Ex 1 $\mathfrak{X}^* = L^\infty[0,h]$ under the pairing

$$\langle \psi | \varphi \rangle = \int_0^h d\theta\, \psi(\theta)\, \varphi(-\theta) \;, \quad (\psi \in L^\infty[0,h],\ \varphi \in L^1[-h,0]).$$

Proof: For

$$\langle \psi | T(t)\varphi \rangle = \int_t^h d\theta\, \psi(\theta)\, \varphi(-\theta + t)$$

$$(\theta = s+t) \quad = \int_0^{h-t} ds\, \psi(s+t)\, \varphi(-s)$$

$$= \int_0^h ds\, T^*(t)\psi(s)\, \varphi(-s)$$

where $T^*(t)\psi \equiv 0$ on $[h-t, h]$, and $\psi(s+t)$ on $\leq h-t$.

So T^* is a translation. Therefore it's generator clearly "acts" by differentiation. Only the domain $D(A^*)$ is to be determined. Since known that $A^*\psi = \psi'$ for $\psi \in D(A^*)$ we have, by def. of the domain,

$$D(A^*) = \{\psi \in L^\infty \mid \exists \psi' \in L^\infty,\ \psi(h) = 0\}$$

$$= \{\psi \in \mathrm{Lip}[0,h] \mid \psi(h) = 0\},$$

where the boundary condition follows from $D(A^*) = R(-A^*)^{-1}$ and

$$(-A^*)^{-1}\psi(\theta) = \int_0^\infty dt\, T^*(t)\psi(\theta) = \int_0^{h-\theta} dt\, \psi(\theta + t)$$

so that $(-A^*)^{-1}\psi(h) = 0$

... Ex 1: We now get $\mathcal{X}^{\odot}$ by the general identity $\underline{\mathcal{X}^{\odot} = \overline{\mathcal{D}(A^*)}}$. Since Lipschitz functions are dense in C, we clearly have

$$\underline{\mathcal{X}^{\odot} = C_0[0, h)}$$

Now $A^{\odot}$ is the part $A^*|_{\mathcal{X}^{\odot}}$ of A^* in $\mathcal{X}^{\odot}$:

$$\mathcal{D}(A^{\odot}) = \{\psi \in C_0[0,h) : \psi' \in C_0[0,h)\}$$

$$= \{\psi \in C_0^1[0,h) : \underline{\psi(0) = \psi'(0) = 0}\}$$

Next, by Riesz-Markov, $\underline{\mathcal{X}^{\odot *} = C_0^*[0,h) = NBV(-h,0]}$, and for $\varsigma \in NBV$, $\psi \in C_0[0,h)$

$$\langle \varsigma \,|\, T^{\odot}(t)\psi \rangle := \int_{-h}^{0} d\varsigma(\theta)\, T^{\odot}(t)\psi(-\theta)$$

$$= \int_{-h+t}^{0} d\varsigma(\theta)\, \psi(-\theta + t)$$

$$\theta = s + t \qquad = \int_{-h}^{-t} d\varsigma(s+t)\, \psi(-s)$$

$$= \int_{-h}^{0} d\big(T^{\odot *}(t)\varsigma(s)\big)\, \psi(-s)$$

where $T^{\odot *}(t)\varsigma \equiv c$ on $[-t, 0]$, $\varsigma(s+t)$ on $[-h, -t]$.

... <u>Ex 1</u>: Since $T^{\odot *}$ is, again, translation, generator $A^{\odot *}$ is, again, derivative on the domain

$$D(A^{\odot *}) = \{ \rho \in NBV[-h,0] : \exists \rho' \in NBV[-h,0] \}$$
$$= \{ \rho : \exists g \in NBV \text{ s.t. } \rho = \int_\theta^0 g \}$$

Each such ρ is uniquely def. by its derivative. In fact we have an isomorphism

$$D(A^{\odot *}) \xrightarrow{i} (NBV[-h,0], \|\cdot\|_{L^1})$$

<u>New norm</u>!

$$i(\rho) := \rho' .$$

i is clearly bijection and it's <u>isometry</u>, for

$$\| \int_\theta^0 g \|_{BV} = V\left(\int_\theta^0 g \right) = \int_{-h}^0 |g| = \|g\|_1 .$$

