Semigroups & Delay eq., Solutions 3.

Ex 1: Since $\|T^*(t)\| = \|T(t)\|$ $\forall t \geq 0$, we see that T^* has the same growth bounds as T.
Therefore $\rho(A^*)$ contains the half-plane $\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda > \omega_0\}$, where $\|T^*(t)\| = \|T(t)\| \leq M e^{\omega_0 t}$.

Now let $\mathrm{Re}\,\lambda > \omega_0$ and $x^* \in \mathcal{X}^*$. Then for all $x \in \mathcal{X}$

$$\langle (\lambda - A^*)^{-1} x^*, x \rangle = \left\langle \int_0^\infty ds\, e^{-\lambda s} T^*(s) x^*, x \right\rangle$$

$$= \int_0^\infty ds\, e^{-\lambda s} \underbrace{\langle T^*(s) x^*, x \rangle}_{\langle x^*, T(s) x \rangle}$$

$$= \langle x^*, \underbrace{\int_0^\infty ds\, e^{-\lambda s} T(s) x}_{(\lambda - A)^{-1} x} \rangle$$

meaning $(\lambda - A)^{-1\,*} = (\lambda - A^*)^{-1}$. Done $\square$

<u>Remark</u>: The above was a nice application of semigroup properties, but it's indeed true in general that $(S^*)^{-1} = (S^{-1})^*$, whenever $S \in \mathcal{B}(\mathcal{X})$ is invertible. For, if $\perp$ denotes the annihilator w.r.t. $\langle (x,y), (a,b) \rangle_{\mathcal{X}\times\mathcal{X}} := \langle x, a \rangle - \langle y, b \rangle$, and "$R(x,y) := (y,x)$", then one can check that

$$\mathcal{G}((A^{-1})^*) = \mathcal{G}(A^{-1})^\perp = (R(\mathcal{G}(A)))^\perp = R(\mathcal{G}(A)^\perp)$$

$$= R\, \mathcal{G}(A^*)$$

$$= \mathcal{G}(A^{*\,-1})$$

$\therefore$ $A^{-1\,*} = A^{*\,-1}$.

Ex 2: Since $f:[a,b] \longrightarrow \mathcal{X}^*$ is weak*-cont., and $[a,b]$ is compact, we see that for each fixed $x \in \mathcal{X}$, the orbits $\{\langle x, f(s)\rangle \in \mathbb{C} : s \in [a,b]\}$ are compact and thus bounded. In view of Banach-Steinhaus this implies the uniform boundedness of $\{f(s) : s \in [a,b]\}$, i.e.

$$\sup_s \|f(s)\|_{\mathcal{X}^*} =: M < \infty .$$

Now, $s \mapsto \langle x, f(s)\rangle$ is cont. so we can Riemann-integrate it and have the estimate

$$\left| \int_a^b \langle x, f(s)\rangle \, ds \right| \leq (b-a) \|x\| \underbrace{\sup_{s \in [a,b]} \|f(s)\|_{\mathcal{X}^*}}_{= M < \infty} , \quad (x \in \mathcal{X}).$$

Thus, if we denote $\Lambda(x) := \int_a^b \langle x, f(s)\rangle \, ds$, $\Lambda \in \mathcal{X}^*$ and we can define

$$\int_a^b f(s)\, ds := \Lambda .$$

Ex 3 : First assume $(\lambda - A)^{-1}$ is $\sigma(\mathcal{X}, \mathcal{X}^{\odot})$-compact, meaning that it maps bounded subsets of $\mathcal{X}$ into $\sigma(\mathcal{X}, \mathcal{X}^{\odot})$-pre-compact sets.

To motivate, note

$$\mathcal{X}^{\odot\odot} = \overline{D(A^{\odot *})} = \overline{R(\lambda - A^{\odot})^{-1 *}}$$

so that if we can show $R(\lambda - A^{\odot})^{-1 *} \subseteq \mathcal{X}$ then, by norm-closedness of $\mathcal{X}$, $\mathcal{X}^{\odot\odot} \subset \mathcal{X}$.

