Semigroups & Delay eq., Solutions

Ex 1. Suppose $\lambda \in \rho(A)$ is such that
$(\lambda - A)^{-1} Y \subseteq Y$.

Let $y \in Y$. Then, since $R(\lambda - A) = \mathcal{X} \supseteq Y$,
we can find $x \in D(A)$ such that

$$(\lambda - A)x = y \iff x = (\lambda - A)^{-1} y \in Y,$$

since $(\lambda - A)^{-1} Y \subseteq Y$. Thus two things follow:

1° $x \in D(A_1)$

2° $(\lambda - A_1)^{-1} y = (\lambda - A)^{-1} y$

$\therefore$ $\lambda \in \rho(A_1)$ and $R(\lambda, A_1) = R(\lambda, A)$. $\square$

Ex 2. By the very definition of the generator of a semigroup

$x^{\odot} \in D(A^{\odot}) \iff \exists \lim_{h \to 0} \frac{T^{\odot}(h) x^{\odot} - x^{\odot}}{h}$ in $X^{\odot}$

But since $T^{\odot} = T^{*}|_{X^{\odot}}$ this is equivalent to

$$\exists \lim_{h \to 0} \frac{T^{*}(h) x^{\odot} - x^{\odot}}{h} =: A^{*} x^{\odot} \quad \underline{\text{and } A^{*} x^{\odot} \in X^{\odot}}$$

$\iff x^{\odot}$ as an element of X^{*} belongs to $D(A^{*})$ and $A^{*} x^{\odot} \in X^{\odot}$.

$\iff x^{\odot} \in D(A^{*}|_{X^{\odot}})$. Done.

$\underline{\rho(A^{*}) \subseteq \rho(A^{\odot})}$: Follows from Ex. 1 with the observation $(\lambda - A^{*})^{-1} X^{\odot} \subseteq X^{\odot}$ for $\lambda \in \rho(A^{*})$

$\underline{\rho(A^{\odot}) \subseteq \rho(A^{*})}$: Let $\lambda \in \rho(A^{\odot})$ and fix $y \in X$.

For small enough $\varepsilon > 0$ $\exists (\mathbb{1} - \varepsilon A^{*})^{-1}$ too, and $(\mathbb{1} - \varepsilon A^{*})^{-1} y \xrightarrow{\|\cdot\|} y$ as $\varepsilon \to 0$.

Now, since $(\mathbb{1} - \varepsilon A^{*})^{-1} y \in D(A^{*}) \subseteq X^{\odot}$ we find $x_{\varepsilon} \in D(A^{\odot})$ such that $x_{\varepsilon} = (\lambda - A^{\odot})^{-1} (\mathbb{1} - \varepsilon A^{*})^{-1} y$. Then, by boundedness of $(\lambda - A^{\odot})^{-1}$ x_{ε} is Cauchy so $\exists \lim_{\varepsilon \to 0} x_{\varepsilon} =: x$

... Ex 2: Now we have that

$(\lambda - A^*) x_\varepsilon = (\lambda - A^{\odot}) x_\varepsilon$ converges (to y)

and $x_\varepsilon = (\lambda - A^{\odot})^{-1} (\mathbb{1} - \varepsilon A^*)^{-1} y$ converges to x

as $\varepsilon \to 0$.

Therefore, by closedness of A^*, we have

$x \in D(A^*)$ and $(\lambda - A^*) x = y$

$\therefore \lambda \in \rho(A^*)$ ~~too~~ $\square$

Ex 3: I use slightly different notation: vector ψ is denoted by $|\psi\rangle$ when it's considered as an element of a "original space", and by $\langle\psi|$ when it's acting as a functional on e.g. X^*.

