Semigroups of Delay eq, Solutions 8.

(1) Let $f \in \mathcal{X} = C_0$, BUC or L^p, $1 \le p \le \infty$, and $\{\mu_t\} \subseteq C_0^*(\mathbb{R}) = \mathcal{M}$, and define

$$T(t)f := f * \mu_t \qquad (t \geq 0)$$

Firstly, when is $f \mapsto f*\mu$ a <u>bounded</u> operator?

<u>Lemma</u>: $\boxed{\|f*\mu\|_p \le \|f\|_p \|\mu\|}$, $f \in L^p$, $1 \le p \le \infty$.

<u>Pf</u>: $p=\infty$ is clear. For $1 \le p < \infty$ we can assume $\|\mu\| = 1$, and then use Jensen's ineq.:

$$\int dx \left| \int d\mu_t(y) f(x-y) \right|^p \le \int dx \left(\int d|\mu_t|(y) |f(x-y)| \right)^p$$

$$\overset{(J)}{\le} \int dx \int d|\mu_t|(y) |f(x-y)|^p$$

$$= \|f\|_p^p \underbrace{\|\mu\|}_{=1} ; \quad \text{Done.}$$

Lemma instantly implies $\underline{T(t) \in \mathcal{B}(L^p)}$ for all p.

It's also easy to see that $T(t)\,\mathrm{BUC} \subseteq \mathrm{BUC}$ so $T(t) \in \mathcal{B}(\mathrm{BUC})$.

Finally one can check that $T(t)C_c \subseteq C_0$, so that $T(t)C_0 = T(t)\overline{C_c} \subseteq C_0$ too, and thus $T(t) \in \mathcal{B}(C_0)$.

Next, when does $T(t+s) = T(t)T(s)$ hold?

Note that the Schwartz-class $\mathcal{S}$ is a subset of $\mathfrak{X}$ for each choice of $\mathfrak{X}$. Therefore if T is a semigroup, then for all $\varphi \in \mathcal{S}$ holds

$$\varphi * \mu_{t+s} = \varphi * \mu_s * \mu_t \qquad \| \mathcal{F} = \wedge$$

$$\Leftrightarrow \hat{\varphi}\,\hat{\mu}_{t+s} = \hat{\varphi}\cdot\hat{\mu}_s\cdot\hat{\mu}_t$$

$$\Leftrightarrow \hat{\mu}_{t+s} = \hat{\mu}_s\cdot\hat{\mu}_t \qquad (t, s \geq 0).$$

Then if we assume measurability of mappings $t \mapsto \hat{\mu}_t(p)$, for each fixed $p \in \mathbb{R}$, it's well known fact that $\hat{\mu}_t(p) = e^{t\omega(p)}$ for some measurable $\omega: \mathbb{R} \to \mathbb{R}$.

But note that Fourier-T of a finite measure is a bounded func. so $\underline{\sup_p \operatorname{Re} \omega(p) < \infty}$.

Hence, a necessary cond. for the family $\{\mu_t\}$ is

$$\boxed{\mu_t = (e^{t\omega})^\vee} \qquad t \geq 0$$

for a measurable $\omega: \mathbb{R} \to \mathbb{R}$ such that $\sup \operatorname{Re} \omega < \infty$.

But once we have this explicit candidate it's easy to check that the above cond. is also sufficient:

$$\mu_t * \mu_s = (e^{t\omega})^\vee * (e^{s\omega})^\vee = (\underbrace{e^{t\omega}\cdot e^{s\omega}}_{e^{(t+s)\omega}})^\vee = \mu_{t+s} \qquad \text{Done}$$

Ex 2: When is T strongly cont. on $\mathcal{X}$?

As always, S.C. is enough to prove on a dense set. Schwartz class $\mathcal{S}$ is dense in L^p, $p<\infty$, and on $(C_0, \|\cdot\|_\infty)$, but not on L^∞ nor $BUC(\mathbb{R})$.