Therefore we also have $D(A^{\odot *}) = (NBV, \|\cdot\|_1)$, and from that it's easy to see

$$X^{\odot\odot} = \|\cdot\|_1\text{-closure}(NBV) = L^1[-h,0] \cong X .$$

Finally,

(As before)

$$D(A^{\odot\odot}) = \{ g \in L^1 : g' \in L^1 ,\ g(0) = 0 \}$$
$$= \{ g \in AC : g(0) = 0 \} .$$

Ex 2: We will make use of this fact:

For any $f \in C_0(\mathbb{R})$ (or even $C(\mathbb{R})$)

$$u(x,y) := P_y * f(x) \quad , \quad (x \in \mathbb{R}, \; y \geq 0)$$

is the unique harmonic extension to the upper half plane. Or, stated other way, any $u \in H(\mathbb{H}_+) \cap C(\overline{\mathbb{R}})$ has repres.

$$u(x,y) = [P_y * u(\cdot, 0)](x), \quad (x \in \mathbb{R}, \; y \geq 0).$$

Now, given $f \in C_0$, define $u(x,t) := P_t * f(x)$, and also $\tilde{u}_s(x,t) = u(x, t+s)$, both harm.

Then

$$\begin{aligned} T(t+s) f(x) &:= u(x, t+s) \\ &= \tilde{u}_s(x, t) \\ &= [P_t * \underbrace{\tilde{u}_s(\cdot, 0)}_{u(\cdot, s) = [P_s * u(\cdot, 0)](\cdot)}](x) \\ &= [P_t * [P_s * \underset{f}{\underset{\shortparallel}{u(\cdot, 0)}}]](x) \\ &= T(t) T(s) f(x) \end{aligned}$$

So, T is a semigroup.

For strong cont. we note that each $f \in C_0(\mathbb{R})$ is <u>unif.</u> cont.

$\therefore$ **Ex 2**: Then we simply estimate

$$|T(t)f(x) - f(x)| = \left| \int P_t(y) f(x-y) - \int P_t(y) f(x) \right|$$

$$\leq \int P_t(y)\, |f(x-y) - f(x)|$$

$$= \underbrace{\int_{-\varepsilon}^{\varepsilon} P_t(y) \underbrace{|f(x-y) - f(x)|}_{\delta_\varepsilon \to 0 \text{ as } \varepsilon \to 0}}_{\leq \delta_\varepsilon \underbrace{\int_{\mathbb{R}} P_t(y)}_{=1}} + \underbrace{\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} P_t(y) \underbrace{|f(x-y) - f(y)|}_{\leq 2\|f\|_\infty}}_{\to 0 \text{ by DCT.}}$$

$\Rightarrow \|T(t)f - f\|_\infty \to 0$. Done.

b) For $\mu \in \mathcal{M}(\mathbb{R}) = C_0^*(\mathbb{R})$ and $f \in C_0$ holds

$$\langle \mu \mid P_t * f \rangle := \int d\mu(x) \int dy\, P_t(x-y) f(y)$$

$$= \int dy\, f(y) \int d\mu(x)\, P_t(x-y)$$

$$= \langle f \mid P_t * \mu \rangle$$

$\therefore$ $\underline{T^*(t)\mu = P_t * \mu}$.

... Exc 2: Next: $C_0(\mathbb{R})^{\odot} = L^1(\mathbb{R}) \cong$ "$\underline{L^1(\mathbb{R})\, dm}$"

First, the abs. cont. measures $L^1 dm \subset \mathcal{M}$ is a $\underline{\text{closed}}$ subset of $\mathcal{M} = C_c^*$. Second, given $\mu \in \mathcal{M}$, $\underline{P_t * \mu \in \text{"}L^1 dm\text{"}}$ $\underset{t>0}{,}$ since $P_t * \mu(x) = u(x,t)$ is actually even a harmonic func. Therefore, if $\mu \in \mathfrak{X}^{\odot}$, then $L^1 dm \ni P_t * \mu \xrightarrow{t \to 0} \mu$, so that $\mu \in L^1 dm$ (by the closedness).

$\therefore\ C_0^{\odot} \subseteq L^1(\mathbb{R}) dm$, the abs. cont. meas.

