Semigroups & Delay eq., Solutions 9

Ex 1 (a): Firstly, let us show that

$$\int_0^t \overline{T_0}^{\odot *}(t-s)\, r^{\odot *}\eta(s)\, ds = \int_0^{\max(0,t+\cdot)} \eta(s)\, ds ,$$

where $\eta \in L^1_{loc}[0,\infty)$, and $r^{\odot *} = \underbrace{(1,\bar{0}) \in \mathbb{C} \times L^\infty[-1,0]}_{}^{\overbrace{\qquad\qquad}^{x^{\odot *}}}$.

Too see this, let $x^\odot = (x^\odot(0), \dot{x}^\odot) \in \mathbb{C} \times L^1[0,1]$ be arbitrary. Then

$$\left\langle \int_0^t \overline{T_0}^{\odot *}(t-s)\, r^{\odot *}\eta(s)\, ds \,\middle|\, x^\odot \right\rangle$$

$$= \int_0^t ds\, \eta(s) \underbrace{\langle r^{\odot *} \mid T_0^\odot(t-s)\, x^\odot \rangle}_{\underbrace{\langle (1,\bar{0}) \mid (x^\odot(t-s), \dot{x}^\odot_{t-s}) \rangle}_{= 1\cdot x^\odot(t-s) + \underbrace{\langle \bar{0} \mid x^\odot_{t-s} \rangle}_{=0}}}$$

$$= \int_0^t ds\, \eta(s) \underbrace{x^\odot(t-s)}_{x^\odot(0) + \int_0^{t-s} \dot{x}^\odot(\sigma)\, d\sigma}$$

$$= x^\odot(0) \int_0^t ds\, \eta(s) + \int_0^t ds \int_0^{t-s} d\sigma\, \eta(s)\, \dot{x}^\odot(\sigma) .$$

Fubini on the second term gives $\displaystyle\int_0^t d\sigma\, \dot{x}^\odot(\sigma) \int_0^{t-\sigma} ds\, \eta(s)$.

... Ex 1(a): To "obtain a pairing" the first integrations must be extended up to h:

$$\int_0^h d\sigma\, \dot{x}^{\odot}(\sigma) \int_0^{\max(0,t-\sigma)} ds\, \eta(s) = \langle \dot{x}^{\odot} \mid \int_0^{\max(0,t+\cdot)} \eta \rangle$$

(pairing of L^1 and L^∞)

Now we see, continuing the previous manipulation,

$$x^{\odot}(0) \int_0^t \eta(s)\, ds + \langle \dot{x}^{\odot} \mid \int_0^{\max(0,t+\cdot)} \eta \rangle$$

$$=: \langle (x^{\odot}(0), \dot{x}^{\odot}) \mid (\int_0^t \eta, \int_0^{\max(0,t+\cdot)} \eta) \rangle$$

Thus, indeed

$$\int_0^t T_0^{\odot *}(t-s)\, r^{\odot *} \eta(s) = (\int_0^t \eta, \int_0^{\max(0,t+\cdot)} \eta),$$

or if we want to use the mapping $j^*(\varphi) = (\varphi(0), \varphi)$, $\varphi \in C[-h, 0]$,

$$j^{*-1} \int_0^t T_0^{\odot *}(t-s)\, r^{\odot *} \eta(s) = \int_0^{\max(0,t+\cdot)} \eta$$

. Done.

... **Ex 1 (a)** To the Business...

Suppose $x : [-h,\infty) \to \mathbb{R}$ is cont. and equals $\varphi \in C[-h,0]$ on $[-h,0]$, and satisfies

$$\dot{x}(t) = F(x_t), \quad t \geq 0, \qquad \text{(DDE)}$$

where $F : C[-h,0] \to \mathbb{R}$ is continuous.

