distributing according to $N_t(z - x)$ (as a function of z). In other words, we are just rewriting $N_t(z - x)$ as the anti-Fourier transform of its characteristic function, which is by definition the Fourier transform of $N_t(z - x)$.
We know that

$$\mathrm{E}(e^{ip\xi}) = \exp\left\{ip\mathrm{E}\xi - \frac{p^2}{2}\mathrm{E}\xi^2\right\} = e^{ipx-p^2t/2} \tag{5}$$

hence we get

$$
\begin{aligned}
\frac{e^{-\frac{x^2}{2\,t}}}{\sqrt{2\,\pi\,t}}\left(\frac{y}{t}\right)^n h_n\,(x\,,t) &= \frac{y^n}{n!}\,\frac{\mathrm{d}^n}{\mathrm{d}z^n}\bigg|_{z=0}\int_{\mathbb{R}}\frac{\mathrm{d}p}{2\pi}e^{-ipz}e^{ipx-p^2t/2} = \\
&= \frac{y^n}{n!}\,\frac{\mathrm{d}^n}{\mathrm{d}z^n}\bigg|_{z=0}\sum_{k=0}^{\infty}\int_{\mathbb{R}}\frac{\mathrm{d}p}{2\pi}\frac{(-ipz)^k}{k!}e^{ipx-p^2t/2} = \\
&= \frac{y^n}{n!}\sum_{k=0}^{\infty}\int_{\mathbb{R}}\frac{\mathrm{d}p}{2\pi}\frac{(-ipz)^k}{k!}e^{ipx-p^2t/2}n!\delta_{n,k} = \\
&= \frac{y^n}{n!}\int_{\mathbb{R}}\frac{\mathrm{d}p}{2\pi}(-ipz)^n e^{ipx-p^2t/2} = \\
&= \frac{(-y)^n}{n!}\,\frac{\mathrm{d}^n}{\mathrm{d}x^n}e^{-\frac{x^2}{2\,t}} \qquad (6)
\end{aligned}
$$

From the last equation we can conclude that indeed

$$
h_n\,(x\,,t) = \frac{(-t)^n}{n!}\,e^{\frac{x^2}{2\,t}}\,\frac{d^n}{dx^n}e^{-\frac{x^2}{2\,t}} \qquad (7)
$$

□

Other two important properties of the Hermite polynomials are stated in the following proposition.

Proposition 2. *(i) The h_n's satisfy the following partial differential equation*

$$
(x\,\partial_x + 2\,t\partial_t)\,h_n(x\,,t) = n\,h_n(x\,,t)
$$

(ii) The expected value of an Hermite polynomial having for argument a Wiener process starting at x is

$$
\mathrm{E}\,h_n\,(w_t + x\,,t) = \frac{x^n}{n!} = h_n(x,0) \qquad (8)
$$

Proof. (i) It is readily verified by performing the derivatives.

(ii) From the definition of expected value we have

$$
\begin{aligned}
\mathrm{E}\,h_n\,(w_t + x\,,t) &:= \int_{\mathbb{R}} dy\,h_n(y,t)\,\frac{e^{-\frac{(y-x)^2}{2\,t}}}{\sqrt{2\,\pi\,t}} = \frac{(-t)^n}{\Gamma(n+1)}\int_{\mathbb{R}} dy\,\frac{e^{-\frac{x^2}{2\,t}+\frac{x\,y}{t}}}{\sqrt{2\,\pi\,t}}\,\frac{\mathrm{d}^n}{\mathrm{d}y^n}e^{-\frac{y^2}{2\,t}} \\
&= \frac{e^{-\frac{(x-y)^2}{2\,t}}}{\sqrt{2\,\pi\,t}}\left\{\left(\frac{\mathrm{d}^{n-1}}{\mathrm{d}y^{n-1}}e^{-\frac{y^2}{2\,t}}\right)e^{\frac{xy}{t}}\bigg|_{y=-\infty}^{y=+\infty} - \int \mathrm{d}y\left(\frac{\mathrm{d}^{n-1}}{\mathrm{d}y^{n-1}}e^{-\frac{y^2}{2\,t}}\right)e^{\frac{xy}{t}}\left(\frac{x}{t}\right)\right\} \qquad (9)
\end{aligned}
$$