With that convention B takes the form

$$B = \sum_j |r_j^{\odot *}\rangle\langle r_j^*| \;:\; X \to X^{\odot *}.$$

It's also rely easy to check that $B^{*'} = B^*|_{X^\odot}$ is

$$B^* = \sum_j |r_j^*\rangle\langle r_j^{\odot *}| \;:\; X^\odot \to X^*.$$

For warm up let us take the case "$M(\lambda)c = c$":

$$A|\psi\rangle = (A_0^{\odot *} + B)|\psi\rangle = \lambda|\psi\rangle$$

for $\lambda \in \rho(A_0^{\odot *})$ if and only if

$$(\lambda - A_0^{\odot *})^{-1} B|\psi\rangle = |\psi\rangle. \quad \circledast$$

This holds iff $|\psi\rangle \in \mathcal{R}((\lambda - A^{\odot *})^{-1}B) = \underline{\operatorname{span}_j |r_j(\lambda)\rangle}$,

which implies the representation

$$|\psi\rangle = \sum_j |r_j(\lambda)\rangle\langle r_j(\lambda)|\psi\rangle,$$

and $\circledast$ holds in the new representation...

... Ex 3 : ....

$$(\lambda - A^{\odot *})^{-1} \sum_i |r_i^{\odot *}\rangle\langle r_i^*| \sum_j |r_j(\lambda)\rangle \overbrace{\langle r_j(\lambda)|\psi\rangle}^{=: c_j}$$

$$= \sum_i \underbrace{(\lambda - A^{\odot *})^{-1}|r_i^{\odot *}\rangle}_{=: |r_i(\lambda)\rangle} \sum_j \underbrace{\langle r_i^* | r_j(\lambda)\rangle}_{=: M(\lambda)_{ij}} c_j$$

$$= \sum_i |r_i(\lambda)\rangle \, (M(\lambda)\bar{c})_i$$

$$= \text{"right side of } \circledast \text{"} = |\psi\rangle = \sum |r_j(\lambda)\rangle c_j .$$

Since $|r_j(\lambda)\rangle$ are independent, this is true iff

$M(\lambda)\bar{c} = \bar{c}$. Done

Now the dual case $A^*|\psi^{\odot}\rangle = \lambda|\psi^{\odot}\rangle$ is similar. As before it's easily checked

$$A^*|\psi^{\odot}\rangle = (A_0^{\odot} + B^*)|\psi^{\odot}\rangle = \lambda|\psi^{\odot}\rangle$$

iff $(\lambda - A_0^{\odot})^{-1} B^* |\psi^{\odot}\rangle = |\psi^{\odot}\rangle$. $(\dagger)$

Again this means $|\psi^{\odot}\rangle \in \mathcal{R}((\lambda - A_0^{\odot})^{-1} B^*) = \underline{\underline{\operatorname{span}_j |r_j^{\odot}(\lambda)\rangle}}$

so that $|\psi^{\odot}\rangle$ is a linear comb.

$$|\psi^{\odot}\rangle = \sum_i |r_i^{\odot}(\lambda)\rangle\langle r_i^{\odot}(\lambda)|\psi^{\odot}\rangle$$

and then $(\dagger)$ becomes ...

Ex 3: ...

$$(\lambda - A^{\odot})^{-1} \sum_i |r_i^*\rangle\langle r_i^{\odot *}| \sum_j |r_j^{\odot}(\lambda)\rangle \overbrace{\langle r_j^{\odot}(\lambda)|\varphi^{\odot}\rangle}^{=: d_j}$$

$$= \sum_i \underbrace{(\lambda - A^{\odot})^{-1}|r_i^*\rangle}_{|r_i^{\odot}(\lambda)\rangle} \sum_j \underbrace{\langle r_i^{\odot *}|r_j^{\odot}(\lambda)\rangle}_{M(\lambda)_{ji}} d_j$$

$$= \sum_i |r_i^{\odot}(\lambda)\rangle (\bar{d}^T M(\lambda))^i$$

$$= \text{"right side of (+)"} = |\varphi^{\odot}\rangle = \sum |r_i^{\odot}(\lambda)\rangle d_i$$

which holds iff $\bar{d}^T M(\lambda) = \bar{d}^T$ □

Remarks: a) For me $\bar{d}$ is not a row vector.