So pick $x^{\odot *} \in \mathcal{X}^{\odot *}$. If there now was $x \in \mathcal{X}$ such that $(\lambda - A^{\odot})^{-1 *} x^{\odot *} = x$ then for all $y^{\odot} \in \mathcal{X}^{\odot}$

$$\underline{\langle x^{\odot *}, (\lambda - A^{\odot})^{-1} y^{\odot} \rangle} = \langle (\lambda - A^{\odot})^{-1 *} x^{\odot *}, y^{\odot} \rangle$$
$$= \underline{\langle x, y^{\odot} \rangle} .$$

As we compare the first and the last expression, we see that this question is non-trivial. However, for any finite set π of $y^{\odot}$, this linear system can be solved for "x_{π}" (And this I take for granted here). The trick then is to pass to the case of all $y^{\odot} \in \mathcal{X}^{\odot}$.

Now we are ready to start. For any finite subset $\pi \subseteq \mathcal{X}^{\odot}$, define

$$F_{\pi} := \{ (\lambda - A)^{-1} x_{\pi} \in \mathcal{X} : \|x_{\pi}\| \le 2 \|x^{\odot *}\| \text{, and for each } y^{\odot} \in \pi$$
$$\langle (\lambda - A^{\odot})^{-1 *} x^{\odot *}, y^{\odot} \rangle = \langle (\lambda - A)^{-1} x_{\pi}, y^{\odot} \rangle \ \}$$

... Ex 3: There is a basic functional analytic result, called Helly's Theorem, which guarantees that indeed $F_\pi \neq \emptyset$ (See, e.g 2.7.8 in Hille, Phillips; Functional analysis and Semigroups).

Then, since $F_{\pi_1} \cap F_{\pi_2} = F_{\pi_1 \cup \pi_2} \neq \emptyset$, the family $\{F_\pi : \pi \subseteq \mathfrak{X}^\odot \text{ finite}\}$ has <u>finite intersection property</u>.

Enter, Hypothesis... Since each F_π is a subset of the <u>image of the same bounded set</u> <u>under $\sigma(\mathfrak{X}, \mathfrak{X}^\odot)$ compact</u> operator $(\lambda - A)^{-1}$, we see that

$$\exists\, x_\infty \in \bigcap_\pi \overline{F_\pi}^{\sigma(\mathfrak{X}, \mathfrak{X}^\odot)} \neq \emptyset \quad (\text{actually } F_\pi \text{ is } \sigma(\mathfrak{X}, \mathfrak{X}^\odot) \text{ closed})$$

For this $x_\infty \in \mathfrak{X}$, then clearly holds

$$\langle (\lambda - A^\odot)^{-1*} x^{\odot *}, y^\odot \rangle = \langle x_\infty, y^\odot \rangle \quad , \quad (\forall y^\odot \in \mathfrak{X}^\odot) \, ,$$

$$\therefore \text{"} x_\infty = (\lambda - A^\odot)^{-1*} x^{\odot *} \text{"} \, . \text{ Done.}$$

<u>For the converse, assume "$\mathfrak{X} = \mathfrak{X}^{\odot\odot}$".</u>

Begin with a general observation: The adjoint T^* of any $T \in B(\mathfrak{X})$ is <u>automatically $\sigma(\mathfrak{X}^*, \mathfrak{X})$-compact</u>. For, as a bounded operator T^* maps bounded sets to bounded sets. But <u>norm-bounded</u> sets of $\mathfrak{X}^*$ are <u>$\sigma(\mathfrak{X}^*, \mathfrak{X})$-compact</u> by Banach-Alaoglu.