Iterating the integrations by parts one gets

$$
\mathrm{E}\,h_n\,(w_t + x\,,t) = \frac{x^n}{n!} \qquad (10)
$$

In order to see that $x^n/n! = h_n(x,0) = \lim_{t\to 0} h_n(x,t)$, we notice that the leading term in t^{-1} of $\partial_x^n e^{-x^2/2t}$ is $e^{-x^2/2t}(-x/t)^n$, hence we get

$$
\lim_{t\to 0} h_n(x,t) = \lim_{t\to 0}\frac{(-t)^n}{n!}\,e^{\frac{x^2}{2\,t}}e^{-\frac{x^2}{2t}}\left(-\frac{x}{t}\right)^n = \frac{x^n}{n!} \qquad (11)
$$

□

The last proposition shows that the expectation value of the Hermite polynomials is conserved. We can see this fact from a different perspective, namely as a consequence of the following proposition.

Proposition 3. *The differential of Hermite polynomials along a realization of the Wiener process is*

$$\mathrm{d}h_n(w_t, t) = \mathrm{d}w_t \partial_{w_t} h_n(w_t, t)$$

where the differential on the right hand side is an Ito differential.

Proof. By Ito lemma we have

$$\mathrm{d}h_n(w_t, t) = \mathrm{d}t \left(\partial_t + \frac{1}{2} \partial^2_{w_t} \right) h_n(w_t, t) + \mathrm{d}w_t \partial_{w_t} h_n(w_t, t)$$

In order to prove the claim we need to show that

$$\left(\partial_t + \frac{1}{2} \partial^2_{w_t} \right) h_n(w_t, t) = 0$$

We can use the representation 1 and the property 8 to get for any $t > 0$

$$0 = \left(\partial_t - \frac{1}{2} \partial^2_x \right) \frac{e^{-\frac{(x-y)^2}{2t}}}{\sqrt{2\pi t}} =$$

$$= \sum_{n=0}^{\infty} \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} \left(\frac{y}{t} \right)^n \frac{1}{t} \left(-n + t\,\partial_t - \frac{t}{2} \partial^2_x + x\,\partial_x \right) h_n = - \sum_{n=0}^{\infty} \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} \left(\frac{y}{t} \right)^n \left(\partial_t + \frac{1}{2} \partial^2_x \right) h_n$$

which implies

$$\left(\partial_t + \frac{1}{2} \partial^2_x \right) h_n = 0$$

as each of these multiply positive definite terms of different order in y. □

Now we can easily recover 8: consider

$$h_n(w_t + x, t) = h_n(x, 0) + \int_0^t dw_t\, \partial_{w_t} h_n(w_t, t)$$

From the property of the Ito integral we get

$$\mathrm{E}\, h_n(w_t + x, t) = \mathrm{E}\, h_n(x, 0) + \int_0^t \mathrm{E}\left[dw_t\, \partial_{w_t} h_n(w_t, t) \right] = h_n(x, 0) = \frac{x^n}{n!}$$

2 Recursion relation over the Wiener process

Proposition 4. *Stochastic integrals over Hermite polynomials satisfy the simple recursion relation*

$$\int_0^t \mathrm{d}w_s\, h_n(w_s, s) = h_{n+1}(w_s, s)$$

Proof. Consider the exponential process:

$$\xi_t = e^{\lambda w_t - \frac{\lambda^2 t}{2}} \tag{12}$$

Applying to it Ito lemma yields

$$\mathrm{d}\xi_t = \lambda\, \xi_t\, \mathrm{d}t + \lambda\, \mathrm{d}w_t \tag{13}$$

or equivalently

$$e^{\lambda w_t - \frac{\lambda^2 t}{2}} = 1 + \lambda \int_0^t dw_s\, e^{\lambda w_s - \frac{\lambda^2 s}{2}}$$