Finally, let us pick $\mu \in L^1 \cdot dm \iff \exists h \in L^1(\mathbb{R})$ s.t. $\underline{d\mu = h \cdot dm}$, dm is the Lebesgue m.

For any $f \in C_0$, we estimate

$$|\langle f | \underbrace{P_t * \mu - \mu}_{\text{"}= P_t * h - h\text{"}} \rangle| = \Big| \int dx\, f(x) \underbrace{\big(P_t * h(x) - h(x)\big)}_{\int dy\, P_t(y)(h(x-y) - h(x))} \Big|$$

$$\leq \int dy\, P_t(y) \underbrace{\int dx\, |f(x)|\, |h(x-y) - h(x)|}_{\leq \|f\|_\infty \|\tau_y h - h\|_{L^1}}$$

... Ex 2: Next recall that translation $\tau_y : L^1 \to L^1$ is cont. so, now invoking the definition $\|\cdot\|_{\mathfrak{X}^c}$-norm, we have

$$\|P_t * \mu - \mu\| = \sup_{\|f\| \le 1} \langle f \mid P_t * \mu - \mu \rangle$$

$$\le \int dy\, P_t(y) \|\tau_y h - h\|_1$$

$$\to 0 \text{ as } t \to 0.$$

$\therefore d\mu = h\, dm \in C_c^{\infty}(\mathbb{R})$.

d) We already proved — well, at least used — this:

$$\|T(t)\mu\|_{L^1} = \|P_t * \mu\|_1$$

$$= \int dx \left| \int d\mu(y)\, P_t(x-y) \right|$$

$$\le \int d|\mu|(y) \underbrace{\int dx\, P_t(x-y)}_{=1}$$

$$= \|\mu\| < \infty$$

so $T(t)\mu \in L^1 = \mathfrak{X}^c$ $\square$

Ex 3: For $T(t)$ defined by $(T(t)x)_n = e^{-nt} x_n$,
it's clear that $T(0) = I$ and $T(t+s) = T(t)T(s)$.

Strong cont. for $1 \le p < \infty$:

$$\|T(t)x - x\|_{\ell^p}^p = \sum_n \underbrace{|e^{-nt} - 1|^p}_{\to 0 \text{ as } t \to 0} |x_n|^p \xrightarrow{t \to 0} 0 \text{ by } \underline{DCT}.$$

Strong cont. for $\mathcal{X} = c_0$:

$$\|T(t)x - x\|_{c_0} \le \underbrace{\sup_{n \le N_\varepsilon} |e^{-nt} - 1||x_n|}_{\xrightarrow{t\to 0} 0, \ \forall \varepsilon} + \underbrace{\sup_{n > N_\varepsilon} 2 \cdot |x_n|}_{\to 0 \text{ as } \varepsilon \to 0}.$$

Nod d) - follows immediately, as we note that
$(T^*(t)x^*)_n = e^{-nt} x_n^*$, $x^* \in \mathcal{X}^* = \ell^{p^*} = \ell^{p'}$.

So that $T^* = T^\odot$ and $\mathcal{X}^* = \mathcal{X}^\odot$.

Also, since $c_0^* = \ell^1$, so that T^* is strongly cont.,
we have $c_0^\odot = c_0^* = \ell^1$.

c) $\underline{\ell^{1\,\odot} = c_0}$

It's very easy to check that the generator is
$(Ax)_n = -nx_n$, $D(A) = \{x \in \mathcal{X} : (-nx_n)_n \in \mathcal{X}\}$.

... Ex 3: So, in particular, if $X = c_0$, $X^* = \ell^\infty$,

$D(A^*) = \{x^* \in \ell^\infty : (n x_n)_n \in \ell^\infty\}$

Claim: $\overline{D(A^*)} = c_0$:

Since c_0 is closed and $D(A^*) \subseteq c_0$, we have "$\subseteq$".

On the other hand $F := \{x^* \in \ell^\infty : x_n$ is eventually zero$\}$ clearly is a subspace of $D(A^*)$, and $c_0 = \overline{F}$ (easy to prove). Thus

$c_0 = \overline{F} \subseteq \overline{D(A^*)}$.

$\therefore (\ell^1)^{\odot} = \overline{D(A^*)} = c_0$ $\square$