Ex 3 Applying our observation to $(\lambda - A^{\odot})^{-1}$, we find that $(\lambda - A^{\odot})^{-1*}$ is $\sigma(\mathcal{X}^{\odot *}, \mathcal{X}^{\odot})$-compact. This implies $\underline{(\lambda - A)^{-1\odot\odot}} = (\lambda - A^{\odot})^{-1\odot} = (\lambda - A^{\odot})^{-1*}\big|_{\mathcal{X}^{\odot\odot}}$, as a restriction, is also $\underline{\sigma(\mathcal{X}^{\odot\odot}, \mathcal{X}^{\odot})\text{-compact}}$. But now, if "$\mathcal{X} = \mathcal{X}^{\odot\odot}$", the $\sigma(\mathcal{X}^{\odot\odot}, \mathcal{X}^{\odot})$-topology on $\mathcal{X}^{\odot\odot}$ can be considered as $\sigma(\mathcal{X}, \mathcal{X}^{\odot})$-topology on $\mathcal{X}$, and $(\lambda - A)^{-1\odot\odot} \in B(\mathcal{X}^{\odot\odot})$ as $(\lambda - A)^{-1}$ on $B(\mathcal{X})$.

Making these identifications, we rephrase the last conclusion as
$(\lambda - A)^{-1}$ is $\sigma(\mathcal{X}, \mathcal{X}^{\odot})$-compact Q.E.D.

Ex 4. Let "$\mathscr{X} = \mathscr{X}^{\odot\odot}$".

As in Ex 3, $(\lambda - A)^{-1*}$ is automatically $\sigma(\mathscr{X}^*, \mathscr{X})$-compact. Moreover, its range $\mathcal{R}(\lambda - A)^{-1*} = \mathcal{D}(A^*)$ lies inside $\mathscr{X}^{\odot}$. Therefore

$(\lambda - A^{\odot})^{-1} = (\lambda - A)^{-1\odot} = (\lambda - A)^{-1*}|_{\mathscr{X}^{\odot}}$

is $\sigma(\mathscr{X}^{\odot}, \mathscr{X})$-compact. But now, since "$\mathscr{X} = \mathscr{X}^{\odot\odot}$", we can identify $\sigma(\mathscr{X}^{\odot}, \mathscr{X})$-topology on $\mathscr{X}^{\odot}$ with $\sigma(\mathscr{X}^{\odot}, \mathscr{X}^{\odot\odot})$-topology, so that $(\lambda - A^{\odot})^{-1}$ is "also" $\sigma(\mathscr{X}^{\odot}, \mathscr{X}^{\odot\odot})$-compact. By Ex 3. $\mathscr{X}^{\odot}$ is $\odot$-reflexive.

Next assume "$\mathscr{X}^{\odot} = \mathscr{X}^{\odot\odot\odot}$":

If it really was the case that $\mathscr{X} \subsetneq \mathscr{X}^{\odot\odot}$, then we could find $x^{\odot\odot*} \in \mathscr{X}^{\odot\odot*}$ such that $x^{\odot\odot*} \neq 0$ and $x^{\odot\odot*}(\mathscr{X}) = \{0\}$.

Actually we can even find such an element from $\mathscr{X}^{\odot\odot\odot}$; just take $z^{\odot\odot\odot} := (\lambda - A^{\odot\odot})^{-1*} x^{\odot\odot*}$. Then indeed

$$z^{\odot\odot\odot}(\mathscr{X}) = x^{\odot\odot*}(\underbrace{(\lambda - A^{\odot\odot})^{-1}\mathscr{X}}_{\subseteq \mathscr{X}}) = 0,$$

for, as one check, $(\lambda - A^{\odot\odot})^{-1} = (\lambda - A)^{-1\odot\odot}$ restricts to $(\lambda - A)^{-1}$ on $\mathscr{X}$.

Now then, by hypothesis, we can identify $\overset{\#}{z}^{\odot\odot\odot} \in \mathscr{X}^{\odot} \subseteq \mathscr{X}^*$ so that we have managed to come up with a non-zero element from $\mathscr{X}^*$ that annihilates the whole original space $\mathscr{X}$. That is non-sense, so it must be the case $\mathscr{X} = \mathscr{X}^{\odot\odot}$ $\square$