Now consider the n-th derivative with respect to λ on both sides:

$$\frac{d^n}{d\lambda^n}\bigg|_{\lambda=0} e^{\lambda w_t - \frac{\lambda^2 t}{2}} = \frac{d^n}{d\lambda^n}\bigg|_{\lambda=0} \lambda \int_0^t dw_s\, e^{\lambda w_s - \frac{\lambda^2 s}{2}} \tag{14}$$

Contrasting the left-hand side of 14 with the expression for the Hermite polynomials in 3, we conclude

$$\frac{d^n}{d\lambda^n}\bigg|_{\lambda=0} e^{\lambda w_t - \frac{\lambda^2 t}{2}} = t^n \frac{\mathrm{d}^n}{\mathrm{d}z^n}\bigg|_{z=0} e^{\frac{z}{t} w_t - \frac{z^2}{2t}} = n!\, h_n(w_t\,, t) \tag{15}$$

where we set $\lambda = z/t$. The right hand side of 14 is

$$\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\bigg|_{\lambda=0} \lambda \int_0^t dw_s\, e^{\lambda w_s - \frac{\lambda^2 s}{2}} &= \frac{\mathrm{d}^{n-1}}{\mathrm{d}\lambda^{n-1}} \frac{\mathrm{d}}{\mathrm{d}\lambda} \lambda \int_0^t dw_s\, e^{\lambda w_s - \frac{\lambda^2 s}{2}}\bigg|_{\lambda=0} = \\
&= \frac{\mathrm{d}^{n-1}}{\mathrm{d}\lambda^{n-1}} \left[\int_0^t dw_s\, e^{\lambda w_s - \frac{\lambda^2 s}{2}} + \lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \int_0^t dw_s\, e^{\lambda w_s - \frac{\lambda^2 s}{2}}\right]\bigg|_{\lambda=0} = \\
&= \frac{\mathrm{d}^{n-1}}{\mathrm{d}\lambda^{n-1}} \int_0^t dw_s\, e^{\lambda w_s - \frac{\lambda^2 s}{2}}\bigg|_{\lambda=0} + \\
&\quad + \frac{\mathrm{d}^{n-2}}{\mathrm{d}\lambda^{n-2}} \frac{\mathrm{d}}{\mathrm{d}\lambda} \left[\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \int_0^t dw_s\, e^{\lambda w_s - \frac{\lambda^2 s}{2}}\right]\bigg|_{\lambda=0}
\end{aligned}$$

By iterating this procedure we get finally

$$\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\bigg|_{\lambda=0} \lambda \int_0^t dw_s\, e^{\lambda w_s - \frac{\lambda^2 s}{2}} = n\, \frac{d^{n-1}}{d\lambda^{n-1}}\bigg|_{\lambda=0} \int_0^t dw_s\, e^{\lambda w_s - \frac{\lambda^2 s}{2}} \tag{16}$$

By looking at 15 we can rewrite the right hand side of the last equation as follows:

$$\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\bigg|_{\lambda=0} \lambda \int_0^t dw_s\, e^{\lambda w_s - \frac{\lambda^2 s}{2}} &= n \int_0^t dw_s\, \frac{d^{n-1}}{d\lambda^{n-1}}\bigg|_{\lambda=0} e^{\lambda w_s - \frac{\lambda^2 s}{2}} = \\
&= n \int_0^t dw_s (n-1)!\, h_{n-1}(w_s\,, s) = \\
&= n! \int_0^t dw_s\, h_{n-1}(w_s\,, s)
\end{aligned} \tag{17}$$

We have therefore proved that

$$h_n(w_t\,, t) = \int_0^t dw_s\, h_{n-1}(w_s\,, s)$$

□

An important consequence is the following. Since

$$h_0(w_t, t) = 1$$

we have that

$$\int_0^t dw_s = \int_0^t dw_s\, h_0(w_s\,, s) = h_1(w_t, t)$$

and

$$\int_0^t dw_{s_1} \int_0^{s_1} dw_{s_0} = h_2(w_t, t)$$

or in full generality

$$\int_0^t dw_{s_1} \prod_{i=1}^{n-1} \int_0^{s_{i-1}} dw_{s_{i-1}} = h_n(w_t, t)$$