Case L^p, $2 \le p \le \infty$:

Riesz-Thorin Teorem implies that for $p \in [1,2]$ $\exists M_p$ s.t.

$\|\check{f}\|_q \le M_p \|f\|_p \quad (f \in L^p)$.

Now let $f \in \mathcal{S}$ so that $f * \mu_t \in \mathcal{S}$ too, meaning we can take it's F-transform $\widehat{f*\mu_t} \in \mathcal{S}$ and apply R-T. to it in the following way: Let $p \in [2,\infty]$

$$\|f*\mu_t - f\|_p \le M_q \|\widehat{f*\mu_t} - f\|_q$$
$$= M_q \|\hat{f}(\hat{\mu} - 1)\|_q$$
$$= M_q \|\hat{f}\,|e^{t\omega} - 1|\,\|_q \qquad (t \ge 0).$$

Now, as $t \to 0$, $|e^{t\omega} - 1| \to 0$ pointwise, and is bounded $|e^{t\omega}-1| \le e^{t \sup \mathrm{Re}\,\omega} + 1$. Since $\hat{f} \in L^q$, we can use DCT to infer

$$\exists \lim_{t\to 0} \|f*\mu_t - f\|_p = 0,$$

proving strong cont. on the set $\mathcal{S}$.

... Ex 2. As $\mathcal{T}$ is dense in spaces L^p, $p<\infty$ and $(C_0, \|\cdot\|_\infty)$, we have proved

T is S.C. on L^p, $p\in[2,\infty)$ and on $(C_0, \|\cdot\|_\infty)$.

As for the spaces L^∞ and BUC, I must admit I don't know. For what it's worth, I say I suspect strong cont. on BUC but not on L^∞.

Cases L^p, $1<p\le 2$

For $2\le p<\infty$, space L^p is *-reflexive.

Thus we know by general theory that

$(L^p)^\odot = (L^p)^* = L^q$,

and $T^\odot = T^*$ is strongly cont. Since, it's easy to check, T^* acts like

$T^*g(x) = \int d\mu_t(y)\, g(x+y) \qquad (x\in\mathbb{R},\ g\in L^q)$,

the original $T(t)$ must also be S.C. on L^q.

$\therefore T$ is S.C. on L^p, $1<p\le 2$.

We also note that $(L^p)^\odot = L^q$, $1<p<\infty$.

Case L^1

By well known fundamental result, S.C. will follow from weak cont.

Let, then $f \in L^1$ and $g \in L^{1*} = L^\infty$. Then

$$\langle f * \mu_t - f \mid g \rangle := \int dx\, g(x) \int d\mu_t(y) \left(f(x-y) - f(x) \right)$$

$$= \int d\mu_t(y) \underbrace{\int dx\, g(x) \left(f(x-y) - f(x) \right)}_{g(\tau_y f - f)},$$

where $g \in L^\infty$ is viewed as a distribution acting like

$$g(\varphi) := \int g(x) \varphi(x) \quad , \quad \varphi \in L^1 .$$

It's sufficiently clear that $y \longmapsto g(\tau_y f - f) =: \phi(y)$ is cont. and 0 at $y = 0$.

Now, since $\hat{\mu}_t = e^{t\omega} \to 1$ pointwise, we expect that $\mu_t \to^* \delta$ in some sense. But for that we need to assume the family $\{\mu_t\}$ is tight, meaning that $\forall \varepsilon > 0\ \exists K_\varepsilon$ compact s.t. $|\mu_t|(K_\varepsilon^c) \leq \varepsilon$ for all $t \geq 0$.

This extra assumption then allows us to assume the func. f is compactly supported (which it is not, but we may now assume that, for simplicity)

Then both μ_t and also ϕ have Fourier transforms which are functions and we can manipulate

$$\int d\mu_t(y)\,\phi(y) = \int d\hat{\mu}_t\,\hat{\phi}$$

$$= \int dp\, e^{t\omega(p)}\,\hat{\phi}(p)$$

$$\xrightarrow{(DCT)} \int dp\,\hat{\phi}(p)$$

$$= \check{\hat{\phi}}(0)$$

$$= \phi(0) = 0.$$

Thus T is weakly, and therefore strongly, continuous. Done.