The (DDE) is equivalent to the integral equation

$$x(t) = \overbrace{x(0)}^{\varphi(0)} + \int_0^t F(x_s)\,ds, \quad t \geq 0.$$

So, if we let $u(t,\theta) := x_t(\theta) = x(t+\theta)$, $t \geq 0$, $\theta \in [-h,0]$,

we see that, for $\underline{t+\theta \geq 0}$

$$u(t,\theta) = x(t+\theta)$$

$$= \varphi(0) + \underbrace{\int_0^{t+\theta} F(\underbrace{x_s}_{u(s,\cdot)})\,ds}_{j^{-1}\int_0^t T_0^{\odot *}(t-s)\, r^{\odot *}\, \underbrace{F(u(s,\cdot))}_{=:\, G(u(s,\cdot))}\,ds}$$

By the "lemma" we proved.

$$= (T_0(t)\varphi)(\theta) + j^{-1}\int_0^t T_0^{\odot *}(t-s)\, G(u(s,\cdot))\,ds,$$

so AIE is satisfied by $u(t,\cdot)$, in this case $\theta + t \geq 0$.

Ex 1(a): For $\theta + t \leq 0$ we have ~~simply~~

$$u(t,\theta) := x_t(\theta) = x(t+\theta) = \varphi(t+\theta)$$

$$= (T_0(t)\varphi)(\theta)$$

$$= (T_0(t)\varphi)(\theta) + \underbrace{j^{-1}\int_0^t T_0^{\odot *}(t-s)\, G(u(s,\cdot))\,ds}_{\text{because this is zero for } \theta + t \leq 0.}$$

Thus $u(t,\cdot)$ indeed satisfies (AIE).

Continuity of $t \mapsto u(t,\cdot) \in C[-h,0]$ can be checked directly. Alternatively, since $s \mapsto G(u(s,\cdot))$ is cont. the general theory of sun-star-perturbations says that

$$t \mapsto \int_0^t T_0^{\odot *}(t-s)\, G(u(s,\cdot))\,ds$$

is cont. $[0,\infty) \longrightarrow X = C[-h,0]$. Done.

Ex 1(b): Now let $u : [0,\infty) \to C[-h,0]$ be cont. and satisfy AIE:

$$u(t,\theta) = (T_0(t)\varphi)(\theta) + \underbrace{\left(j^{-1}\int_0^t T_0^{\odot *}(t-s)\, r^{\odot *} F(u(s,\cdot))\, ds\right)(\theta)}_{\int_0^{\max(0,\,t+\theta)} F(u(s,\cdot))\, ds},$$

so that, for $t+\theta \le 0$

$$u(t,\theta) = (T_0(t)\varphi)(\theta) + 0$$
$$= \varphi(t+\theta),$$

and for $t+\theta \ge 0$ $(T_0(t)\varphi)(\theta) = \varphi(0)$ and

$$u(t,\theta) = \varphi(0) + \int_0^{t+\theta} F(u(s,\cdot))\, ds.$$

Now define

$$x(t) := \begin{cases} u(t,0), & t \ge 0 \\ \varphi(t), & t \in [-h,0]. \end{cases}$$

Then if $t+\theta \le 0$ we have

$$x_t(\theta) := x(t+\theta) = \varphi(t+\theta) = u(t,\theta),$$

and also if $t+\theta \ge 0$

$$x_t(\theta) = x(t+\theta) = u(t+\theta, 0) = u(t,\theta)$$

Ex 1 (b):

So $\quad x_t(\cdot) = u(t,\cdot)\,,\ t \geq 0\,.$

Finally we see, for $t \geq 0$,

$$x(t) = u(t,0)$$

$$= \varphi(0) + \int_0^t F(\underbrace{u(s,\cdot)}_{x_s(\cdot)})\,ds$$

so x indeed satisfies

$$\begin{cases} \dot{x}(t) = F(x_t) & t \geq 0 \\ x(\theta) = \varphi(\theta) & \theta \in [-h,0]\,. \end{cases}$$

Lastly, $t \mapsto x_t$ is cont. $\mathbb{R} \to C[-h,0]$ (by unif. cont.), and F is cont. Therefore

$t \mapsto F(x_t)$ is cont.

$\Rightarrow t \mapsto \dot{x}(t)$ is cont for $t \geq 0$.