b) I defined $M(\lambda)_{ji}$ as $\langle r_i^* | r_j(\lambda)\rangle$ which is reversed order of i and j as compared to the lectures.

c) $\langle r_i^{\odot *} | r_j^{\odot}(\lambda)\rangle$ is $M(\lambda)_{ji}$ because, formally,

$$\langle r_i^{\odot *} | (\lambda - A^*)^{-1} r_j^* \rangle = \langle (\lambda - A^{\odot *})^{-1} r_i^{\odot *} | r_j^* \rangle$$
$$= \langle r_i(\lambda) | r_j^* \rangle$$
$$= \langle r_j^* | r_i(\lambda) \rangle$$
$$= M(\lambda)_{ji} \quad \text{(in my book)}.$$

Exc. 4 First define

$$q_{ij}(t) := \langle \gamma^{-1} \int_0^t T_0^{\odot *}(t-s) r_j^{\odot *} ds, r_i^* \rangle, \quad (t \geq 0).$$

Then q_{ij} is locally Lipschitz, for

$$|q_{ij}(t) - q_{ij}(\tilde{t})| \leq \int_{\tilde{t}}^t \underbrace{\| T_0^{\odot *}(t-s) r_j^{\odot *} \|}_{\leq M e^{\omega(t-s)} \| r_j^{\odot *} \|} ds \, \| r_i^* \|$$

(pick $\omega \neq 0$) $\quad \leq M \dfrac{e^{\omega \tilde{t}} - e^{\omega t}}{\omega} \| r_j^{\odot *} \| \| r_i^* \|$.

Thus q_{ij} is abs. cont. so $\exists \, k_{ij} \in L^1_{loc}$ s.t.

$$q_{ij}(t) = \int_0^t k_{ij}(s) \, ds,$$

and in fact k_{ij} must be L^∞_{loc} or otherwise q_{ij} is not Lipschitz.

Now let $\eta \in L^1_{loc}$. We show that

$$\langle \gamma^{-1} \int_0^s T_0^{\odot *}(s-\tau) r_j^{\odot *} \eta(\tau) d\tau, r_i^* \rangle = \int_0^s k_{ij}(s-\tau) \eta(\tau) d\tau$$

by showing the integrals $\int_0^t \ldots ds$ of both sides coincide for all $t \geq 0$:

Right side:

$$\int_0^t ds \int_0^s k_{ij}(s-\tau) \eta(\tau) d\tau = \int_0^t d\tau \left(\int_\tau^t k_{ij}(s-\tau) ds \right) \eta(\tau)$$

$$= \int_0^t d\tau \, q_{ij}(t-\tau) \eta(\tau) .$$

... **Ex 4:** Left side:

$$\int_0^t ds \left\langle j^{-1} \int_0^s T_0^{\odot *}(s-\tau)\, r_j^{\odot *}\, \eta(\tau)\, d\tau \,,\, r_i^* \right\rangle$$

$$= \left\langle j^{-1} \int_0^t d\tau\, \eta(\tau) \int_\tau^t ds\, T_0^{\odot *}(s-\tau)\, r_j^{\odot *} \,,\, r_i^* \right\rangle$$

$$= \int_0^t d\tau\, \eta(\tau)\, q_{ij}(t-\tau)$$

Done.

Lastly, take the Laplace T.:

$$\int_0^\infty dt\, e^{-\lambda t} \left\langle j^{-1} \int_0^t ds\, T_0^{\odot *}(t-s)\, r_j^{\odot *}\, \eta(s) \,;\, r_i^* \right\rangle$$

$$= \left\langle j^{-1} \int_0^\infty ds\, e^{-\lambda s}\, \eta(s) \int_s^\infty dt\, e^{-\lambda(t-s)}\, T_0^{\odot *}(t-s)\, r_j^{\odot *} \,,\, r_i^* \right\rangle$$

$$= \underbrace{\int_0^\infty ds\, e^{-\lambda s}\, \eta(s)}_{\hat{\eta}(\lambda)} \left\langle j^{-1} (\lambda - A^{\odot *})^{-1} r_j^{\odot *} \,,\, r_i^* \right\rangle$$