Ex 3 $\mathcal{X}^{\odot}$ for C_0 and L^1.

$C_0^* = \mathcal{M}$ = "finite measures"

$\cong NBV(\mathbb{R}) = \{ f \in BV(\mathbb{R}) : \exists \lim_{x \to -\infty} f(x) = 0 \}$.

I'm fairly confident that $\mathcal{X}^{\odot} \cong L^1 dm = AC$,
but all I can really prove is that T^* is S.C. on $L^1 dm$.

T^* is S.C. on $L^1 dm \cong L^1$.

We regard C_0^* as NBV so that T^* acts on functions:

$$T^*(t) g = g \tilde{*} \mu_t$$
$$= \int d\mu_t(y)\, g(x+y) . \qquad (g \in NBV).$$

It's quick to check that $g \tilde{*} \mu_t \in NBV$ and that

$$d(T^*(t) g) = d(g \tilde{*} \mu_t)$$
$$= (g' \tilde{*} \mu_t) dx$$

IF g is AC ($dg = g' dx$).

Therefore, since $\| F \|_{NBV} = \| F' \|_{L^1}$ whenever $F \in AC$,

we see

$$\| T^*(t) g - g \|_{NBV} = \| g' \tilde{*} \mu_t - g' \|_{L^1} \to 0 \quad \text{as } t \to 0$$

by the S.C. on L^1. $\therefore T^*$ is S.C. $L^1 \cong AC$

Finally, I suspect $\underline{(L^1)^\odot = BUC}$, but, once again, I fail to give rigorous justification. It seems, at least when w is not bounded, that T^* is not S.C. on $L^\infty \setminus BUC$ but on BUC we indeed have:

$\underline{T_t^* \text{ and } T_t \text{ is S.C. on } BUC}$:

Let $f \in BUC(\mathbb{R})$. Then

$$\|f * \mu_t - f\|_\infty = \sup_{\|g\|_1 \leq 1} |\langle f * \mu_t - f \,|\, g \rangle|$$

$$= \sup \left| \int dx\, g(x) \int d\mu_t(y) \big(f(x-y) - f(x)\big) \right|$$

(Fubini)

$$= \sup \left| \int d\mu_t(y)\, g(\tau_y f - f) \right|$$

$$\leq \int d\mu_t(y) \|g\|_1 \|\tau_y f - f\|_\infty$$

$$\leq \int d|\mu_t|(y) \|\tau_y f - f\|_\infty$$

Now, translation is cont on BUC (contrast with L^∞!), so that, by the arguments identical to above, we get

$$\exists \lim_{t \to 0} \int d|\mu_t|(y) \|\tau_y f - f\|_\infty = 0.$$ Done.

Ex 4: Basic key observation is to note that, for each x,

$$C_0 \ni f \longmapsto Tf(x)$$

defines a bounded linear functional on C_0. By Riesz, we know then that

$$\boxed{Tf(x) = \mu_x(f)} \quad (f \in C_0)$$

for some family $\{\mu_x\}_{x \in \mathbb{R}}$ of measures.

Now, the fact that

$$T(G(y)f)(x) = (G(y)Tf)(x) = (Tf)(x+y)$$

translates into

$$\boxed{\mu_x \circ G(y) = \mu_{x+y}} \quad (x, y \in \mathbb{R})$$

In particular, if $x=0$, $\mu_0 \circ G(y) = \mu_y$, i.e.

$$\begin{aligned}(Tf)(x) &= \mu_x(f) \\ &= \mu_0(G(x)f) \\ &= \int d\mu_0(y)\, f(x+y) \\ &= \tilde{\mu}_0 * f(x)\end{aligned}$$

, where "$\tilde{\mu}(E) := \mu(-E)$". Done.