On the other hand, the Laplace T. of $k_{ij} * \eta$ is $\hat{k}_{ij}(\lambda)\, \eta(\lambda)$, so, since $\eta \in L^1_{loc}$ is arbitrary,

$$\hat{k}_{ij}(\lambda) = \langle \underbrace{j^{-1}(\lambda - A^{\odot *})^{-1} r_j^{\odot *}}_{r_j(\lambda)} \,,\, r_i^* \rangle = M(\lambda)_{ij}$$

(using the notation convention of the lectures) □

Ex 5: Given φ, the solution to

$$u(t) = T_0(t)\varphi + \int_0^t T_0^{\odot *}(t-s)\big(Bu(s) + f(s)\big)\, ds \qquad (1)$$

is unique. For, if u_1, u_2 are two solutions then

$$\|u_1(t) - u_2(t)\| \le \int_0^t \| T_0^{\odot *}(t-s)\, B(u_1(s) - u_2(s))\|\, ds$$

$$\le \int_0^t M e^{\omega(t-s)} \|B\| \|u_1(s) - u_2(s)\|$$

so $u_1(t) \equiv u_2(t)$ by Grönwall's lemma.

Therefore, all we have to do is show that solution u to

$$u(t) = T(t)\varphi + \int_0^t T^{\odot *}(t-s)\, f(s)\, ds \qquad (2)$$

satisfies (1) too.

Plugging (2) to the left of (1) gives

$$\underline{T_0(t)\varphi} + \int_0^t T_0^{\odot *}(t-s)\, B\, \underline{\underline{T(s)\varphi}} + \int_0^t T^{\odot *}(t-s)\, f(s)\, ds .$$

(Recall $T(t)\varphi = T_0\varphi + \int_0^t T_0^{\odot *}(t-s)\, B\, T(s)\varphi$).

Plugging (2) to right of (2) gives

$$\underline{T_0(t)\varphi} + \int_0^t T_0^{\odot *}(t-s)\, B\Big\{\underline{\underline{T(s)\varphi}} + \int_0^s T^{\odot *}(s-\tau)\, f(\tau)\, d\tau\Big\} + \int_0^t T_0^{\odot *}(t-s)\, f(s)$$

Making the underlined comparisons and cancellations it comes down to showing...

...Ex 5 ... that following holds

$$\int_0^t T^{\odot *}(t-s) f(s)\,ds = \underbrace{\int_0^t T_0^{\odot *}(t-s) B \int_0^s T^{\odot *}(s-\tau) f(\tau)\,d\tau}_{\int_0^t T_0^{\odot *}(\xi) B \int_0^{t-\xi} T^{\odot *}(t-\xi-\tau) f(\tau)\,d\tau} + \int_0^t T_0^{\odot *}(t-s) f(s) .$$

Since for each $x \in X$ and $t \geq 0$ $T(t)$ obeys

$$T(t)x = T_0(t)x + \int_0^t T_0^{\odot *}(t-s) B\, T(s) x ,$$

so it looks "formally right". The trouble, of course, is that $f(s) \in X^{\odot *}$.

Remedy is to show instead that the integrals $\int_0^{\tilde{t}} \dots dt$ over both sides match for all $\tilde{t} \geq 0$, and then use integrated semigroups. For example left side becomes

$$\int_0^{\tilde{t}} \int^t T^{\odot *}(t-s) f(s)\,ds = \int_0^{\tilde{t}} W(\tilde{t}-s) f(s)\,ds ,$$

where $W(\tilde{t}) := \int_0^{\tilde{t}} T^{\odot *}(t)\,dt$.

This approach works because W have a "smoothing effect" that $W : X^{\odot *} \to X$. The details are somewhat tedious I'm afraid. The remaining part of the proof/calculation can be found on page 75, prop. 2.5, in Perturbation Theory for dual Semigroups III.